Integrals

Exercises (Solutions)

1.

$$\int (2x^3 - 3\sin x + 5\sqrt{x})\, dx = \int 2x^3\, dx + \int 3\sin x\, dx + \int 5\sqrt{x}\, dx$$

$$= 2\int x^3 dx - 3\int \sin x dx + 5\int x^{\frac{1}{2}} dx$$

$$= 2\cdot\frac{x^{3+1}}{3+1} - 3(-\cos x) + 5\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{2}x^4 + 3\cos x + \frac{10}{3}\, x\sqrt{x} + C\,, \quad C \in \mathbb{R}.$$

2.

$$\int\left(\frac{3}{\sqrt[3]{x}} + \frac{1}{2\sqrt{x}} + x\sqrt[4]{x}\right) dx = 3\int x^{-\frac{1}{3}} dx + \frac{1}{2}\int x^{\frac{1}{2}} dx + \int x^{\frac{3}{4}} dx$$

$$= 3\cdot\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + \frac{1}{2}\cdot\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + C$$

$$= \frac{9}{2}\sqrt[3]{x^2} + \sqrt{x} + \frac{4}{9}x^2\cdot + \sqrt[4]{x} + C\,, \quad C \in \mathbb{R}.$$

3.

Denote $t = x + 3$, then $\frac{dt}{dx} = 3$ and the integrand is of the form $\frac{\frac{dt}{dx}}{t(x)}$. Therefore

$$\int \frac{dx}{x+3} = \ln|x+3| + C\ , \ C \in \mathbb{R}.$$

4.

$$t = 7x\,, \qquad \frac{dt}{dx} = 7\,, \qquad dx = \frac{dt}{7}.$$

$$\int \cos 7x dx = \int \cos t\cdot\frac{dt}{7} = \frac{1}{7}\int \cos t dt = \frac{1}{7}\sin t + C = \frac{1}{7}\sin 7x + C\,, \quad C \in \mathbb{R}$$

5.

1.

Since

$$x^2 + 2x + 2 = (x+1)^2 + 1,$$

we have

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{1 + (1+x)^2}.$$

Let

$$u = 1 + x,$$

so

$$dx = du.$$

The we have

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{du}{1+u^2} = \arctan u + C$$

$$= \arctan(1+x) + C, \qquad C \in \mathbb{R}$$

2.

Let $u = \ln x$, so $du = -2x \cdot dx$. Then we have

$$\int \frac{x \cdot dx}{4 - x^2} = -\frac{1}{2}\int \frac{du}{u} + C$$

$$= -\frac{1}{2}\ln\left(4 - x^2\right) + C$$

$$= \ln \frac{1}{\sqrt{4 - x^2}} + C, \qquad C \in \mathbb{R}.$$

3.

Let $u = 1 + e^x$, so $du = e^x \cdot dx$ and hence $dx = \dfrac{du}{u-1}$. Then we have

$$\int \frac{dx}{1+e^x} = \int \frac{du}{u \cdot (u-1)} = \int \left(\frac{1}{u-1} - \frac{1}{u} \right) \cdot du$$

$$= \int \left(\frac{du}{u-1} \right) - \int \frac{du}{u} = \int \frac{d(u-1)}{u-1} - \int \frac{du}{u}$$

$$= \ln(u-1) - \ln u + C$$

$$= \ln e^x - \ln\left(1+e^x\right) + C$$

$$= \ln \frac{e^x}{1+e^x} + C = \ln\left(\frac{1+e^x}{e^x}\right)^{-1} + C$$

$$= -\ln\left(1+e^{-x}\right) + C, \qquad C \in \mathbb{R}.$$

4.

Let $u = \ln(x+1)$, so $du = \dfrac{dx}{x+1}$. Then, we have

$$\int \frac{\sqrt[3]{\ln(x+1)}}{x+1} dx = \int u^{\frac{1}{3}} du = \frac{3}{4} u^{\frac{4}{3}} + C$$

$$= \frac{3}{4}\left[\ln(x+1)\right]^{\frac{4}{3}} + C, \quad C \in \mathbb{R}.$$

6.

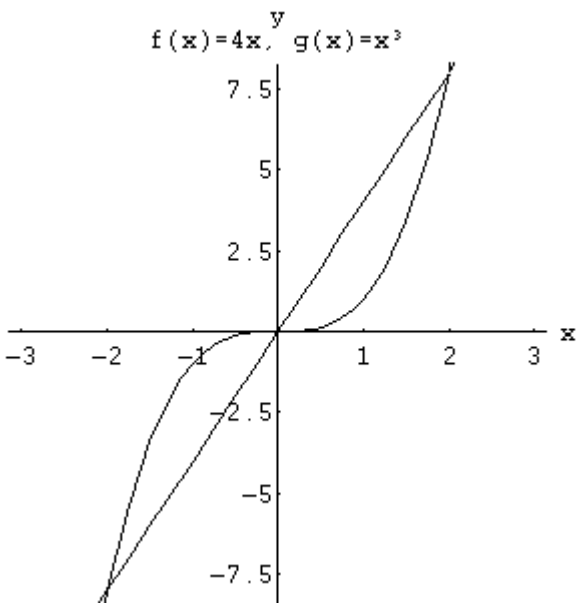
Denote $t = x^2 + 3x + 5$, then $\dfrac{dt}{dx} = 2x + 3$ and the integrand is of the form $\dfrac{\frac{dt}{dx}}{2 \cdot \sqrt{t(x)}}$.

Therefore

$$F(x) = \sqrt{x^2 + 3x + 5} + C, \qquad C \in \mathbb{R}.$$

7.

First, we draw the graphs of the two functions:



The points of intersection occur when $f(x) = g(x)$. So $4x = x^3$ implies $x = -2,\ 0, 1$ and hence $(-2,-8)$, $(0,\ 0)$, $(2,\ 8)$ are the points of intersection.
Observe that

$$g(x) \geq f(x), \quad \forall x[-2,\ 0] \quad \text{and} \qquad f(x) \geq g(x), \qquad \forall x[0,\ 2]\ .$$

Therefore, the required area is

$$A = \int_{-2}^{2} \left| f(x) - g(x) \right| dx = A_1 + A_2$$

$$= \int_{-2}^{0} \left(x^3 - 4x\right) dx + \int_{0}^{2} \left(4x - x^3\right) dx$$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} + \left[2x^2 - \frac{x^4}{4}\right]_{0}^{2}$$

$$= 0 - \left[\frac{(-2)^4}{4} - 2\cdot(-2)^2\right] + \left[2\cdot(2)^2 - \frac{2^4}{4}\right] = 8\ .$$

8.

$$\int_{2}^{\infty} \frac{dx}{x^2 - 1} = \lim_{t\to+\infty} \int_{2}^{t} \frac{dx}{(x-1)\cdot(x+1)}$$

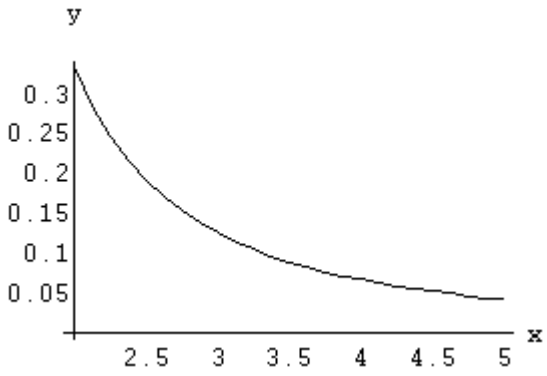
$$= \lim_{t\to+\infty} \int_{2}^{t} \left[\frac{1}{2\cdot(x-1)} - \frac{1}{2\cdot(x+1)}\right] \cdot dx \qquad \text{(by partial fraction)}$$

$$= \lim_{t\to+\infty} \left[\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right]_{2}^{t}$$

$$= \frac{1}{2}\lim_{t\to+\infty}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_{2}^{t} = \frac{1}{2}\lim\left[\ln\left|\frac{t-1}{t+1}\right| - \ln\frac{1}{3}\right]$$

$$= \frac{1}{2}\lim_{t\to+\infty}\ln\left|\frac{1-\frac{1}{t}}{1+\frac{1}{t}}\right| - \frac{1}{2}\lim_{t\to+\infty}\ln\frac{1}{3}$$

$$= \frac{1}{2}\cdot(0) + \frac{1}{2}\ln\left(\frac{1}{3}\right)^{-1} = \frac{1}{2}\ln 3 = \ln\sqrt{3}\ .$$



(To be continued.)