Differentiable Functions

Exercises

1.
Prove that $f(x) = x^2$ is differentiable at $x_0 = 2$ and that $f'(2) = 4$.

2.
Show that the implication

$$f \text{ and } g \text{ are differentiable at } x_0 \;\Rightarrow\; f + g \text{ is differentiable at } x_0$$

can not be reversed (see T. 6. 1.)
.

3.
Show that the implication

$$f \text{ and } g \text{ are differentiable at } x_0 \;\Rightarrow\; f \cdot g \text{ is differentiable at } x_0$$

can not be reversed (see T. 6. 2.).

4.
Give an example to illustrate the theorem T .6. 3.

5.
Give an example to illustrate the theorem T .6. 4.

6.
Given the function $y = x^2$, find its derivative y'

1. at an arbitrary point x,
2. at $x = 3$.

7.
Given the function $y = \frac{1}{x}$, find its derivative y'.

8.
Find the tangents of the angles of inclination α_1, α_2 of the line tangent to the curve $y = x^2$ at the points $P_1\left(\frac{1}{2}, \frac{1}{4}\right)$ and $P_2(-1, 1)$.

9.
Find the slope of the tangent line to the curve

$$y = f(x) = \sqrt{x}$$

at the point $(4,\ 2)$ and write down the equation of the tangent line that passes through $(4,\ 2)$.

10.
Let

$$f(x) = \frac{3+x}{3-x}, \quad x \neq 3.$$

Evaluate $f'(2)$.

11.
Differentiate the function $f(x) = \ln(x^2+1)$ using the corollary C. 1. 1.

12.
A function $f(x)$ is defined on an interval $[0, \ 2]$ as follows:

$$f(x) := \begin{cases} x & when \quad 0 \le x \le 1 \\ 2x-1 & when \quad 1 < x \le 2 \end{cases}$$

Show that $f(x)$ has no derivative at $x = 1$, although it is continuous at this point.

13.
The function $f(x) = \sqrt[3]{x}$ is obviously defined at $x = 0$. Find out whether this function has a derivative at $x = 0$

14.
Find the derivatives of the following functions:

1. $y = \dfrac{1}{x\sqrt{x}}, \quad x > 0$
2. $y = 3 \cdot \dfrac{1}{\sqrt{x}}, \quad x > 0$
3. $y = 3x^4 - \dfrac{1}{\sqrt[3]{x}}, \quad x > 0$
4. $y = x^2 \cdot \sin x$
5. $y = \sqrt{x} \cdot \sin x \cdot \cos x$
6. $y = \dfrac{x^3}{\cos x}, \quad x \neq \dfrac{\pi}{2} + k \cdot \pi, \ k = 1, \ 2 \ldots$

15.
Find the derivatives of the following functions:

1. $y = \sin\left(x^2\right)$
2. $y = (\ln x)^2$
3. $y = \sin\left[(\ln x)^3\right]$

16.
Find the derivatives of the following functions:

1. $y = x^x$
2. $y = e^{x^2}$
3. $y = (\sin x)^{x^2}$
4. $y = \dfrac{(x+1)^2 \cdot \sqrt{x-1}}{(x+4)^3 \cdot e^x}, \qquad x \geq 1$

17.
Find the derivatives of the following functions:

1. $y = \arcsin e^x$
2. $y = \left(\arcsin \dfrac{1}{x}\right)^2, \quad x \neq 0$
3. $y = \arccos(\tan x)$.
4. $y = (\arctan x)^4$

18.
Find the differential dy and the increment Δy of the function $y = x^2$

1. for arbitrary values of x and dx;
2. for $x = 20, \ \Delta x = 0.1$.

19.
Approximate the value of $\sin 46^\circ$.

20.
Given a function $y = e^{kx}$ $(k = const)$ find the expression of its derivative at any order n.

21.
Let

$$f(x) = x^3 - x$$

Show that f satisfies all the conditions of Rolle's theorem on $[0, \ 1]$ and find all $c \in]0, \ 1[$ such that $f'(c) = 0$.

22.
Let

$$f(x) = x^3 - 3x + 1, \qquad x \in [0, \ 2].$$

Find all numbers $c \in [0,$ satisfying the conclusion of the Mean Value Theorem..

23.
Examine the following function for maximum and minimum

$$y = 2\sin x + \cos 2x$$

24.
Test for maximum and minimum the function

$$y = 1 - x^4 .$$

25.
Test for maximum and minimum the function

$$y = x^6 .$$

26.
Test for maximum and minimum the function

$$y = (x-1)^3$$

27.
Determine the maximum and minimum of the function

$$y = x^3 - 3x + 3$$

on the interval $\left[-3, \frac{3}{2}\right]$.

28.
Test the following function for maximum and minimum:

$$f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1 .$$

29.
If

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0 ,$$

find the values of a, b, c and d so that f has a local maximum at 4, a local minimum at 0, and passing through points $(0,\ 5)$ and $(4, 33)$.

30.
Find the points of inflection and determine the intervals of convexity and concavity of the curve

$$y = e^{-x^2} \qquad \text{(Gaussian curve)}$$

31.
Test the curve

$$y = x^4$$

for points of inflection.

32.
Investigate the following curve for points of inflection:

$$y = (x-1)^{\frac{1}{3}}$$

33.

Find the asymptotes of the curve

$$y = \frac{x^2 + 2x - 1}{x}.$$

34.

Find the asymptotes of the curve

$$y = e^{-x} \cdot \sin x + x.$$

35.

Investigate the function

$$y = f(x) = \frac{x}{1+x^2}$$

(Last revised: 12.02.05)