Chapter VII

Integrals

<u>D. 7. 1.</u>(Primitive of a Function)
Let f and F be two functions defined on the interval I . F is said to be a *primitive of* f on I if F is differentiable on I and $F' = f$.

<u>T. 7. 1.</u>
Let f and F be two functions defined on the interval I . If F is a primitive of f , then the functions $F + C$, with constant C , are all primitives of f .

<u>D. 7. 2.</u>(Indefinite Integral)
All the primitives of the function f will be denoted by the *indefinite integral*

$$\int f(x)\, dx$$

<u>R. 7. 1.</u> (Integration by Substitution)
Let f and g be two functions, each being differentiable on its own domain. By theorem T. 6. 5. $g \circ f$ is differentiable and we have

$$(g \circ f)' = g' \circ f \cdot f' .$$

It follows that:

$$\int (g \circ f)'(x)\, dx = \int (g' \circ f)(x) \cdot f'(x)\, dx$$

Now denote $t = f(x)$, then $dt = f'(x)\, dx$ and the above equation can be written as follows:

$$\int (g \circ f)'(x)\, dx = \int g'(f(x)) \cdot f'(x)\, dx = \int g'(t)\, dt$$

<u>R. 7. 2.</u> (Integration by Parts)
The so-called *integration by parts* is a direct consequence of the rule for derivation of a product.
Let f and g be two functions, each being differentiable on its own domain. Then we have:

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx'$$

<u>R. 7. 3.</u> (Development into Partial Fractions)

(1) A *rational fraction* is an expression of the form

$$r(x) = \frac{f(x)}{g(x)},$$

where $f(x)$ and $g(x)$ are two polynomials.

(2) If $\deg f(x) < \deg g(x)$, the rational fraction is *proper, otherwise it is improper*.
(3) An improper rational fraction is the sum of a polynomial and a proper rational fraction.
(4) A rational fraction will be decomposed according to the following rules:

1. If the denominator $r(x)$ contains a factor $ax+b$ of multiplicity, then there is a single fraction of the form $\dfrac{\alpha}{ax+b}$ (α : a constant) in the desired development.
2. If the denominator $r(x)$ contains a factor $ax+b$ of multiplicity m, then the decomposition of $r(x)$ contains a sum of fractions of the form $\sum_{i=1}^{m} \dfrac{\alpha_i}{(ax+b)^i}$ ($\alpha_i, i = 1, 2, ..., m$:constants).
3. If the denominator $r(x)$ contains a factor $ax+b$ of multiplicity m, then the decomposition of $r(x)$ contains a sum of fractions of the form $\sum_{i=1}^{m} \dfrac{\alpha_i}{(ax+b)^i}$ ($\alpha_i, i = 1, 2, ..., m$:constants)
4. If the denominator $r(x)$ contains an irreducible factor ax^2+bx+c) of multiplicity 1, then there is a single fraction of the form $\dfrac{\alpha x+\beta}{ax^2+bx+c}$ in the desired development (α, β :constants).
5. If the denominator $r(x)$ contains an irreducible factor ax^2+bx+c) of multiplicity m, then the decomposition of $r(x)$ into partial fractions contains a sum of fractions of the form $\sum_{i=1}^{m} \dfrac{\alpha_i x+\beta}{(ax^2+bx+c)^i}$ ($\alpha_i, \beta_i, i = 1, 2, ..., m$:constants).

<u>D. 7. 3.</u> (Definite Integral)
Let f be a function defined on the interval $[a,b]$ with the primitive F on $[a,b]$. The (*definite*) *integral of* f *over* $[a,b]$ is the number

$$\int_a^b f(x)\,dx := [F(x)]_a^b = F(b) - F(a)\,.$$

<u>T. 7. 2.</u>

$$\int_a^a f(x)\,dx = 0\,.$$

<u>T. 7. 3.</u>
Let f be integrable on an interval $I := [a,b]$ and let $a, b, c \in I$ $(a < c < b)$. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

T. 7. 4.
Let f and g be integrable on an interval $I := [a, b]$. Then $f + g$ is integrable on $I := [a, b]$ and

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx .$$

T. 7. 5.
Let f be integrable on an interval $I := [a, b]$. Then $\alpha \cdot f$ is integrable on $I := [a, b]$ and

$$\int_a^b \alpha \cdot f(x)\, dx = \alpha \cdot \int_a^b f(x)\, dx .$$

T. 7. 6.
Let f be integrable on an interval $I := [a, b]$. Then

$$f(x) \geq 0 \text{ on } I := [a, b] \qquad \Rightarrow \qquad \int_a^b f(x)\, dx \geq 0$$

C. 7. 1.
Let f and g be integrable on an interval $I := [a, b]$. Then

$$f(x) \geq g(x) \text{ on } I := [a, b] \qquad \Rightarrow \qquad \int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$$

C. 7. 2.
Let f be integrable on an interval $I := [a, b]$. Then $|f|$ is also integrable on $I := [a, b]$ and

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

T. 7. 7. (Substitution Method for Definite Integrals)
Let
1. a and b be two real numbers such that $a < b$;
2. g be a function having a continuous first derivative on $[a, b]$ such that

$$a \leq x \leq b \qquad \Rightarrow \qquad m \leq g(t) \leq M \ .$$

Then

$$f \text{ is a continuous function on } [m, M]$$
$$\Rightarrow$$
$$\forall u \in [a, b], \quad \int_{g(a)}^{g(u)} f(t)\, dt = \int_a^u f[g(x)] \cdot g'(x)\, dx$$

In particular

$$\int_{g(a)}^{g(b)} f(t)\,dt = \int_a^b f[g(x)]\cdot g'(x)\,dx\,.$$

<u>T. 7. 8.</u>
Let f be a function defined on $[a,b]$. Suppose that

1. f has a primitive on $[a,b]$;
2. $\forall x \in [a,b],\ f(x) \geq 0$.

Then the area of the region defined by $a \leq x \leq b$ and $0 \leq y \leq f(x)$ is equal to $\int_a^b f(x)\,dx$.

<u>R. 7. 4.</u>
Let G_1 and G_2 be the respective graphs of the functions f_1 and f_2, both defined and continuous on $[a,b]$. Suppose that

$$\forall x \in [a,b],\ f_1(x) \leq f_2(x).$$

Then the area of the region by G_1 and G_2, and by the lines whose equations are $x = a$ and $x = b$ is given by

$$A = \int_a^b [f_2(x) - f_1(x)]dx$$

<u>D. 7. 4.</u> (Smooth Curve)
The function f is *smooth* on $I := [a,b]$ if the following conditions are fulfilled:

1. f is differentiable on I;
2. f' is continuous on I.

If f is smooth, its graph is called a *smooth curve*.

<u>D. 7. 5.</u> (Length of a Graph)
Let f be a smooth function on $I := [a,b]$. The *length of the graph* f will be defined as

$$L := \int_a^b \sqrt{1 + f'(x)^2}\,dx\,.$$

<u>T. 7. 9.</u>
Let f be a continuous function on $I := [a,b]$. Suppose that for any $x \in [a,b]$ $f(x) \geq 0$. The volume of the solid of revolution obtained by revolving about the x-axis the region between the x-axis and the graph of f is equal to

$$V = \pi \int_a^b [f(x)]^2\,dx\,.$$

<u>T. 7. 10.</u> (The Fundamental Theorem of Calculus)
Let f be a continuous function on $I := [a, b]$. Let

$$\forall x \in I,\ F(x) := \int_{x_0}^{x} f(t)\, dt, \quad x_0 \in I\ .$$

Then F is differentiable on I and $F' = f$.

(With another notation.

$$\frac{d}{dx}\left(\int_{x_0}^{x} f(t)\, dt \right) = f(x)$$

).

<u>D. 7. 6.</u> (Improper Integrals of the First Type)
If there exists a finite limit

$$\lim_{b \to +\infty} \int_{a}^{b} f(x)\, dx\ ,$$

then this limit is called the *improper integral* of the function of the function $f(x)$ in the interval $[a, +\infty[$ and is denoted by the symbol

$$\int_{a}^{+\infty} f(x)\, dx\ .$$

Thus, by definition, we have

$$\int_{a}^{+\infty} f(x)\, dx = \lim_{b \to +\infty} \int_{a}^{b} f(x)\, dx\ .$$

In this case it is said that the improper integral $\int_{a}^{+\infty} f(x)\, dx$ *exists* or *converges*.

If $\int_{a}^{b} f(x)\, dx$ as $b \to +\infty$ does not have a finite limit, one says that $\int_{a}^{+\infty} f(x)\, dx$ *does not exist* or *diverges.*

We similarly define the improper integrals of other infinite intervals:

$$\int_{-\infty}^{a} f(x)\, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\, dx$$

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{+\infty} f(x)\, dx\ .$$

T. 7. 11.
If for all $x\,(\geq a)$ the inequality

$$0 \leq f(x) \leq g(x)$$

is fulfilled and if $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ also converges, and

$$\int_a^{+\infty} f(x)\,dx \leq \int_a^{+\infty} g(x)\,dx$$

T. 7. 12.
If for all $x\,(\geq a)$ the inequality

$$0 \leq g(x) \leq f(x)$$

is fulfilled and if $\int_a^{+\infty} g(x)\,dx$ diverges, then $\int_a^{+\infty} f(x)\,dx$ also diverges.

T. 7. 13.
If the integral $\int_a^{+\infty} |f(x)|\,dx$ converges, then the integral $\int_a^{+\infty} f(x)\,dx$ also converges.

(In this case the integral $\int_a^{+\infty} f(x)\,dx$ is called an *absolutely convergent integral*.)

D. 7. 6. (Improper Integral of the Second Type)
Let f be a function continuos on $[a, t]$ and unbounded on $]t, b[$, where $a \leq t < b$.

1. If the integral $\int_a^t f(x)\,dx$ as $t \to b^-$ has a finite limit, then this limit is called the *improper integral* of the function $f(x)$.
   Thus, by definition, we have

   $$\int_a^t f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx\,.$$

   In this case it is said that the improper integral $\int_a^b f(x)\,dx$ *exists* or *converges*.

2. If the integral $\int_a^t f(x)\,dx$ as $t \to b^-$ does not have a finite limit, one says that $\int_a^b f(x)\,dx$ *does not exist* or *diverges.*

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D. 7. 7.
Let f be a function continuos on $[t, b]$ and unbounded on $]a, t[$, where $a < t \leq b$.

1. If the integral $\int_t^b f(x)\,dx$ as $t \to a^+$ has a finite limit, then this limit is called the *improper integral* of the function $f(x)$.
   Thus, by definition, we have

$$\int_t^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx .$$

   In this case it is said that the improper integral $\int_a^b f(x)\,dx$ *exists* or *converges*.

2. If the integral $\int_t^b f(x)\,dx$ as $t \to a^+$ does not have a finite limit, one says that $\int_a^b f(x)\,dx$ *does not exist* or *diverges.*

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<u>D. 7. 8.</u>

Let $\int_a^c f(x)\,dx$ and $b\int_c^c f(x)\,dx$ ($a < c < b$) be convergent improper integrals. Then the integral

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

will also said to be *convergent*.

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