Chapter IV

Limits of Functions

<u>D. 4. 1.</u>(Limit at One Real Point)
Let $l \in \mathbb{R}$ and $x_0 \in \mathbb{R}$.

$$\lim_{x \to x_0} f(x) := l \quad \Leftrightarrow \quad \forall \varepsilon > 0, \exists \delta > 0 : |x - x_0| < 0 \Rightarrow |f(x) - l| < \varepsilon$$

$$\lim_{x \to x_0} f(x) := \infty \quad \Leftrightarrow \quad \forall A > 0, \exists \delta > 0 : |x - x_0| < 0 \Rightarrow f(x) > A$$

$$\lim_{x \to x_0} f(x) := -\infty \quad \Leftrightarrow \quad \forall A < 0, \exists \delta > 0 : |x - x_0| < 0 \Rightarrow f(x) < A$$

<u>R. 4. 1.</u>
If the function f has a (finite or infinite) limit at x_0, this limit is unique.

<u>D. 4. 2.</u>(Finite Limit at Infinity)
Let $l \in \mathbb{R}$.

$$\lim_{x \to +\infty} f(x) := l \quad \Leftrightarrow \quad \forall \varepsilon > 0, \exists A > 0 : x > A \Rightarrow |f(x) - l| < \varepsilon$$

$$\lim_{x \to -\infty} f(x) := l \quad \Leftrightarrow \quad \forall \varepsilon > 0, \exists A < 0 : x < A \Rightarrow |f(x) - l| < \varepsilon$$

<u>D. 4. 2.</u>(Infinite Limit at Infinity)

$$\lim_{x \to +\infty} f(x) := +\infty \quad \Leftrightarrow \quad \forall A > 0, \exists B > 0 : x > B \Rightarrow f(x) > A$$

$$\lim_{x \to +\infty} f(x) := -\infty \quad \Leftrightarrow \quad \forall A < 0, \exists B > 0 : x > B \Rightarrow f(x) < A$$

$$\lim_{x \to -\infty} f(x) := +\infty \quad \Leftrightarrow \quad \forall A > 0, \exists B < 0 : x < B \Rightarrow f(x) > A$$

$$\lim_{x \to -\infty} f(x) := -\infty \quad \Leftrightarrow \quad \forall A < 0, \exists B < 0 : x < B \Rightarrow f(x) < A$$

<u>D. 4. 3.</u>(One-Sided Limit at One Point)

$$\lim_{x \to -x_0^+} f(x) := l \quad \Leftrightarrow \quad \forall \varepsilon > 0, \exists \delta > 0 : x_0 < x < x_0 + \delta \Rightarrow |f(x) - l| < \varepsilon$$

(Right limit at x_0 *)*

$$\lim_{x \to -x_0^-} f(x) := l \quad \Leftrightarrow \quad \forall \varepsilon > 0, \exists \delta > 0 : x_0 - \delta < x < x_0 \Rightarrow |f(x) - l| < \varepsilon$$

(Left limit at x_0 *)*

<u>T. 4. 1.</u>

Let f be a function defined on a pointed neighbourhood of x_0. The function f has a limit l when x is arbitrarily close to x_0 if and only if for any convergent series $\{x_n\}$ having x_0 as limit, we have

$$\lim_{n\to+\infty} f(x_n) = f(l).$$

<u>T. 4. 2.</u>

Let f_1 and f_2 be given functions with

$$\lim_{n\to x_0} f_1(x) = l_1, \qquad \lim_{n\to x_0} f_2(x) = l_2.$$

Then

$$\lim_{n\to x_0}[f_1(x) + f(x_2)] = l_1 + l_2,$$
$$\lim_{n\to x_0}[f_1(x) - f(x_2)] = l_1 - l_2,$$
$$\lim_{n\to x_0}[c\cdot f_1(x)] = c\cdot l_1$$
$$\lim_{n\to x_0}[f_1(x)\cdot f(x_2)] = l_1\cdot l_2.$$

Under the additional condition that $f_2(x) \neq 0$ for all x from a pointed neighbourhood of x_0 and that $l_2 \neq 0$, we have

$$\lim_{x\to x_0} \frac{f_1(x)}{f_2(x)} = \frac{l_1}{l_2}.$$

<u>T. 4. 3.</u>

Let f be a given function with

$$\lim_{n\to x_0} f\ (x) = 0.$$

Then

1.

$f(x) > 0$ for all x from a pointed neighbourhood of x_0

$$\Rightarrow$$

$$\lim_{x\to x_0} \frac{1}{f(x)} = +\infty;$$

2.

.

$f(x) < 0$ for all x from a pointed neighbourhood of x_0

$$\Rightarrow$$

$$\lim_{x\to x_0} \frac{1}{f(x)} = -\infty$$

<u>R. 4. 2.</u>

In a neighbourhood of $-\infty$ resp. $+\infty$, a polynomial function has the same limit as its term with highest power.

R. 4. 3.
In a neighbourhood of $-\infty$ resp. $+\infty$, a rational function has the same limit as the quotient of the term with highest power in the numerator by the term with highest power in the denominator.

T. 4. 3. (Sandwich Theorem)
Let the function f be bounded in a (pointed) neighbourhood of x_0 and that $\lim_{x \to x_0} g(x) = 0$.
Then

$$\lim_{x \to x_0} f(x) \cdot g(x) = 0 .$$

D. 4. 4. (**Vertical Asymptote**)
Let f be a function defined on a (pointed) neighbourhood of x_0. If f has an infinite (one-sided) limit at x_0, then the line whose equation is $x = x_0$ is a *vertical asymptote* to the graph of f.

D. 4. 5.
Let f be a function defined on a neighbourhood of $+\infty$. Let L be a line with equation $y = ax + b$.
The line L is an *asymptote* to the graph of f if

$$\lim_{x \to +\infty} \left[f(x) - (ax + b) \right] = 0 .$$

(There is a similar definition for a neighbourhood of $-\infty$.)

(Last revised: 28.10.04)