Limits of Functions

Exercises (Solutions)

1.
Here, we have $\varepsilon = 0.001$. To find an appropriate δ we need to obtain a connection between $|(4x-7)-1|$ and $|x-2|$.
Since

$$|(4x-7)-1| = |4x-8| = 4\cdot|x-2| < 0.001,$$

we obtain

$$|x-2| < \frac{0.001}{4} = 0.00025.$$

Thus, if we take $\delta = 0.00025$, we will always have

$$0 < |x-2| < 0.00025 \quad \Rightarrow \quad |(4x-7)-1| < 0.001.$$

2.
The inequality $\left|\left(\frac{1}{x^2}\right)-1\right| < 0.001$ has the following equivalences:

$$\begin{aligned}
\left|\left(\frac{1}{x^2}\right)-1\right| < 0.001 \quad &\Leftrightarrow \quad -0.001 < \frac{1}{x^2}-1 < 0.001 \\
&\Leftrightarrow \quad 1-0.001 < \frac{1}{x^2}-1 < 1+0.001 \\
&\Leftrightarrow \quad \frac{1}{1+0.001} < x^2 < \frac{1}{1-001} \\
&\Leftrightarrow \quad \frac{1}{\sqrt{1+0.001}}-1 < x-1 < \frac{1}{\sqrt{1-0.001}}-1 \\
&\Leftrightarrow \quad -0.00049961 < x-1 < 0.0005004.
\end{aligned}$$

This implies

$$|x-1| < \min\{|-0.0004996|, |0.005004|\}.$$

Thus, if we take $\delta = 0.0004996$, we have

$$|x-1| < 0.0004996 \quad \Rightarrow \quad \left|\frac{1}{x^2}-1\right| < 0.001.$$

3.
We are required to show that

$$\forall \varepsilon > 0,\ \ \exists \delta > 0 \text{, such that } 0 < |x-1| < \delta \ \Rightarrow\ \left|\left(x^3 + 5x^2\right) - 6\right| < \varepsilon .$$

Sine the value of δ depends on ε, we need to determine a connection between $\left|\left(x^3 + 5x^2\right) - 6\right|$ and $|x-1|$. So we have

$$\begin{aligned}\left|\left(x^3 + 5x^2\right) - 6\right| &= \left|(x-1)^3 + 8\cdot(x-1)^2 + 13\cdot(x-1)\right| \\ &\le \left|(x-1)^3\right| + \left|8\cdot(x-1)^2\right| + \left|13\cdot(x-1)\right| \\ &= |x-1|^3 + 8\cdot|x-1|^2 + 13\cdot|x-1| .\end{aligned}$$

Now, we choose $0 < \delta < 1$. If $n \in \mathbb{N}$ and $|x-1| < \delta$, we the have $|x-1|^n < \delta$. So we get

$$\begin{aligned}\left|x^3 + 5x^2 - 6\right| &\le |x-1|^3 + 8\cdot|x-1|^2 + 13\cdot|x-1| \\ &< \delta + 8\delta + 13\delta = 22\delta .\end{aligned}$$

No, if we choose $22\delta \le \varepsilon$, we have $\delta \le \dfrac{\varepsilon}{22}$.

Apparently, if we take $\delta < \min\{1,\ \dfrac{\varepsilon}{22}\}$, we have:

$$\forall \varepsilon > 0,\ \ \exists \delta > 0 \text{, such that } 0 < |x-1| < \delta \ \Rightarrow\ \left|\left(x^3 + 5x^2\right) - 6\right| < \varepsilon .$$

This proves that $\lim\limits_{x\to 1}\left(x^3 + 5x^2\right) = 6$.

4.
Let an arbitrary $\varepsilon > 0$ be given. For the inequality

$$|(3x+1) - 7| < \varepsilon$$

to be fulfilled it is necessary to have the following inequalities fulfilled:

$$|3x - 6| < \varepsilon ,\quad |x-2| < \frac{\varepsilon}{3},\quad -\frac{\varepsilon}{3} < x - 2 < \frac{\varepsilon}{3}$$

Thus, given any $\varepsilon > 0$, for all values of x satisfying the inequality

$$|x-2| < \frac{\varepsilon}{3} = \delta ,$$

the value of the function $f(x) := 3x + 1$ will differ from 7 by less than ε. And this means that 7 is the limit of the function as $x \to 2$.

5.
Here, the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

is not defined for $x = 2$. It is necessary to prove that for any arbitrary $\varepsilon > 0$, there will be a $\delta > 0$ such that the following inequality will be fulfilled:

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon \qquad \text{if} \qquad |x - 2| < \delta .$$

But when $x \neq 2$ the inequality $\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon$ is equivalent to the inequality

$$\left| \frac{(x+2)\cdot(x-2)}{x-2} - 4 \right| = |(x+2) - 4| < \varepsilon$$

or

$$|x - 2| < \varepsilon .$$

Thus for an arbitrary $\varepsilon > 0$, the inequality $\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon$ will be fulfilled if the inequality $|x - 2| < \varepsilon$ is fulfilled (here, $\delta = \varepsilon$), which means that the given function has the number 4 as its limit as $x \to 2$.

6.
It is necessary to prove that, for any arbitrary $\varepsilon > 0$, the following inequality will be fulfilled:

$$\left| \left(1 + \frac{1}{x}\right) - 1 \right| < \varepsilon ,$$

provided $|x| < N$, where N is determined by the choice of ε.

The above inequality is equivalent to the inequality $\left| \frac{1}{x} \right| < \varepsilon$, which will be fulfilled if

$$|x| > \frac{1}{\varepsilon} = N .$$

And this means that $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right) = 1$.

7.
(i)

$$\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\lim_{x\to 0}\frac{\left(\sqrt{x+a}-\sqrt{a}\right)\cdot\left(\sqrt{x+a}+\sqrt{a}\right)}{x\cdot\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$=\lim_{x\to 0}\frac{x+a-a}{x\cdot\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$=\lim_{x\to 0}\frac{1}{\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$=\frac{1}{2\sqrt{a}}.$$

(ii)

$$\lim_{x\to 1}\frac{1-x^3}{2-\sqrt{x^2+3}}=\lim_{x\to 1}\frac{\left(1-x^3\right)\cdot\left(2+\sqrt{x^2+3}\right)}{\left(2-\sqrt{x^2+3}\right)\cdot\left(2+\sqrt{x^2+3}\right)}$$

$$=\lim_{x\to 1}\frac{(1-x)\cdot\left(1+x+x^2\right)\cdot\left(2+\sqrt{x^2+3}\right)}{4-\left(x^2+3\right)}$$

$$=\lim_{x\to 1}\frac{(1-x)\cdot\left(1+x+x^2\right)\cdot\left(2+\sqrt{x^2+3}\right)}{(1-x)\cdot(1+x)}$$

$$=\lim_{x\to 1}\frac{\left(1+x+x^2\right)\cdot\left(2+\sqrt{x^2+3}\right)}{(1+x)}$$

$$=\frac{(1+1+1)\cdot(2+2)}{1+1}=6.$$

7.
For any $A>0$ we have

$$\frac{1}{(1-x)^2}>A,$$

provided

$$(1-x)^2<\frac{1}{A},\qquad |1-x|<\frac{1}{\sqrt{A}}=\delta$$

(The function assumes only positive values.)

8.
For any $A>0$ we have

$$\left|-\frac{1}{x}\right|>A,$$

provided

$$|x| = |x - 0| < \frac{1}{A} = \delta \ .$$

(Here $\left(-\frac{1}{x}\right) > 0$ for $x < 0$ and $\left(-\frac{1}{x}\right) < 0$ for $x > 0$

9.

1. $\lim\limits_{x \to \pm\infty} (4x^5 - 18x^4 + 7x^4 - x + 56) = \pm\infty$
2. $\lim\limits_{x \to +\infty} \frac{3x^3 - 5x + 2}{4x^2 - x + 6} = \lim\limits_{x \to +\infty} \frac{3x^3}{4x^2} = \lim\limits_{x \to +\infty} \frac{3}{4} x = +\infty$

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