Chapter VI

Differentiable Functions

<u>D. 6. 1.</u> (Differentiability at a Point)
Let f be a function defined on a neighbourhood of $x_0 \in D(f)$. The function f is differentiable at x_0 if

$$\frac{f(x) - f(x_0)}{x - x_0}$$

has a finite limit for $x \in N_\varepsilon(x_0)$.
This limit is called the *first derivative of* f *at* x_0 and is denoted by $f'(x_0)$:

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x) - f(x_0)}{h}$$

<u>R. 6. 1.</u>
Sometimes we write $\frac{df(x_0)}{dx}$ for $f'(x_0)$.

<u>T. 6. 1.</u>

f and g are differentiable at $x_0 \Rightarrow f + g$ is differentiable at x_0

(The converse is not true!)

and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0)$$

<u>T. 6. 2.</u>

f and g are differentiable at $x_0 \Rightarrow f \cdot g$ is differentiable at x_0

(The converse is not true!)

and

$$(f.g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0) \quad (\textit{Leibniz's Formula})$$

T. 6. 3.

$$g \text{ is differentiable at } x_0 \text{ and } g(x_0) \neq 0 \Rightarrow \frac{1}{g} \text{ is differentiable at } x_0$$

(The converse is not true!)

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{(g(x_0))^2}$$

T. 6. 4.

$$f \text{ and } g \text{ are differentiable at } x_0 \text{ and } g(x_0) \neq 0 \Rightarrow \frac{f}{g} \text{ is differentiable at } x_0$$

(The converse is not true!)

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\cdot g(x_0) - f(x_0)\cdot g'(x_0)}{(g(x_0))^2}$$

T. 6. 5.

$$f \text{ is differentiable at } x_0 \text{ and } g \text{ is differentiable at } y_0 = f(x_0)$$
$$\Rightarrow$$
$$g \circ f \text{ is differentiable at } x_0$$

(The converse is not true!)

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

($(g'(f(x_0))$: *outer derivative;* $f'(x_0)$ *: inner derivative*)

R. 6. 2. (Chain Rule)
Using the notation of R. 6. 1. with the following setting

$$x \xrightarrow{f} y \xrightarrow{g} z$$

we can write

$$\frac{dz}{dx} = \frac{dx}{dy} \cdot \frac{dy}{dz}$$

C. 1. 1.

Let f be a differentiable Function on $D(f')$. The $\ln|f|$ is also differentiable on $D(f')$ and we have

$$(\ln|f|)' = \frac{f'}{f}$$

<u>D. 6. 2.</u> (One-Sided Derivatives at a Point)
The Function f is differentiable on the left at x_0 if

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. This limit is called *the first derivative on the left of* f *at* x_0 and is denoted by

$$f_-' := \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The Function f is differentiable on the right at x_0 if

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. This limit is called *the first derivative on the right of* f *at* x_0 and is denoted by

$$f_+' := \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

<u>T. 6. 6.</u>
The function f is differentiable at x_0 if, and only if, it satisfies the following three conditions:

1. f is differentiable on the left at x_0
2. f is differentiable on the right at x_0
3. $f_-'(x_0) = f_+'(x_0)$

<u>T. 6. 7.</u>

f is differentiable at $x_0 \Rightarrow f$ is continuous at x_0

(The converse is not true!)

<u>D. 6. 3.</u> (Differentiability on an Interval)

1. The function f is differentiable on $]a, b[$ if it is differentiable at each point of I .
2. The function f is differentiable on $[a, b]$ if it is differentiable
   (1) at each point of $I =]a, b[$;
   (2) on the right at a ;
   (3) on the left at b .

<u>T. 6. 8.</u>
The differentiability can similarly be defined for other interval forms.

<u>R. 6. 3.</u>
If the function f is differentiable at x_0, then its graph has a tangent at the point $P(x_0, f(x_0))$. The slope of the tangent is equal to $f'(x_0)$.

(The equation of the tangent to the graph of f at $P(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

)

<u>T. 6. 9.</u> (Rolle's Theorem)
Let f be a function, continuous on $[a, b]$ and differentiable on $[a, b]$. If $f(a) = f(b)$, the there exists at least a point $c \in]a, b[$ such that $f'(c) = 0$.

<u>R. 6. 3.</u>

This means that, at least at one point, the graph of the function f has a tangent parallel to the x -axis.

<u>T. 6. 10.</u> (Lagrange's Theorem, First Form)

Let f be a function, continuous on $[a, b]$ and differentiable on $[a, b]$. There exists at least a point $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(c)}{b - a}.$$

<u>R. 6. 4.</u>

This means that, at least at one point, the graph of the function f has a tangent parallel to the cord connecting $(a, f(a))$ and $(b, f(b))$.

<u>T. 6. 11.</u> (Lagrange's Theorem, Second Form)
Let f be a function differentiable on a neighbourhood $N_\varepsilon(x_0)$ of x_0. Then for any real number h, there exists a $\theta \in]0, 1[$ such that

$$f(x) = f(x_0) + f'(x_0 + \theta h)$$

T. 6. 12.

Let f be a function defined and differentiable on an interval I

1. $\forall x \in I, \quad f'(x) = 0 \quad \Rightarrow \quad f$ is constant on I ;
2. $\forall x \in I, \quad f'(x) > 0 \quad \Rightarrow \quad f$ is strictly increasing on I ;
3. $\forall x \in I, \quad f'(x) \geq 0 \quad \Rightarrow \quad f$ is not decreasing on I ;
4. $\forall x \in I, \quad f'(x) < 0 \quad \Rightarrow \quad f$ is strictly decreasing on I ;
5. $\forall x \in I, \quad f'(x) \leq 0 \quad \Rightarrow \quad f$ is not increasing on I .

T. 6. 13.

f is differentiable at x_0 and has an extremun at x_0

$$\Rightarrow$$

$$f'(x_0) = 0$$

(The converse is not true!)

D. 6. 4.(Derivatives of Higher Order)

Let f be a function differentiable on a neighbourhood $N_\varepsilon(x_0)$ of x_0. If f' is differentiable at x_0, its first derivative $(f')'(x_0)$ is called *the second derivative of* f at x_0. By induction we define the $n-$th derivative of the function f at a point x_0, denoted by $f^{(n)}(x_0)$.

D. 6. 5.(Concavity and Convexity)

Let f be a function defined and differentiable, at least twice, on an interval I .

1. $\forall x \in I, \quad f''(x) < 0 \quad \Rightarrow \quad f$ is concave on I ;
2. $\forall x \in I, \quad f''(x) > 0 \quad \Rightarrow \quad f$ is convex on I ;

D. 6. 6.(Point of Inflexion)

Let f be a function, defined on an interval I , and let $x_0 \in \operatorname{int} I$. If concavity and convexity interchange at I , we say that x_0 is an *inflexion point of* f .

R. 6. 5.

Denote by G the graph of the function f and by P_0 a point with the coordinates $(x_0, f(x_0))$.
If f is convex at x_0 , then the tangent to G at P_0 is over G in a neighbourhood of P_0 .
If f is concave at x_0 , then the tangent to G at P_0 is under G in a neighbourhood of P_0 .

T. 6. 14.

Let f be a function defined on $N_\varepsilon(x_0)$. Suppose that f is (at least) twice differentiable at x_0 and $f'(x_0) = 0$.

1. If $f''(x_0) < 0$, then f has a maximum at x_0 .
2. If $f''(x_0) > 0$, then f has a minimum at x_0 .
3. If $f''(x_0) = 0$, then f has a point of inflexion at x_0 .

<u>T. 6. 15.</u>(L'Hospital's Rule)
Let f and g be two functions fulfilling the following conditions:

1. f and g are both differentiable on $N_\varepsilon(x_0)$
2. $\lim\limits_{x \to x_0} g'(x) \neq 0$
3. $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

<u>R. 6. 6.</u>
The result in T. 6. 15. remains valid if we replace condition 3. by $\lim\limits_{x \to x_0} f(x) = \infty$ and $\lim\limits_{x \to x_0} g(x) = \infty$.

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