Continuous Functions

Exercises (Solutions)

1.
For example, take

$$f(x) := \begin{cases} -1 & for \quad x \le 0 \\ 1 & for \quad x > 0 \end{cases}, \qquad g(x) := \begin{cases} 1 & for \quad x \le 0 \\ -1 & for \quad x > 0 \end{cases}$$

f and g are discontinuous at 0, but $f + g$ is continuous at 0 ($f + g$ is constant on $\mathbb{R}$)

2.
For example, take

$$f(x) := \begin{cases} -1 & for \quad x \le 0 \\ 1 & for \quad x > 0 \end{cases}, \qquad g(x) := \begin{cases} 1 & for \quad x \le 0 \\ -1 & for \quad x > 0 \end{cases}$$

f and g are discontinuous at 0, but $f \cdot g$ is continuous at 0 ($f \cdot g$ is constant on $\mathbb{R}$)

3.
Indeed,

$$y_0 = x_0^2, \qquad y_0 + \Delta y = (x_0 + \Delta x)^2, \quad \Delta y = (x_0 + \Delta x)^2 - x_0^2 = 2x_0 \cdot \Delta x + \Delta x^2$$

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \left(2x_0 \cdot \Delta x + \Delta x^2\right) = 2x \cdot \lim_{\Delta x \to 0} \Delta x + \lim_{\Delta x \to 0} \Delta x \cdot \lim_{\Delta x \to 0} \Delta x = 0$$

4.
Indeed,

$$y_0 = \sin x_0, \qquad y_0 + \Delta y = \sin(x_0 + \Delta x),$$
$$\Delta y = \sin(x_0 + \Delta x) - \sin x_0 = 2 \cdot \sin \frac{\Delta x}{2} \cdot \cos\left(x_0 + \frac{\Delta x}{2}\right)$$

It can be shown that $\lim_{\Delta x \to 0} \sin \frac{\Delta x}{2} = 0$. The function $\cos\left(x + \frac{\Delta x}{2}\right)$ is bounded. Therefore,

$$\lim_{\Delta x \to 0} \Delta y = 0$$

(To be continued.)