# Chapter III

# Sequences

**<u>D. 3. 1.</u> (Sequence)**

An (infinite) *sequence* of real numbers is a mapping whose domain is an infinite subset of $\mathbb{N}$ and whose range is $\mathbb{R}$:

$$\{a_n\}, \quad n \in \mathbb{N}$$

**<u>R. 3. 1.</u>**

The general term of a sequence can be given by

1. an explicit formula
2. a recurrence formula

**<u>Ex. 3. 1.</u>**

1.

$$\{a_n\},\ n \in \mathbb{N}, \quad a_n = \frac{n}{n+1} \qquad \text{(explicit)}$$

2.

$$\{a_n\},\ n \in \mathbb{N}, \quad a_0 = 1, \qquad a_n = \frac{1}{2}\left(a_{n-1} + \frac{10}{a_{n-1}}\right), \qquad \text{(recurrent)}$$

3.

$$\{f_n\},\ n \in \mathbb{N}, \quad f_0 = f_1 = 1, \qquad f_{n+2} = f_{n+1} + f_n, \qquad n = 0, 1, \ldots \qquad \text{(recurrent)}$$

(This is the so-called Fibonacci sequence.)

**<u>D. 3. 2.</u> (Partial Sequence)**

Let $\{a_n\},\ n \in \mathbb{N}$ be a sequence of real numbers and $k_1, k_2, \ldots$ a sequence of natural number with $k_1 < k_2 < \ldots$

The sequence $\{a_{k_n}\}$, $k_n \in \mathbb{N}$ is called a *partial sequence* of the sequence of the sequence $\{a_n\}, \quad n \in \mathbb{N}$

**<u>D. 3. 3.</u> (Monotonic Sequences)**

The sequence $\{a_n\},\ n \in \mathbb{N}$, is

1. *constant* if $\quad a_{n+1} - a_n = 0, \quad \forall n \in \mathbb{N}$
2. *non-decreasing* if $\quad a_{n+1} - a_n \geq 0, \quad \forall n \in \mathbb{N}$

3. *non-increasing* if $a_{n+1} - a_n \le 0, \quad \forall n \in \mathbb{N}$

4. *increasing* if $a_{n+1} - a_n > 0, \quad \forall n \in \mathbb{N}$

5. *decreasing* if $a_{n+1} - a_n < 0, \quad \forall n \in \mathbb{N}$

6. *monotonic* if it is either (non-)decreasing or (non-)increasing.

**<u>D. 3. 4.</u> (Sequence Bounds)**

Let $\{a_n\}, \; n \in \mathbb{N}$, be a sequence.

1. $\{a_n\}, \; n \in \mathbb{N}$ is bounded above if $\exists \; \bar{B} \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, \; a_n \le \bar{B}$
2. $\{a_n\}, \; n \in \mathbb{N}$ is bounded below if $\exists \; \underline{B} \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, \; a_n \ge \underline{B}$
3. $\{a_n\}, \; n \in \mathbb{N}$ is bounded if it is bounded below and above.

   (i. e. if $\exists \; B_{\max} := \max\left\{\left|\underline{B}\right|, \; \left|\bar{B}\right|\right\} \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, \; |a_n| \le B_{\max}$ ).

**<u>D. 3. 6.</u> (Infimum, Supremum)**

Let $\{a_n\}, \; n \in \mathbb{N}$, be a sequence.

1. $\{a_n\}, \; n \in \mathbb{N}$ has a supremum *sup*, if

   i) $\exists \, sup \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, \; a_n \le sup$

   ii) $\forall \varepsilon > 0, \; \exists a_N : \; a_N > sup - \varepsilon$

2.. $\{a_n\}, \; n \in \mathbb{N}$ has an infimum *inf*, if

   iii) $\exists \, sup \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, \; a_n \ge sup$

   iv) $\forall \varepsilon > 0, \; \exists a_N : \; a_N > inf + \varepsilon$

**<u>D. 3. 7.</u> (Convergent and Divergent Sequences)**

The sequence $\{a_n\}, \; n \in \mathbb{N}$, is *convergent* if there exists a number $l \in \mathbb{R}$ fulfilling the following statement:

$$\forall \varepsilon > 0, \exists N_0 \in \mathbb{N} : \; n \ge N_0 \;\; \Rightarrow \;\; |a_n - l| < \varepsilon \quad (\textit{Cauchy's inequality}).$$

We write

$$\lim_{n \to \infty} a_n = l \, .$$

Otherwise is the sequence *divergent*.

**<u>T. 3. 1.</u>**
A convergent sequence is bounded.

*<u>Proof</u>*
Let $\{a_n\}$, $n \in \mathbb{N}$, be a convergent series with the limit sequence $l$. Then

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N_0 \quad \Rightarrow \quad |a - l| < \varepsilon .$$

Let

$$\underline{B} := \min\{a_1, a_2, ..., a_N, a - \varepsilon\}$$

$$\bar{B} := \max\{a_1, a_2, ..., a_N, a + \varepsilon\}.$$

Then

$$\forall \varepsilon > 0 \quad \Rightarrow \quad \underline{B} \leq a_n \leq \bar{B} .$$

Thus, the sequence is bounded.

**<u>R. 3. 2.</u>**
The converse is not true.

**<u>D. 3. 8.</u> (Arithmetic Sequences)**
The sequence $\{a_n\}$, $n \in \mathbb{N}$ is *arithmetic*, if there exists a number $d$, called the *difference of the sequence*, such that

$$a_{n+1} - a_n = d, \quad \forall n \in \mathbb{N}.$$

**<u>D. 3. 9.</u> (Geometric Sequences)**
The sequence $\{a_n\}$, $n \in \mathbb{N}$ is *geometric*, if there exists a number $q$, called the *ratio of the sequence*, such that

$$\frac{a_{n+1}}{a_n} = q, \quad a_n \neq 0, \quad \forall n \in \mathbb{N}$$

**<u>T. 3. 2.</u>**
Let $\{a_n\}$, $n \in \mathbb{N}$ be an arithmetic sequence whose difference is equal to $d$.

1. $d = 0 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is constant.
2. $d > 0 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is increasing and $\lim\limits_{n \to \infty} a_n = +\infty$ .
3. $d < 0 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is decreasing and $\lim\limits_{n \to \infty} a_n = -\infty$ .

**T. 3. 3.**

Let $\{a_n\}, n \in \mathbb{N}$ be a geometric sequence whose quotient is equal to $q$.

1. $q = 1 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is constant.
2. $q = 2 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is constant from its second term.
3. $|q| < 1 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is convergent and $\lim\limits_{n\to\infty} a_n = 0$.
4. $|q| > 1 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is divergent and
   $\lim\limits_{n\to\infty} a_n = -\infty$ if $a_1 < 0$,
   $\lim\limits_{n\to\infty} a_n = +\infty$ if $a_1 > 0$,
5. $q < -1 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is divergent and has no limit.
6. $q = -1 \quad \Rightarrow \quad \{a_n\}, n \in \mathbb{N}$ is periodic, with period equal to 2.

**D. 3. 10.**

1. The sequence $\{a_n\}, n \in \mathbb{N}$ is said to have the limit $+\infty$, if

$$\forall A > 0, \exists N_0 \in \mathbb{N}: \; n \geq N_0 \Rightarrow a_n > A \; .$$

We write $\lim\limits_{n\to\infty} a_n = +\infty$.

2. The sequence $\{a_n\}, n \in \mathbb{N}$ is said to have the limit $-\infty$, if

$$\forall A < 0, \exists N_0 \in \mathbb{N}: \; n \geq N_0 \Rightarrow a_n < A \; .$$

We write $\lim\limits_{n\to\infty} a_n = -\infty$.

**T. 3. 4.**

Let $\{a_n\}$ and $\{b_n\}$ $n \in \mathbb{N}$, be given with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Then

1. $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot a$
2. $\lim\limits_{n\to\infty} (a_n \pm b_n) = a \pm b$,
3. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = a \cdot b$,
4. $\lim\limits_{n\to\infty} \dfrac{1}{a_n} = \dfrac{1}{a}$, if $\forall n \in \mathbb{N}, a_n \neq 0$ and $\lim\limits_{n\to\infty} a_n \neq 0$
5. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$, if $\forall n \in \mathbb{N}, a_n \neq 0$ and $\lim\limits_{n\to\infty} b_n \neq 0$

**T. 3. 5.**

$$\{a_n\},\ a_n \geq 0,\ n \in \mathbb{N} \text{ is convergent with } \lim_{n\to\infty} a_n = l$$

$$\Rightarrow$$

$$l \geq 0$$

**C. 3. 1.**

Let $\{a_n\}$ and $\{b_n\},\ n \in \mathbb{N}$, be given with $\forall n,\ a_n \leq b_n$. Then

$$\{a_n\} \text{ and } \{b_n\}\ n \in \mathbb{N} \text{ are convergent with } \lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

$$\Rightarrow$$

$$a \leq b$$

**T. 3. 6.(Sandwich Theorem)**

Let $\{a_n\}$ and $\{b_n\},\ \{c_n\},\ n \in \mathbb{N}$, be given with $\forall n,\ a_n \leq c_n \leq b$ . Then

$$\{a_n\} \text{ and } \{b_n\}\ n \in \mathbb{N} \text{ are convergent with } \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l.$$

$$\Rightarrow$$

$$\{c_n\},\ n \in \mathbb{N} \text{ is convergent with } \lim_{n\to\infty} c_n = l$$

**C. 3. 2.**

$$\{a_n\},\ n \in \mathbb{N} \text{ has the limit } l \quad \Rightarrow \quad \{|a_n|\} \text{ has the limit } |l|$$

(The converse is not necessarily true!)

**T. 3. 7.**

Let $\{a_n\}, n \in \mathbb{N}$ be sequence. Then

1.

$$\{a_n\}, n \in \mathbb{N}, \text{ is increasing and bounded above}$$

$$\Rightarrow$$

$$\{a_n\}, n \in \mathbb{N}, \text{ is convergent}$$

2.

$\{a_n\}, n \in \mathbb{N}$, is decreasing and bounded below

$$\Rightarrow$$

$\{a_n\}, n \in \mathbb{N}$, is convergent

*(Last revised: 27.12.04)*