Elementary Mathematical Logic

D. 1. (*Proposition*)
A *proposition* is a statement, which is either *true* (T) or *false* (F).

Ex. 1.
Say whether each of the following sentences is a proposition. In case of a proposition, determine its truth value:

1. The capital of Germany is Berlin.
2. How old is your father?
3. $5 \cdot 2 = 20$.
4. Switch the radio on.
5. Every even number greater than 2 is the sum of two primes.
6. $x > 13,\ x \in R$

Solution:
1. A proposition with the truth value T.
2. No proposition.
3. A proposition with the truth value F.
4. No proposition.
5. A proposition whose truth value is not known, the so-called *Goldbach's conjecture*.
6. No proposition. Substituting a real number for x will turn the statement into a proposition having a truth value.
   (Such a proposition is sometimes called a *proposition form*.)

R. 1. (*Two Principles*)

1. A proposition is either true or false.
2. A Proposition cannot be at the same time true and false.

(The two principles will later be formulated as assertions and proved.)

D. 2. (*Logical Quantifiers*)
We have the following *quantifiers*:

1. The *universal quantifier*: $\forall$ means "for all", "for every".
2. The *existence quantifier*: $\exists$ means "there exists at least one".
3. The *extended existence qualifier*: $\exists!$ "there exists exactly one".

R. 2. (*Logical Connectors*)
We have the following *connectors*:

1. Negation
2. Disjunction
3. Conjunction
4. Implication
5. Equivalence

D. 3. (*Negation*)
The *negation* of p is the proposition $\neg p$ which is true if, and only if, p is false:

p	$\neg p$
T	F
F	T

Ex. 2.

p: "Everybody knows Einstein."
$\neg p$: "At least one person does not know Einstein."

D. 4. (*Disjunction*)
The proposition p *or* q is true if, and only if, at least one of the two propositions is true. The *disjunction* will be denoted by $\vee$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex. 3.
Denote by

p: "$20:4=5$"
q: "$2>10$".

The (compound) proposition "$20:4=5$ or $2>10$" is true.

D. 5. (*Conjunction*)
The proposition p *and* q is true if, and only if, both propositions are true. The *conjunction* will be denoted by $\wedge$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex. 4.
Denote by

p: "$20:4=5$"
q: "$2>10$".

The (compound) proposition "$20:4=5$ and $2>10$" is false.

D. 6. (*Implication*)
The proposition p *implies* q is false if, and only if, p is true and q is false. The *implication* will be denoted by $\Rightarrow$.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex. 5.
Denote by

p: "$20:4=5$"
q: "$2>10$".

1. The (compound) proposition "If $20:4=5$ then $2>10$" is false.
2. The (compound) proposition "If $2>10$ then $20:4=5$" is true.

R. 2. (*Sufficient Condition, Necessary Condition*)
In the implication

$$p \Rightarrow q$$

p is the *sufficient condition* for q to be fulfilled; q is the *necessary condition* for p to be fulfilled.

D. 7. (*Equivalence*)
The *equivalence* $p \Leftrightarrow q$ is true whenever p and q have the same logical value.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex. 6.
Denote by

p: "$20:4=6$"
q: "$2>10$".

The compound proposition "$20:4=6$ if and only if $2>10$" is true.

R. 3. (*Sufficient and Necessary Condition*)
In the equivalence

$$p \Leftrightarrow q$$

p is both *sufficient and necessary condition* for q to be fulfilled; q is both *necessary and sufficient condition* for p to be fulfilled.

R. 4. (***Or Exclusion***)
In electronics and some programming languages, there is also an *or exclusion* (the so called *xor connector* defined as follows:
Let p and q be two propositions. The connection $p \; xor \, q$ is true if, and only if, p and q have different logical values.

p	q	p *xor* q
T	T	F
T	F	T
F	T	T
F	F	F

R. 5.
The following *truth table* summarises the above-mentioned logical connectors for the two propositions p and q:

Truth Table

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$	p *xor* q
T	T	F	T	F	T	T	F
T	F	F	T	F	F	F	T
F	T	F	T	F	T	F	T
F	F	T	F	T	T	T	F

D. 8. (***Tautology***)
A proposition which is always true is called a *tautology*.

D. 9. (***Contradiction***)
A proposition which is always false is called a *contradiction*.

Th. 1. (***See R.1.***)

1. A proposition is either true or false.
2. A Proposition cannot be at the same time true and false.

Proof:
Let p be a proposition.

1.
We prove the compound proposition:

$$p \vee \neg p$$

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

We have thus proved that $p \vee \neg p$ is a tautology.

2.
We prove the compound proposition:

$$\neg(p \wedge \neg p)$$

p	$\neg p$	$p \wedge \neg p$	$\neg(p \wedge \neg p)$
T	F	F	T
F	T	F	T

We have thus proved that $p \wedge \neg p$ is a contradiction and $\neg(p \wedge \neg p)$ is a tautology.

The above rules are also known as *rules of complementation*.

R. 6. (***Some Rules***)
Let p, q, and r be three propositions. Then we have the following rules:

1. *Commutative*

   $p \vee q \Leftrightarrow q \vee p$
   $p \wedge q \Leftrightarrow q \wedge p$

2. *Associative*

   $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
   $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

3. *Distributive*

   $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
   $p \vee (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

4. *Idempotent*

   $(p \vee p) \Leftrightarrow p$
   $(p \wedge p) \Leftrightarrow p$

5. *Absorption*

   $p \vee (p \wedge q) \Leftrightarrow p$
   $p \wedge (p \vee q) \Leftrightarrow p$

6. *Involution*

   $\neg\neg p \Leftrightarrow p$

7. De Morgan

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$
$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

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