Chapter VIII

Series of Real Numbers

<u>D. 8. 1.</u> (Series)
Let be given a sequence $\{a_n\}, n \in \mathbb{N}$ of real numbers. The expression

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \ldots + a_n + \ldots$$

is called the (*infinite*) *series* with the *general term* a_n.

<u>D. 8. 2.</u> (Sequence of Partial Sums)
The sequence $\{s_n\}, n \in \mathbb{N}$ whose general term is

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

is called the *sequence of partial sums* of the given series $\sum_{i=1}^{\infty} a_i$.

<u>D. 8. 3.</u> (Convergence and Divergence)
The series $\sum_{i=1}^{\infty} a_i$ is said to be *convergent* if is sequence of partial sums is convergent; otherwise it is *divergent*.
If the sequence of partial sums converges, its limit is called the *sum of the series*.

<u>D. 8. 3.</u> (Geometric Series)
A *geometric series* $\sum_{i=1}^{\infty} a_i$ is a series, where $\{a_n\}, n \in \mathbb{N}$ is a geometric series.

<u>D. 8. 4.</u> (Ratio of the Geometric Series)
The ratio of the sequence $\{a_n\}, n \in \mathbb{N}$ is called *the ratio of the geometric series*.

<u>R. 8. 1.</u>
Denote by $a_1 \neq 0$ be the first term of a geometric series and q its ratio. Because of

$$s_n = \sum_{i=1}^{n} a_i = a_1 \cdot \frac{1 - q^{n+1}}{1 - q}$$

we have

$$|q| < 1 \quad \Rightarrow \quad \text{the series is convergent with the sum } s = \frac{1}{1-q}$$

$$|q| \geq 1 \quad \Rightarrow \quad \text{the series is divergent}$$

T. 8. 1.

Let $A = \sum_{i=1}^{+\infty} a_i$ and $B = \sum_{i=1}^{+\infty} b_i$ be two convergent series. Then the series $\sum_{i=1}^{+\infty} (a_i + b_i)$ is convergent and its sum is equal to $A + B$.
(The converse is not true!).

T. 8. 2.

Let $A = \sum_{i=1}^{+\infty} a_i$ be a convergent series and $\alpha \in \mathbb{R}$. Then the series $\sum_{i=1}^{+\infty} \alpha \cdot a_i$ is convergent and its sum is equal to $\alpha \cdot A$.
(The converse is only true for $\alpha \neq 0$!).

R. 8. 2.

Together the last two theorems mean that the set of convergent series of real numbers is a real vector space.

R. 8. 3.

Adding or deleting from a series does not change the convergence/divergence. For a convergent series, its changes the value of the sum.

T. 8. 3. (Direct Comparison)

Let $\sum_{i=1}^{+\infty} a_i$ be a series of numbers such that $\forall i \in \mathbb{N}$, $a_i \geq 0$.

1.

Suppose that there exists a convergent series $\sum_{i=1}^{+\infty} b_i$ such that for a given natural number N, $i \geq N \Rightarrow 0 \leq a_i \leq b_i$. Then the series $\sum_{i=1}^{+\infty} a_i$ is convergent.

2.

Suppose that there exists a divergent series $\sum_{i=1}^{+\infty} c_i$ such that for a given natural number N, $i \geq N \Rightarrow 0 \leq c_i \leq a_i$. Then the series $\sum_{i=1}^{+\infty} a_i$ is divergent.

T. 8. 4. (Limit Comparison)

et $\sum_{i=1}^{+\infty} a_i$ and $\sum_{i=1}^{+\infty} b_i$ be two series of real numbers, such that $\forall i \in$, $a_i, b_i \geq 0$

1.

If there exists a real number L such that $\lim_{i \to +\infty} \frac{a_i}{b_i} = L$, then the two series either are both convergent or both divergent.

2.

If $\lim_{i \to +\infty} \frac{a_i}{b_i} = 0$ and $\sum_{i=1}^{+\infty} b_i$ is convergent, then $\sum_{i=1}^{+\infty} a_i$ is convergent too.

3.

If $\lim_{i\to+\infty} \frac{a_i}{b_i} = +\infty$ and $\sum_{i=1}^{+\infty} b_i$ is divergent, then $\sum_{i=1}^{+\infty} a_i$ is divergent too.

<u>T. 8. 5.</u> (Ratio Test, d'Alembert Test)

Let $\sum_{i=1}^{+\infty} a_i$ such that $\forall n \in \mathbb{N}, a_n \geq 0$. Suppose that

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l .$$

Then

1.	$l < 1$	$\Rightarrow$	$\sum_{i=1}^{+\infty} a_i$ is convergent
2.	$l > 1 \quad \vee \quad \mathrm{l} = +\infty$	$\Rightarrow$	$\sum_{i=1}^{+\infty} a_i$ is divergent
3.	$l = 1$	$\Rightarrow$	The test gives non conclusion. is convergent

<u>T. 8. 6.</u> (Root Test)

Let $\sum_{i=1}^{+\infty} a_i$ such that $\forall n \in \mathbb{N}, a_n \geq 0$. Suppose that

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = l.$$

Then

1.	$l < 1$	$\Rightarrow$	$\sum_{i=1}^{+\infty} a_i$ is convergent
2.	$l > 1 \quad \vee \quad \mathrm{l} = +\infty$	$\Rightarrow$	$\sum_{i=1}^{+\infty} a_i$ is divergent
3.	$l = 1$	$\Rightarrow$	The test gives non conclusion. is convergent

<u>D. 8. 5.</u> (Alternating Series)

An *alternating series* is a series where the terms of the series are alternatively positive and negative.

<u>T. 8. 7.</u> (Leibniz)

Let $\{a_n\}$, $n \in \mathbb{N}$, $a_n > 0$, $a_n \in \mathbb{R}$, be given. Suppose:

1. $\lim_{n\to+\infty} a_n = 0$;
2. $\exists N_0 > 0 : \forall n \geq N_0, a_{n+1} > a_n$

Then the alternating series $\sum_{i=1}^{+\infty} (-1)^{n+1} a_i$ is convergent.

T. 8. 8.

Take a convergent series $\sum_{i=1}^{+\infty}(-1)^{n+1}a_i$. Then

1.

For any $n \geq N_0$, the nth partial sum $s_n = \sum_{i=1}^{n\infty}(-1)^{n+1}a_i$ is an approximation of the sum of the series s, with an error which is less than $|a_{n+1}|$.

2.

The remainder $s - s_n$ and the first unused term a_{n+1} have the same sign.

D. 8. 6. (Absolute Convergence)

A series $\sum_{i=1}^{+\infty}a_i$ is *absolutely convergent*, if the series $\sum_{i=1}^{+\infty}|a_i|$ is convergent.

R. 8. 4.

If a series is absolutely convergent, it is convergent.

(The converse is not true!)

T. 8. 9.

Let $\sum_{i=1}^{+\infty}a_i$ and $\sum_{i=1}^{+\infty}b_i$, where the sequence $\{b_i\}$ is a rearrangement of the sequence $\{b_i\}$.

If $\sum_{i=1}^{+\infty}a_i$ is absolutely convergent, then $\sum_{i=1}^{+\infty}b_i$ is absolutely convergent, and the two series have the same sum.

R. 8. 5.

If the series $\sum_{i=1}^{+\infty}a_i$ is conditionally convergent, we can rearrange its terms so that it will show any predefined behaviour: divergence or convergence (with an arbitrary limit!)

(Last revised: 05.11.04)