Chapter II

Functions

<u>D. 2. 1.</u> (Mapping)
Let X and Y be two sets. Any subset $f \subseteq X \ x Y$ is called a *mapping* (or an *application*) from X to Y.

<u>D. 2. 2.</u> (Domain and Range)
Let X and Y be two sets and $f \subseteq X \ x Y$.
The set

$$D(f) := \{x \in X \mid \exists y \in Y : (a,b) \in f\}$$

is called the *domain* of f. x is a *pre-image* of y.
The set

$$R(f) := \{y \in Y \mid \exists x \in X : (a,b) \in f\}$$

is called the *Range* of f. ´y is an *image* of x.
We write

$$\begin{aligned} f : \ & X \to Y \\ & x \mapsto y \end{aligned}$$

<u>D. 2. 3.</u> (Equal Mappings)
Let f and g be two mappings.

$$f := g \quad \Leftrightarrow \quad D(f) = D(g) \ \wedge \ R(f) = R(g) \wedge \forall x \in X, \ f(x) = g(x).$$

<u>D. 2. 4.</u> (Composition of Mappings)
Let X, Y, Z be sets, $f \subseteq X \ x Y$ and $g \subseteq Y \ x Z$. Let further $R(f) \subseteq D(g)$.
The set

$$f \circ g := \{(x,y) \in X \ x Y \mid \exists \mathrm{y} \in Y : (x,y) \in f \wedge (y,z) \in g\}$$

is called the *composition* of f with g.

<u>R. 2. 1.</u>
Generally

$$f \circ g :\neq g \circ f,$$

even when both compositions are well defined.

<u>D. 2. 5.</u> (Injection)
$f : X \to Y$ is an *injection* (or is *injective*) $\Leftrightarrow \forall x_1 \in X, \forall x_2 \in X, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$,

or equivalently

$$\forall x_1 \in X, \forall x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

<u>R. 2. 2.</u>
Let $f : X \to Y$ and $g : Y \to Z$ be two mappings.

1.

f and g are injectives $\Rightarrow$ $g \circ f : X \to \mathrm{Z}$ is an injection.

2.

$g \circ f : X \to \mathrm{Z}$ is an injective $\Rightarrow$ f is injective

<u>D. 2. 6.</u> (Surjection)

$f : X \to Y$ is a *surjection* (or is *surjective*) $\Leftrightarrow \forall y \in Y, \exists x \in X : y = f(x)$,

<u>R. 2. 3.</u>
Let $f : X \to Y$ and $g : Y \to Z$ be two mappings.

1.

f and g are surjectives $\Rightarrow$ $g \circ f : X \to \mathrm{Z}$ is a surjection.

2.

$g \circ f : X \to \mathrm{Z}$ is a surjection $\Rightarrow$ f is surjective

<u>D. 2. 7.</u> (Bijection)
A function $f : X \to Y$ is *a bijection* (or is *bijective*) if it is both injective and surjective, i. e. . if every element of Y has exactly one pre-image in X .
Equivalently

$$f : X \to Y \text{ is a bijection } \Leftrightarrow \forall y \in Y, \exists x \in X : y = f(x)$$

<u>C. 2.1.</u>
Let $f : X \to Y$ and $g : Y \to Z$ be two mappings.

f and g are bijectives $\Rightarrow$ $g \circ f : X \to \mathrm{Z}$ is a bijection.

<u>D. 2. 8.</u> (Identity)
If X is a set, the *identity* of X is the mapping $Id_X : X \to X$ such that $\forall x \in X, Id(x) = x$.

<u>T. 2. 1.</u>

$$f : X \to Y \text{ is bijective } \Leftrightarrow \exists \text{ a mapping } g : Y \to X : g \circ f = Id_X \wedge f \circ g = Id_Y$$

<u>D. 2. 9.</u> (Inverse Mapping)
The mapping g in the theorem T. 2. 1. is called the *inverse* of f and is denoted $g := f^{-1}$.
(and $f := g^{-1}$).

<u>D. 2. 10.</u> (Function)
The mapping $f \subseteq X \times Y$ is *unique*, if

$$\forall x \in X, \forall y, z \in Y, ((x, y) \in f \wedge (x, z) \in f) \Rightarrow y = z.$$

A unique mapping will be called a *function.*

<u>D. 2. 11.</u> (Monotonous Functions)
Let f denote a function defined on an interval I.

1. f is *strictly increasing* on I if $\forall x_1 \in I, \forall x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
2. f is *increasing* on I if $\forall x_1 \in I, \forall x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
3. f is *strictly decreasing* on I if $\forall x_1 \in I, \forall x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
4. f is *decreasing* on I if $\forall x_1 \in I, \forall x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
5. f is (*strictly*) *monotonous* on I if it is either (strictly) increasing on I or (strictly) decreasing .
6. f is *constant* on I if $\forall x_1 \in I, \forall x_2 \in I,\ f(x_1) = f(x_2)$

<u>D. 2. 12.</u> (Extremal Points)
Let f denote a function defined on a domain $D(f)$. Let $x_0 \in D(f)$.

1. f has an *absolute* (or *global*) *maximum* at x_0 if $\forall x \in D(f),\ f(x) \leq f(x_0)$
2. f has a *relative*(or *local*) *maximum* at x_0 if $\forall x \in D(f) \cap U_\varepsilon(x_0),\ f(x) \leq f(x_0)$
3. f has an *absolute* (or *global*) *minimum* at x_0 if $\forall x \in D(f),\ f(x) \geq f(x_0)$
4. f has a *relative*(or *local*) *minimum* at x_0 if $\forall x \in D(f) \cap U_\varepsilon(x_0),\ f(x) \geq f(x_0)$
5. A maximum and a minimum are called *extremal points* (singular: *extremum*).