Differentiable Functions

Exercises (Solutions)

1.

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{x^2-4}{x-2}=x+2$$

$$f'(2)=\lim_{x\to 2}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to 2}(x+2)=4\,.$$

2.
Let us, for example, take

$$f(x):=\begin{cases}-1 & for \quad x\le 0\\ 1 & for \quad x>0\end{cases}$$

and

$$g(x):=\begin{cases}1 & for \quad x\le 0\\ -1 & for \quad x>0\end{cases}$$

The functions f and g are not differentiable at 0 (as they are not continuous at 0), but $f+g$ is differentiable at 0 ($f+g$ is constant on $\mathbb{R}$).

3.
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The functions f and g are not differentiable at 0 (as they are not continuous at 0), but $f\cdot g$ is differentiable at 0 ($f\cdot g$ is constant on $\mathbb{R}$).

4.
Let $f(x)=\dfrac{1}{\cos x}$ and $x_0=0$. The cosine function is differentiable at 0 and $\cos 0=1$. Thus f is differentiable at 0 and

$$f'(0) = -\frac{\sin 0}{\cos 0} = 0$$

5.

Let $f(x) = \tan x = \frac{\sin x}{\cos x}$. Sine the cosine and cosine functions are differentiable at 0 and $\cos 0 \neq 0$ and the tangent function is differentiable at 0, we have

$$(\tan)'(0) = \frac{\cos 0 \cdot \cos 0 - \sin 0 \cdot (-\sin 0)}{\cos^2 0} = \frac{1+0}{1} = 1$$

6.

1.

$$y + \Delta y = (x + \Delta x)^2, \quad \Delta y = (x + \Delta x)^2 - x^2 = 2x \cdot \Delta x + (\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = \frac{2x \cdot \Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

2.

$$y'(3) = 6$$

7.

$$y = \frac{1}{x}; \qquad y + \Delta y = \frac{1}{x + \Delta x},$$

$$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = -\frac{\Delta x}{x \div (x + \Delta x)}; \quad \frac{\Delta y}{\Delta x} = -\frac{1}{x \cdot (x + \Delta x)};$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[-\frac{1}{x \cdot (x + \Delta x} \right] = -\frac{1}{x^2}$$

8.

$$\tan \alpha_1 = y'\left(\frac{1}{2}\right) = 1; \qquad \tan \alpha_2 = y'(-1) = -2$$

9.

Let

$$y = f(x) = \sqrt{x}, \qquad x \in \mathbb{R}^+ \cup \{0\}.$$

Then the slope of the tangent line to the curve $y = f(x) = \sqrt{x}$ at point $(4, \ 2)$ it

$$m_T = \lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$$

$$= \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{\left(\sqrt{x}-2\right)\cdot\left(\sqrt{x}+2\right)}{(x-4)\cdot\left(\sqrt{x}+2\right)}$$

$$= \lim_{x\to 4}\frac{x-4}{(x-4)\cdot\left(\sqrt{x}+2\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x}+2}$$

$$= \frac{1}{4}.$$

The equation of the tangent line T to the curve passing through $(4,\ 2)$ is

$$y-2 = m_T\cdot(x-4)$$

$$= \frac{1}{4}(x-4).$$

10.

First solution method:

$$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2}\frac{\frac{3+x}{3-x}-\frac{3+2}{3-2}}{x-2}$$

$$= \lim_{x\to 2}\frac{6\cdot(x-2)}{(3-x)\cdot(x-2)}$$

$$= \lim_{x\to 2}\frac{6}{3-x} = 6.$$

Second solution method:

$$f'(2) = \lim_{\Delta x\to 0}\frac{f(2+\Delta x)-f(2)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\frac{3+(2+\Delta x)}{3-(2+\Delta x)}-\frac{3+2}{3-2}}{x-2}$$

$$= \lim_{\Delta x\to 0}\frac{6\cdot\Delta x}{\Delta x\cdot(1-\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{6}{1-\Delta x} = 6$$

11.

The function $f(x) = \ln(x^2+1)$ is defined and differentiable on $\mathbb{R}$. We have

$$\forall x \in \mathbb{R}, \quad f'(x) = \frac{2x}{x^2+1}$$

12.

When $\Delta x > 0$ we have

$$\lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[2(1+\Delta x-1] - [2\cdot 1 - 1]}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2\,.$$

When $\Delta x < 0$ we get

$$\lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[1+\Delta x] - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1\,.$$

Thus, this limit depends on the sign of Δx, and this means that the function has no derivative at the point $x = 1$.

Geometrically, this is in accord with the fact that at the point $x = 1$ the given „curve“ does not have a definite line tangent.

13.

We have

$$f(0) = 0\,, \qquad f(0+\Delta x) = \sqrt[3]{(\Delta x)}\,, \quad y + \Delta x = \sqrt[3]{(\Delta x)}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{(\Delta x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt[3]{\Delta x^2}} = +\infty$$

Consequently, the function is not differentiable at the point $x = 0$.

Geometrically, the line tangent to the curve at this point forms, with the x – axis, an angle $\frac{\pi}{2}$, which means that it coincides with the y – axis

14.

1.

$$y = \frac{1}{x\sqrt{x}} = x^{-\frac{3}{2}}, \qquad x > 0$$

$$y' = -\frac{3}{2} x^{-\frac{3}{2}-1} = -\frac{3}{2} x^{-\frac{5}{2}} = -\frac{3}{2x^2\sqrt{x}}, \quad x > 0$$

2.

$$y' = -3 \cdot \left(\frac{1}{\sqrt{x}} \right)' = 3 \cdot \left(x^{-\frac{1}{2}} \right)' = 3 \cdot \left(-\frac{1}{2} \right) \cdot x^{-\frac{1}{2}-1} = -\frac{3}{2} x^{-\frac{3}{2}},$$

or

$$y' = -\frac{3}{2x\sqrt{x}}, \qquad x > 0$$

3.

$$y' = 3(x^4)' - \left(x^{-\frac{1}{3}} \right)' = 3 \cdot 4x^3 - \left(-\frac{1}{3} \right) x^{-\frac{1}{3}-1};$$

$$y' = 12x^3 + \frac{1}{3} \cdot \frac{1}{x \cdot \sqrt[3]{x}}, \qquad x > 0$$

4.

$$y' = \left(x^2 \right) \cdot \sin x + x^2 \cdot (\sin x)'$$

$$y' = 2x \cdot \sin x + x^2 \cdot \cos x$$

5.

$$\begin{aligned} y' &= \left(\sqrt{x} \right) \cdot \sin x \cdot \cos x + \sqrt{x} \cdot (\sin x)' \cdot \cos x + \sqrt{x} \cdot \sin x \cdot (\cos x)' \\ &= \frac{1}{2\sqrt{x}} \cdot \sin x \cdot \cos x + \sqrt{x} \cdot \cos x \cdot \cos x + \sqrt{x} \cdot \sin x \cdot (-\sin x) \\ &= \frac{1}{2\sqrt{x}} \cdot \sin x \cdot \cos x + \sqrt{x} \cdot \left(\cos^2 x - \sin^2 x \right) \\ &= \frac{\sin 2x}{4\sqrt{x}} + \sqrt{x} \cdot \cos 2x, \qquad x > 0 \end{aligned}$$

6.

$$y' = \frac{\left(x^3 \right)' \cdot \cos x - x^3 \cdot (\cos x)'}{\cos^2 x} = \frac{3x^2 \cdot \cos x + x^3 \cdot \sin x}{\cos^2 x}, \qquad x \neq \frac{\pi}{2} + k \cdot \pi, \quad k = 1,\ 2 \ldots$$

15.

1.

$$u = x^2, \qquad \frac{du}{dx} = 2x;$$

$$y = \sin u, \qquad \frac{dy}{du} = \cos u;$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du} = 2x \cdot \cos u = 2x \cdot \cos\left(x^2 \right)$$

2.

$$y = (\ln x)^3$$

$$u = \ln x, \qquad \frac{du}{dx} = \frac{1}{x};$$

$$y = u^3, \qquad \frac{dy}{du} = 3u^2$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du} = \frac{3u^2}{x} = \frac{3(\ln x)^2}{x}.$$

3.

$$y = \sin\left[(\ln x)^3\right]$$

$$u = \ln x, \qquad \frac{du}{dx} = \frac{1}{x}$$

$$v = u^3, \qquad \frac{dv}{du} = 3u^2$$

$$y = \sin v, \qquad \frac{dy}{dv} = \cos v$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dv}{du} \cdot \frac{dy}{dv} = \frac{1}{x} \cdot 3u^2 \cdot \cos v = \frac{1}{x} \cdot 3 \cdot (\ln x)^2 \cdot \cos\left[(\ln x)^3\right].$$

16.

1.

$$y = x^x$$

$$\ln y = \ln x^x = x \cdot \ln x$$

$$\frac{y'}{y} = \ln x + \frac{x}{x} = \ln x + 1$$

$$y' = (\ln x + 1) \cdot x^x$$

2.

$$y = e^{x^2}$$

$$\ln y = x^2 \cdot \ln e = x^2$$

$$\frac{y'}{y} = 2x$$

$$y' = 2x \cdot e^{x^2}$$

3.

$$y = (\sin x)^{x^2}$$

$$\ln y = x^2 \cdot \ln(\sin x)$$

$$y' = x^2 \cdot (\sin x)^{x^2-1} \cdot (\sin x)' + (\sin x)^{x^2} \cdot (x^2)' \cdot \ln \sin x$$

$$= x^2 \cdot (\sin x)^{x^2-1} \cdot \cos x + (\sin x)^{x^2} \cdot 2x \cdot \ln \sin x$$

4.

$$\ln y = 2\ln(x+1) + \frac{1}{2}\ln(x-1) - 3\ln(x+4) - x)$$

$$\frac{y'}{y} = \frac{2}{x+1} + \frac{1}{2\cdot(x-1)} - \frac{3}{x+4} - 1, \qquad x > 1$$

$$y' = \frac{(x+1)^2 \cdot \sqrt{x-1}}{(x+4)^3 \cdot e^x} \cdot \left[\frac{2}{x+1} + \frac{1}{2\cdot(x-1)} + \frac{3}{x+4} - 1\right], \quad x > 1.$$

17.

1.

$$y' = \frac{1}{\sqrt{1-(e^x)^2}} \cdot (e^x)' = \frac{e^x}{\sqrt{1-e^{2x}}}$$

2.

$$y' = 2\arcsin\frac{1}{x} \cdot \frac{1}{\sqrt{1-\frac{1}{x_2}}} \cdot \left(\frac{1}{x}\right)'$$

$$= -2\arcsin\frac{1}{x} \cdot \frac{1}{x\cdot\sqrt{x^2-1}}, \quad -1 < x < +1 \ \wedge \ x \neq 0$$

3.

$$y' = -\frac{1}{\sqrt{1-\tan^2 x}} \cdot (\tan x)' = -\frac{1}{\sqrt{1-\tan^2 x}} \cdot \frac{1}{\cos^2 x}$$

4.

$$y' = 4\cdot(\arctan x)^3 \cdot (\arctan x)' = 4\cdot(\arctan x)^3 \cdot \frac{1}{1+x^2}$$

18

1.

$$\Delta y = (x+\Delta x)^2 - x^2 = 2x\Delta x + \Delta x^2,$$

$$dy = (x^2)\cdot\Delta x = 2x\Delta x$$

2.

If $x = 20, \Delta x = 1$, then

$$\Delta y = 2 \cdot 20 \cdot 0.1 + (0.1)^2 = 4.01 .$$

$$dy = 2 \cdot 20 \cdot 0.1 = 4.00$$

Replacing Δy by dy yields an error of 0.1. In many cases, it may be considered compared to $\Delta y = 4.01$ and therefore disregarded.

19.
Let $f(x) = \sin x$, then $f'(x) = \cos x$. We put

$$x = 45^\circ = \frac{\pi}{4}, \quad \Delta x = 1^\circ = \frac{\pi}{180}, \quad 45^\circ + 1^\circ = \frac{\pi}{4} + \frac{\pi}{180} .$$

Substituting into

$$\sin(x + \Delta x) \approx \sin x + \cos x \cdot dx$$

we get

$$\sin 46^\circ = \sin\left(\frac{\pi}{4} + \frac{\pi}{180}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} \cdot \frac{\pi}{180}$$

or

$$\sin 46^\circ = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{180} = 0.7071 + 0.7071 \cdot 0.017 = 0.7194 .$$

20.

$$y' = k \cdot e^{kx}, \quad y'' = k^2 \cdot e^{kx}, \ldots, \ y^{(n)} = k^n \cdot e^{kx}$$

21.
Since $f(x) = x^3 - x$ is a polynomial, f is continuous on $[0, 1]$ and differentiable on $]0, 1[$ Here, we note that $f(0) = f(1) = 0$. Thus, f satisfies all the conditions of Rolle's Theorem.

$$f'(x) = 3x^2 - 1 = \left(\sqrt{3}x - 1\right) \cdot \left(\sqrt{3}x + 1\right)$$

implies

$$f'\left(\frac{1}{\sqrt{3}}\right) = f'\left(-\frac{1}{\sqrt{3}}\right) = 0 .$$

Since $\frac{1}{\sqrt{3}} \in]0, 1[$, $c = \frac{1}{\sqrt{3}}$ is the desired answer.

22.
Since

$$f'(x) = 3x^2 - 3 \qquad \text{and} \qquad \frac{f(2) - f(0)}{2 - 0} = \frac{3-1}{2} = 1,$$

therefore,

$$3c^2 - 3 = 0$$

or, equivalently,

$$c = \pm\frac{2}{\sqrt{3}} \text{ -}$$

Since $c \in [0, \quad 2]$, we have

$$c = \frac{2}{\sqrt{3}}$$

as our desired solution.

23.
Since the function periodic with a period of 2π , it is sufficient to investigate it in the interval $[0, 2\pi]$.

$$y' = 2\cos x - 2\sin 2x = 2 \cdot (\cos x - 2\sin\cos x) = 2\cos x \cdot (1 - 2\sin x)$$

$$2\cos x \cdot (1 - 2\sin x) = 0,$$

$$x_1 = \frac{\pi}{6}; \qquad x_2 = \frac{\pi}{2}, \qquad x_3 = \frac{5\pi}{6}, \qquad x_4 = \frac{3\pi}{2}.$$

$$y'' = -2\sin x - 4\cos 2x$$

$$(y'')_{x=\frac{\pi}{6}} = -2 \cdot \frac{1}{2} - 4 \cdot \frac{1}{2} = -3 < 0.$$

Hence, at the point $x_1 = \frac{\pi}{6}$, we have maximum:

$$(y)_{x=\frac{\pi}{6}} = 2 \cdot \frac{1}{2} + \frac{1}{2} = \frac{3}{2}.$$

Further,

$$(y'')_{x=\frac{\pi}{2}} = -2 \cdot 1 + 4 \cdot 1 = 2 > 0.$$

An so at the point at the point $x_2 = \frac{\pi}{2}$, we have minimum:

$$(y)_{x=\frac{\pi}{2}} = 2\cdot 1 - 1 = 1.$$

At $x_3 = \dfrac{5\pi}{6}$ we have

$$(y'')_{x=\frac{5\pi}{6}} = -2\cdot\frac{1}{2} - 4\cdot\frac{1}{2} = -3 < 0\,.$$

Thus at $x_3 = \dfrac{5\pi}{6}$ the function has a maximum:

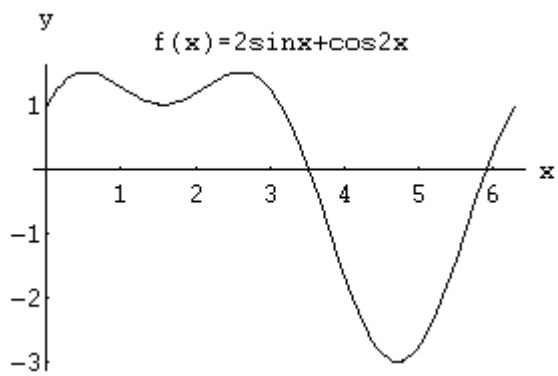
$$(y)_{x=\frac{5\pi}{6}} = 2\cdot\frac{1}{2} + \frac{1}{2} = \frac{3}{2}\,.$$

Finally,

$$(y'')_{x=\frac{3\pi}{2}} = -2\cdot(-1) - 1\cdot(-1) = 6 > 0\,.$$

Consequently, at $x_4 = \dfrac{3\pi}{2}$ we have a minimum:

$$(y)_{x=\frac{3\pi}{2}} = 2\cdot(-1) - 1 = -3\,.$$



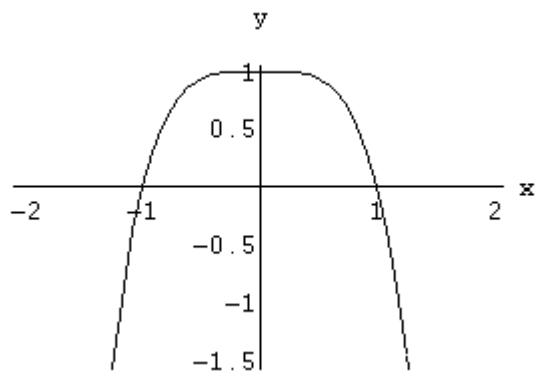
24.

$$y' = -4x^3, \qquad -4x^3 = 0, \qquad x = 0\,.$$

$$y'' = -12x^2, \quad (y'')_{x=0} = 0$$

$$(y')_{x<0} > 0\,, \quad (y')_{x>0} < 0\,,$$

Consequently, at $x = 0$ the function has a maximum, namely $(y)_{x=0} = 1$.



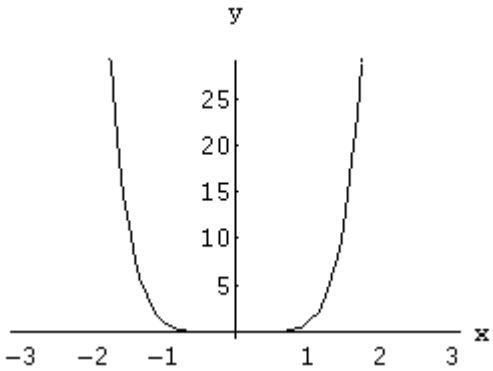
25.

$y' = 6x^5$, $\quad y' = 6x^5 = 0$, $\quad x = 0$;

$y'' = 30x^4$, $\quad (y'')_{x=0} = 0$

$(y')_{x<0} < 0$, $\quad (y')_{x>0} > 0$.

Therefore, at $x = 0$ the function has a minimum:



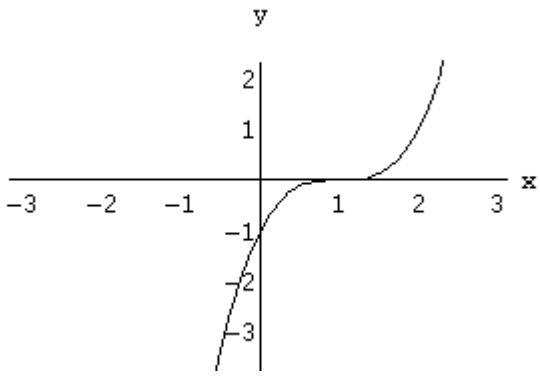
26.

$y' = 3 \cdot (x-1)^2$, $\quad 3 \cdot (x-1)^2 = 0$, $\quad x = 1$;

$y'' = 6 \cdot (x-1)$, $\quad (y'')_{x=1} = 0$

$(y')_{x<1} > 0$, $\quad (y')_{x>1} > 0$.

Consequently, at $x = 1$ the function does not have either a maximum or a minimum:



27.

$y' = 3x^2 - 3$, $\quad 3x^2 - 3 = 0$, $\quad x_1 = -1$, $\quad x_2 = 1$.

$$y'' = 6x$$

$$(y'')_{x=-1} = -6 < 0 .$$

Thus, at $x = -1$, we have a maximum:

$$(y)_{x=-1} = 5 .$$

Further

$$(y'')_{x=1} = 6 > 0$$

And so at $x = 1$, we have a minimum:

$$(y)_{x=1} = 5 .$$

We now determine the value of the function at the end points of the interval:

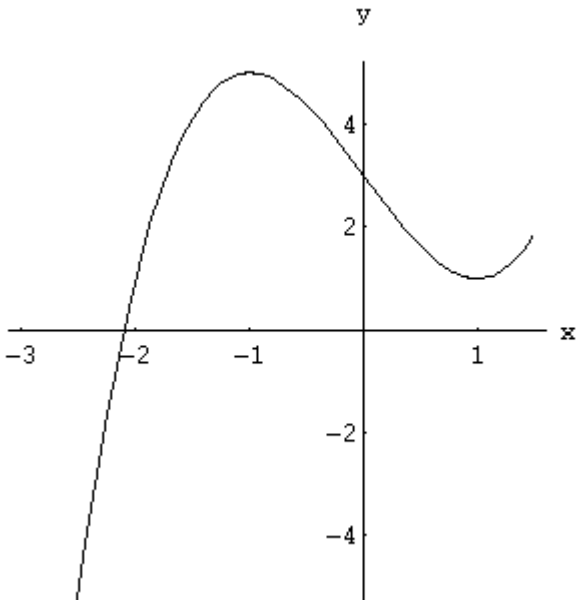
$$(y)_{x=-3} = -15 , \qquad (y)_{x=\frac{3}{2}} = \frac{15}{8} .$$

Thus, the greatest value of the function on the interval $\left[-3, \frac{3}{2}\right]$ is

$$(y)_{x=1} = 5$$

and the smallest value is

$$(y)_{x=-3} = -15$$



28.
Let us find the critical values of the function:

$$f'(x) = 4x^3 - 12x^2 + 12x - 4 = 4 \cdot (x^3 - 3x^2 + 3x - 1).$$

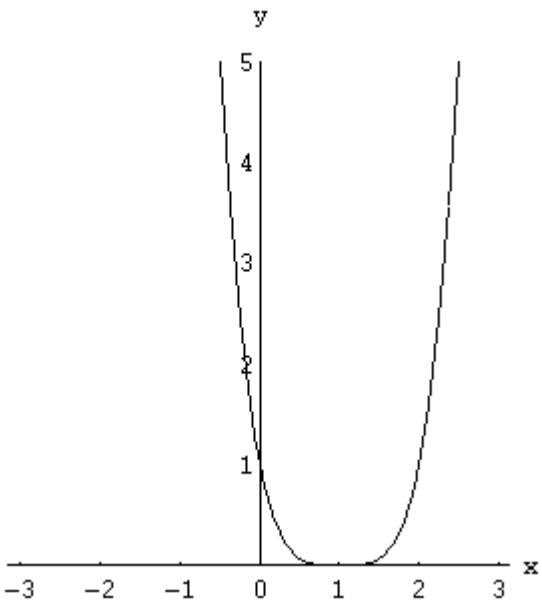
From the equation

$$4 \cdot (x^3 - 3x^2 + 3x - 1) = 0$$

we obtain the only critical point $x = 1$ (since this equation has only one real root).
We now investigate the character of the critical point $x = 1$:

$$f''(x) = 12x^2 - 24x + 12, \qquad f''(1) = 0$$
$$f'''(x) = 24x - 24, \qquad f'''(1) = 0$$
$$f^{(4)}(x) = 24 > 0.$$

Consequently, for $x = 1$ the function $f(x)$ has a minimum.



29.

$$f(0) = 5 \quad \Rightarrow \quad d = 5$$

$$f(4) = 33 \quad \Rightarrow \quad 64a + 16b + 4c = 28$$

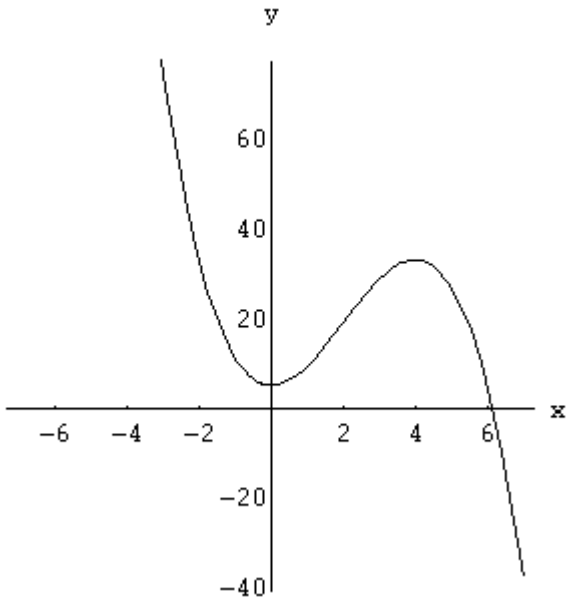
$$f'(x) = 3ax^2 + 2bx + c;$$

$$f'(0) = 0, \quad f'(4) = 0 \quad \Rightarrow \quad c = 0, \qquad 48a + 8b = 0$$

$$\begin{cases} 6a + b = 0 \\ 16a + 4b = 7 \end{cases} \quad \Rightarrow \quad a = -\frac{7}{8}, \quad b = \frac{21}{4}.$$

Thus, we have

$$f(x) = -\frac{7}{8} \cdot x^3 \frac{21}{4} \cdot x^2 + 5$$



30.

$$y' = -2x \cdot e^{-x^2}; \qquad y'' = 2e^{-x^2} \cdot (2x^2 - 1).$$

$$2e^{-x^2} \cdot (2x^2 - 1) = 0 \quad \Rightarrow \quad x_1 = -\frac{1}{\sqrt{2}}, \quad x_2 = \frac{1}{\sqrt{2}}.$$

For $x < -\frac{1}{\sqrt{2}}$ we have $y'' > 0$.

For $x > -\frac{1}{\sqrt{2}}$ we have $y'' < 0$;

The second derivative changes sign when passing through the point x_1. Hence, for $x_1 = -\frac{1}{\sqrt{2}}$, there is a point of inflection on the curve; its coordinates are: $\left(-\frac{1}{\sqrt{2}},\ e^{-\frac{1}{2}}\right)$.

For $x < \frac{1}{\sqrt{2}}$ we have $y'' < 0$.

For $x > \frac{1}{\sqrt{2}}$ we have $y'' > 0$;

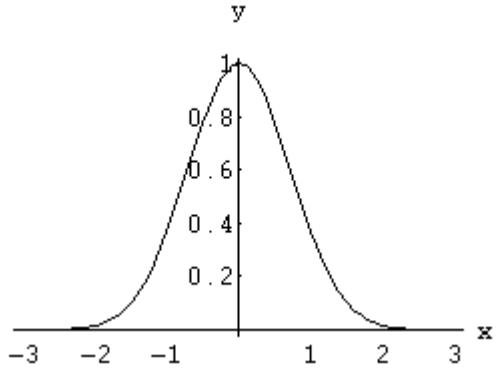
Thus, there is also a point of inflection on the curve for $x_2 = \frac{1}{\sqrt{2}}$; its coordinates are: $\left(\frac{1}{\sqrt{2}},\ e^{-\frac{1}{2}}\right)$.

Incidentally, the existence of the second point of inflection follows directly from the symmetry of the curve about the y – axis.
From the foregoing it follows that

for $-\infty < x < -\frac{1}{\sqrt{2}}$ the curve is convex;

for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ the curve is concave;

for $\frac{1}{\sqrt{2}} < x < +\infty$ the curve is convex.



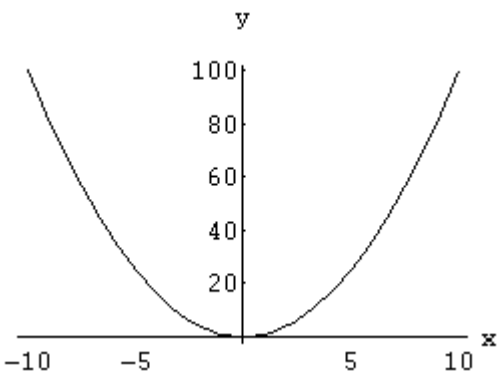
31.

$y'' = 4x^3$; $\qquad y'' = 12x^2$

$y'' = 0$; $\qquad y'' = 12x^2 = 0$; $\qquad x = 0$.

for $x < 0$, $\quad y'' > 0$, $\quad$ the curve is convex ;
for $x > 0$, $\quad y'' > 0$, $\quad$ the curve is convex .

Thus, the curve has no points of inflection.



32.

$y' = \frac{1}{3}(x-1)^{-\frac{2}{3}}$; $\qquad y'' = -\frac{2}{9}(x-1)^{-\frac{5}{3}}$

The second derivative does not vanish anywhere, but at $x = 1$ it does not exist ($y'' \to \pm\infty$).
Let us investigate the value $x = 1$:

for $x < 1$ $\quad y'' > 0$, the curve is convex;
for $x > 1$ $\quad y'' < 0$, the curve is concave.

Consequently, at $x = 1$ there is a point of inflection (1 ; 0).
It will be noted that for $x \to 1$ $\quad y' \to +\infty$; the curve at this point has a vertical tangent.

33.
1)
Look for vertical asymptotes:

when $x \to 0^-$ $\quad y \to +\infty$
when $x \to 0^+$ $\quad y \to -\infty$

Therefore, the straight line $x = 0$ is a vertical asymptote.

2)
Look for inclined asymptotes:

$$k = \lim_{x \to \pm\infty} \frac{y}{x} = \lim_{x \to \pm\infty} \frac{x^2 + 2x - 1}{x} = \lim_{x \to \pm\infty} \left[1 + \frac{2}{x} - \frac{1}{x^2} \right] = 1,$$

that is

$$b = \lim_{x \to \pm\infty} [y - x] = \lim_{x \to \pm\infty} \left[\frac{x^2 + 2x - 1}{x} - x \right] = \lim_{x \to \pm\infty} \left[\frac{x^2 + 2x - 1 - x^2}{x} \right] = \lim_{x \to \pm\infty} \left[2 - \frac{1}{x} \right] = 2,$$

or, finally,

$$b = 2 \quad .$$

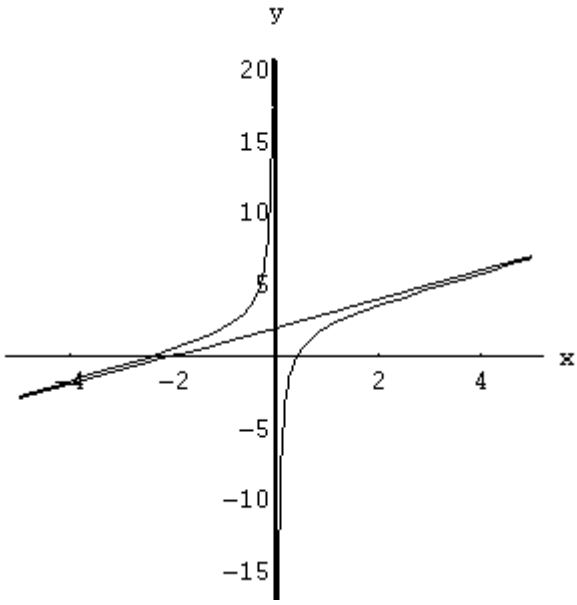
Therefore, the straight line

$$y = x + 2$$

is an inclined asymptote to the given curve.
To investigate the mutual position of a curve and an asymptote, let us consider the difference of the ordinates of the curve and the asymptote for one and the same value of x:

$$\frac{x^2 + 2x - 1}{x} - (x + 2) = -\frac{1}{x}.$$

This difference is negative for $x > 0$, and positive for $x < 0$; and so for $x > 0$ the curve lies below the asymptote, and for $x < 0$, it lies above the asymptote.



34.

1)

It is obvious that there are no vertical asymptotes.

2)

Look for inclined asymptotes:

$$k = \lim_{x\to+\infty} \frac{y}{x} = \lim_{x\to+\infty} \frac{e^{-x}\sin x + x}{x} = \lim_{x\to+\infty}\left[\frac{e^{-x}\cdot\sin x}{x} + 1\right] = 1.$$

$$b = \lim_{x\to+\infty}\left[e^{-x}\cdot\sin x + x - x\right] = \lim_{x\to+\infty} e^{-x}\cdot\sin x = 0\,.$$

Hence, the straight line

$$y = x$$

is an inclined asymptote as $x \to +\infty$.

The given curve has no asymptote as $x \to -\infty$. Indeed, the limit $\lim\limits_{x\to-\infty}\frac{y}{x}$ does not exist, since $\frac{y}{x} = \frac{e^{-x}\cdot\sin x}{x} + 1$. (Here, the first term increases without bound as $x \to -\infty$.)

35.

$$y = f(x) = \frac{x}{1+x^2}$$

1. Domain

$D(f) = \mathbb{R}$.

2. Continuity

The function is continuous in $D(f)$.

3. Symmetry

The function is odd, since

$$-f(-x) = -\frac{-x}{1+(-x)^2} = \frac{x}{1+x^2} = f(x), \qquad \forall x \in D(f).$$

4. Points of Intersection with the Axes

$$x = 0 \quad \Leftrightarrow \quad y = 0$$

5. Domains of Decrease and Increase

$$f'(x) = \frac{1-x^2}{\left(1+x^2\right)^2}$$

$$f'(x) < 0 \;\Rightarrow\; 1-x^2 < 0 \;\Rightarrow\; (1-x)\cdot(1+x) < 0 \;\Rightarrow\; x \in]-\infty, -1[\, \cup]1, +\infty[$$

$f(x)$ decreases in $]-\infty, -1[\, \cup]1, +\infty[$ and increases in $]-1, +1[$.

6. Extremal Points

$$f'(x) = \frac{1-x^2}{\left(1+x^2\right)^2} := 0 \;\Rightarrow\; x_1 = -1, \qquad x_2 = +1 \quad \text{(critical points).}$$

$$f''(x) = \frac{2x\cdot\left(x^2-3\right)}{\left(1+x^2\right)^3}$$

$$f'(-1) > 0, \qquad f'(+1) < 0.$$

Hence, at $x_1 = -1$ the function $f(x)$ has a relative minimum and at $x_1 = +1$ a relative maximum:

$$f(-1) = -1, \qquad f(+1) = +1$$

7. Domains of Convexity and Concavity

$$f''(x) = \frac{2x\cdot\left(x^2-3\right)}{\left(1+x^2\right)^3} > 0 \;\Rightarrow\; x\cdot\left(x-\sqrt{3}\right)\cdot\left(x+\sqrt{3}\right) > 0$$

$$\Rightarrow$$

$f(x)$ is convex in $]-\infty, -\sqrt{3}[\, \cup]0, \sqrt{3}[$, concave in $]-\sqrt{3}, 0[\, \cup]3, +\infty[$.

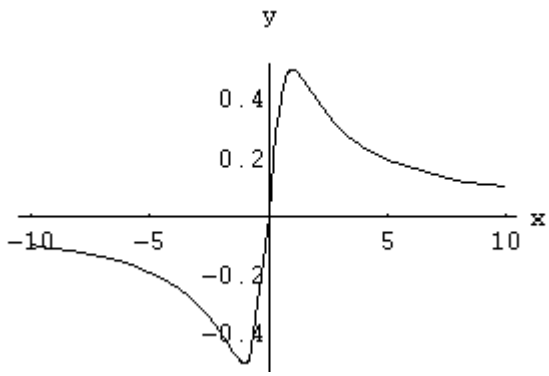
It hat the following *points of inflection*:

$$\left(-\sqrt{3},\ -\frac{\sqrt{3}}{4}\right), \qquad (0,\ 0), \qquad \left(\sqrt{3},\ \frac{\sqrt{3}}{4}\right).$$

8. Asymptotes

For $x \to +\infty \quad y \to 0$, for $x \to -\infty \quad y \to 0$. Consequently, the straight line $y = 0$ is the only inclined asymptote. The curve has no vertical asymptotes because the function does not approach infinity for a single finite value of x.

9. Graph of the Curve



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