Analysis in Economics (II)

Solutions

1.
Since revenue is the selling price times the number of tickets sold, the revenue function for the concert is

$$R(x_1, x_2) = 15x_1 + 25x_2.$$

This means that when 125 tickets are sold fort he matinee show and 200 tickets are sold fort he evening show, the music hall generates revenue of

$$R(125, 200) = 15 \cdot 125 + 25 \cdot 200 = 6875\,€.$$

2.

$$f(2000,1500) = 10 \cdot 2000^{0.25} \cdot 1500^{0.75} \approx 16119.$$

3.

$$\frac{\partial y}{\partial L}(L,C) = 0.8L^{-0.6}C^{0.6}, \qquad \frac{\partial y}{\partial C}(L,C) = 1.2L^{0.4}C^{-0.4}$$

4.

i)

$$\frac{\partial y}{\partial L}(L,C) = 72L^{-0.2}C^{0.2} \qquad \frac{\partial y}{\partial C}(L,C) = 18L^{0.8}C^{-0.8}$$

$$\frac{\partial y}{\partial L}(1000;\ 200) = 52.1841; \qquad \frac{\partial y}{\partial C}(1000;\ 200) = 65.2302$$

An increase of labour from 1000 units to 1001 units, while keeping the capital at 200 units, will lead to an approximate increase of production by 52.1841 units.
An increase of capital from 200 units to 201 units, while keeping the labour at 1000 units, will lead to an approximate increase of production by 65.2302 units.

ii)

$$C = 8L \quad \text{or} \quad L = \frac{1}{8}C$$

$$\frac{\partial y}{\partial L}(1,\ 8) = 72 \cdot 8^{0.2} = 109.1316, \qquad \frac{\partial y}{\partial C}\left(\frac{1}{8}, 1\right) = 18 \cdot \left(\frac{1}{8}\right)^{0.8} = 3.4104$$

An increase of labour by 1 unit, while holding the capital constant, will lead to an approximate increase of production by 109.1316 units.

An increase of capital by 1 unit, while holding the labour constant, will lead to an approximate increase of production by 3.4104 units.

5.

$$\frac{\partial y}{\partial L} = 0.8L^{-0.2}C^{0.2} > 0;\ \frac{\partial y}{\partial C} = 0.2L^{0.8}C^{-0.8} > 0\ .$$

Holding the capital (labour) constant, the output increases with increasing labour (capital).

$$\frac{\partial^2 y}{\partial L^2} = -0.16L^{-1.2}C^{0.2} < 0; \qquad \frac{\partial^2 y}{\partial C^2} = -0.16A^{0.8}C^{-1.8} < 0$$

The marginal products of labour and capital are diminishing.

$$\frac{\partial^2 y}{\partial L \partial C} = \frac{\partial^2 y}{\partial C \partial L} = 0.16L^{-0.2}C^{-0.8} > 0\ .$$

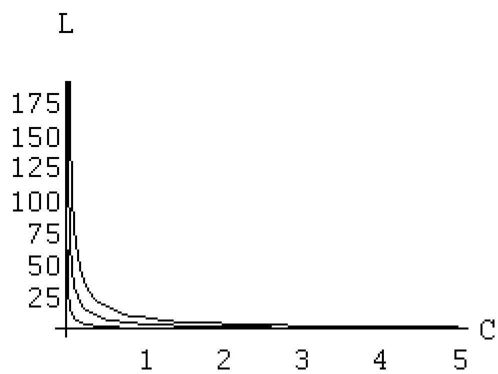
The positive sign of theses “cross” derivatives tells us that capital and labour are complement in production.

6.

$$Q = 1 \ \Rightarrow\ L = \frac{1}{C}\ ,$$

$$Q = 2 \ \Rightarrow\ L = \frac{4}{C}$$

$$Q = 3 \ \Rightarrow\ L = \frac{9}{C}$$



The curves form an “isoquant map”. Each isoquant gives all possible equivalent combinations of labour and capital ensuring the same quantity of production.

7.

$$Q_{r_1}(r_1, r_2) = 4 - 2r_1 + 3r_2 := 0$$

$$Q_{r_2}(r_1, r_2) = 10 + 3r_1 - 5r_2 := 0$$

$$\Rightarrow \quad r_1 = 50, \quad r_2 = 32$$

$$Q_{r_1r_1}(r_1, r_2) = -2, \quad Q_{r_1r_2}(r_1, r_2) = 3, \quad Q_{r_2r_2}(r_1, r_2) = -5 .$$

For the factor combination $r_1 = 50,\ r_2 = 32$ the production will be maximal, since

$$Q_{r_1r_1}(r_1, r_2) \cdot Q_{r_2r_2}(r_1, r_2) - \left[Q_{r_1r_2}(r_1, r_2)\right]^2 = (-2)\cdot(-5) - 3^2 = 1 > 0 \text{ and } Q_{r_1r_1}(r_1, r_2) = -2 < 0 .$$

The firm will then have a maximum profit of $Q(50, 32) = 700$.

8.

1.

$$P_{\mathrm{Pr}_1}(p_1, p_2) = p_1 x_1 - C_1(x_1)$$
$$= -2p_1^2 - p_1 p_2 + 104 p_1 + 2 p_2 - 320$$

$$P_{\mathrm{Pr}_2}(p_1, p_2) = p_2 x_2 - C_2(x_2)$$
$$= -3p_2^2 - p_1 p_2 + 2 p_1 + 126 p_2 - 360$$

$$P(p_1, p_2) = -2p_1^2 - 2p_1 p_2 - 3p_2^2 + 106 p_1 + 128 p_2 - 680$$

2.

$$P_{\mathrm{Pr}_1}(p_1, p_2) = -4p_1 - 2p_2 + 106 = 0$$
$$P_{\mathrm{Pr}_2}(p_1, p_2) = -2p_1 - 6p_2 + 128 = 0$$

$$p_1 = 19, \quad p_2 = 15$$

$$P_{p_1p_1} = -4, \qquad P_{p_1p_2} = -2, \qquad P_{p_2p_2} = -6$$

Because of $P_{p_1p_1} \cdot P_{p_2p_2} - (P_{p_1p_2})^2 = (-4)\cdot(-6) - (-2)^2 > 0, \quad P_{p_1p_2} = \text{-}4 < 0$ the maximum profit will be attained for $p_1 = 19, \quad p_2 = 15$; it will be equal to $P(19,\ 15) = 1287$.

3.

$$P_1(p_1,\ 16) = -2p_1^2 + 88p_1 - 288 =: \tilde{P}_1(p_1)$$

$$\tilde{P}_1'(p_1) = -4p + 88 = 0 \quad \Leftrightarrow \quad p_1 = 22$$

$$\tilde{P}_1''(p_1) = -4 < 0$$

Therefore, P_1 will be maximal for $p_1 = 22$; it will be equal to 680.

4.
It will be advantageous for the consumers if the producers put an end to their „price war“, since the price p_1 has increased from 19 to 22 .

9.
1.

$$R(x_1, x_2) = x_1 \cdot (35 - 3x_1 + 4x_2) + x_2 \cdot (20 - 2x_1 + x_2)$$

$$= -3x_1^2 + x_2^2 + 2x_1 \cdot x_2 + 35x_1 + 20x_2 .$$

$$R(10,18) = 1094 .$$

2.

$$P(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$$

$$= -3x_1^2 + x_2^2 + 2x_1 \cdot x_2 + 35x_1 + 20x_2 - (8.5x_1 + 6x_2 + 400)$$

$$= -3x_1^2 + x_2^2 + 2x_1 \cdot x_2 + 26.5x_1 + 14x_2 - 400$$

$$P(10,18) = 501 .$$

3.

$$P_{x_1}(x_1, x_2) = -6x_1 + 2x_2 + 26.5 ; \qquad P_{x_1}(10,18) = 2.5 .$$

An increase of x_1 by 1 unit, while holding x_2 unchanged, will lead to an approximate increase of profit by 2.5 units.

$$P_{x_2}(x_1, x_2) = 2x_1 + 2x_2 + 14 ; \qquad P_{x_2}(10,18) = 70 .$$

An increase of x_2 by 1 unit, while holding x_1 unchanged, will lead to an approximate increase of profit by 70 units.

4.

$$R_{x_1}(x_1, x_2) = -6x_1 + 2x_2 + 35 ; \qquad R_{x_1}(10,\ 18) = 11 ,$$

$$\varepsilon_{R,x_1}(x_1, x_2) = \frac{x_1}{R(x_1, x_2)} \cdot R_{x_1}(x_1, x_2) ; \qquad \varepsilon_{R,x_1}(10,\ 18) = \frac{10}{1094} \cdot 11 \approx 0.1\ \%.$$

An increase of x_1 by 1 %, while holding x_2 unchanged, will lead to an approximate increase of revenue by 0.1 %. The revenue function at this point is inelastic.

$$R_{x_2}(x_1, x_2) = 2x_1 + 2x_2 + 20 ; \qquad R_{x_2}(10,\ 18) = 66$$

$$\varepsilon_{R,x_2}(x_1,x_2)=\frac{x_2}{R(x_1,x_2)}\cdot R_{x_2}(x_1,x_2)\,; \qquad \varepsilon_{R,x_2}(10,\ 18)=\frac{18}{1094}\cdot 66\approx 1.1\ \%.$$

An increase of x_2 by 1 %, while holding x_1 unchanged, will lead to an approximate increase of revenue by 1.1 %. The revenue function at this point is elastic.

10.

$$R(p,q)=x_1\cdot p+x_2\cdot q$$
$$=(850-36p+15q)\cdot p+(1075+20p-25q)\cdot q$$
$$R(p,q)==-36p^2-25q^2+35p\cdot q+850p+1075q$$

$$C(x_1,x_2)=30x_1+45x_2$$

$$C(p,q)=30\cdot(850-36p+15q)+45\cdot(1075+20p-25q)$$

$$C(p,q)=-180p-675q+73875$$

$$P(p,q)=R(p,q)-C(p,q)$$

$$P(p,q)=-36p^2-25q^2+35p\cdot q+1030p+1750q-73875.$$

$$P_p(p,q)=-72p+35q+1030$$
$$P_q(p,q)=\ \ 35p-50q+1750$$

$$\begin{cases}-72p+35q=-1030\\ \ \ 35p-50q=-1750\end{cases}$$

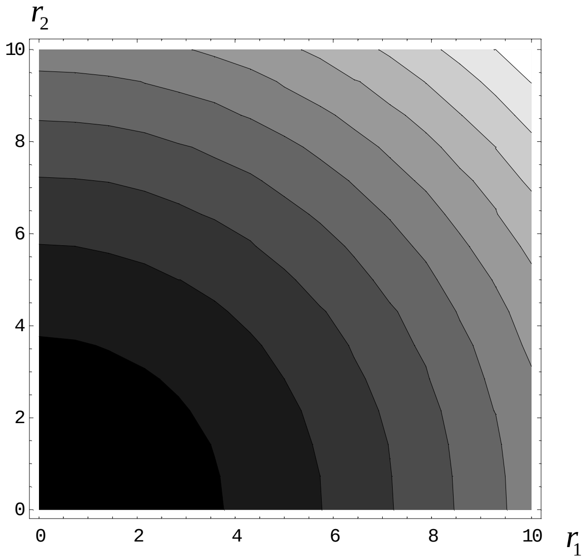
$$\Rightarrow \quad p=47.47368421\approx 47.47, \quad q=68.23157895\approx 68.23$$

Thus, because of

$$H=\begin{pmatrix}-72 & 35\\ 35 & -50\end{pmatrix}, \qquad \det H=(-72)\cdot(-50)+35^2>0\,, \qquad P_{p,p}(p,q)=-72<0$$

to maximise profit, the shoe company should sell a pair of aerobic shoes for 47.47 € and a pair of running shoes for 68.23 €. Its maximum profit will amount to 10276.58 €

11.



12.

$$P(x_1,x_2) = -3x_1^2 - 2x_2^2 - 2x_1x_2 + 160x_1 + 120x_2 - 18$$

$$P_{x_1}\left(x_1,x_2\right) = -6x_1 - 2x_2 + 160$$
$$P_{x_2}\left(x_1,x_2\right) = -2x_1 - 4x_2 + 120$$

$$\begin{cases} 6x_1 + 2x_2 = 160 \\ 2x_1 + 4x_2 = 120 \end{cases} \quad \Rightarrow \quad x_1 = x_2 = 20$$

$$P_{x_1x_1} = -6, \quad P_{x_1x_2} = P_{x_2x_1} = -2, \quad P_{x_2x_2} = -4$$

$$\det H = \det\begin{pmatrix} -6 & -2 \\ -2 & -4 \end{pmatrix} = 20 > 0, \qquad P_{x_1x_1} < 0\,.$$

Hence, the firm’s profit will be maximised for $x_1 = x_2 = 20$ with $P\left(20,20\right) = 2782$.

13.

$$x_1 = 14 - 0.25p_1 \quad \Rightarrow \quad p_1 = 56 - 4x_1$$
$$x_2 = 24 - 0.5p_2 \quad \Rightarrow \quad p_2 = 48 - 2x_2$$

$$R(x_1,x_2) = p_1 \cdot x_1 + p_2 \cdot x_2$$

$$= (56 - 4x_1)x_1 + (48 - 2x_2)x_2$$

$$P(x_1,x_2) = R(x_1,x_2) - C(x_1,x_2)$$

$$P(x_1,x_2)=5x_1^2-3x_2^2-5x_1\cdot x_2+56x_1+48x_2$$

$$P_{x_1}\left(x_1,x_2\right)=-10x_1-5x_2+56$$
$$P_{x_2}\left(x_1,x_2\right)=-5x_1-6x_2+48$$

$$\begin{cases}10x_1+5x_2=56\\ \ 5x_1+6x_2=44\end{cases} \qquad \Rightarrow \quad x_1=2.75, \quad x_2=5.7$$

$$P_{x_1x_1}=-10, \quad P_{x_1x_2}=P_{x_2x_1}=-5, \quad P_{x_2x_2}=-6$$

$$\det H=\det\begin{pmatrix}-10 & -5\\ -5 & -6\end{pmatrix}=35>0, \qquad P_{x_1x_1}<0\,.$$

Hence, the producer's profit will be maximised for $x_1=2.75,\ x_2=5.7$ with

$$P\left(2.75,5.70\right)=213.94\,.$$

14.

$$P\left(x_1,x_2\right)=\left(144-2x_1\right)x_1+\left(120-x_2\right)x_2-\left(x_1^2+x_1x_2+x_2^2\right)$$

$$P\left(x_1,x_2\right)=-3x_1^2-2x_2^2-x_1x_2+144x_1+120x_2-35$$

$$L\left(x_1,x_2;\lambda\right)=-3x_1^2-2x_2^2-x_1x_2+144x_1+120x_2-35-\lambda\left(x_1+x_2-40\right)$$

$$L_{x_1}\left(x_1,x_2;\lambda\right)=-6x_1-x_2+144-\lambda$$
$$L_{x_2}\left(x_1,x_2;\lambda\right)=-x_1-4x_2+120-\lambda$$
$$L_{\lambda_1}\left(x_1,x_2;\lambda\right)=x_1+x_2-40$$

(1)

$$\begin{cases}-6x_1 \ -x_2+144-\lambda=0\\ \ -x_1-4x_2+120-\lambda=0\\ \quad x_1 \ +x_2-40 \qquad =0\end{cases} \qquad \Rightarrow \ x_1=18, \quad x_2=22, \quad \lambda=14\,.$$

(The sufficiency condition will not be tested.)

(2)

$$p_1=144-2\cdot 18=108, \quad p_2=120-22=98.$$

(3) $\quad P\left(18,\ 22\right)=2861.$

15.

$$L\left(L,K;\lambda\right)=L^{0.5}\cdot K^{0.4}-\lambda\left(4L+3K-108\right)$$

$$L_L\left(L,K;\lambda\right)=0.5L^{-0.5}K^{0.4}-4\lambda=0$$
$$L_K\left(L,K;\lambda\right)=0.4L^{0.5}K^{-0.6}-3\lambda=0$$
$$L_\lambda\left(L,K;\lambda\right)=4L+3K-108=0$$

$$\frac{0.4L^{0.5}K^{-0.6}}{0.5L^{-0.5}K^{0.4}}=\frac{-3\lambda}{-4\lambda},\qquad 0.8K^{-1}\cdot L^{1}=0.75,\qquad \frac{L}{K}=\frac{0.75}{0.80},\qquad L=0.9375K,$$

$$3K+4\cdot 0.9375K=108,$$

$$K=16,\quad L=15,\quad \lambda=0.097839083.$$

(The sufficiency condition will not be tested.)

16.

1.

$$L(x_1,x_2;\lambda)=5x_1^2+3x_2^2+2x_1\cdot x_2+800-\lambda\left(x_1+x_2-39\right)$$

$$L_{x_1}(x_1,x_2;\lambda)=10x_1+2x_2-\lambda=0$$
$$L_{x_2}(x_1,x_2;\lambda)=2x_1+6x_2-\lambda=0\qquad\Rightarrow\quad x_1=13,\quad x_2=26,\quad \lambda=182,\quad C\left(13,26\right)=4349.$$
$$L_\lambda(x_1,x_2;\lambda)=x_1+x_2=39.$$

2.

Since $\lambda=182$, an increased production quota will lead to additional costs of approximately 182.

17.

$$L(x_1,x_2;\lambda)=-3x_1^2-2x_2^2-2x_1\cdot x_2+110x_1+140x_2-\lambda\left(x_1-2x_2\right)$$

$$L_{x_1}(x_1,x_2;\lambda)=-6x_1-2x_2+110-\lambda=0$$
$$L_{x_2}(x_1,x_2;\lambda)=-2x_1-4x_2+140+2\lambda=0$$
$$L_\lambda(x_1,x_2;\lambda)=x_1-2x_2=0.$$

$$\Rightarrow\quad x_1=20,\quad x_2=10,\quad \lambda=30,\quad C\left(20,10\right)=1800.$$

18.

$$L_{x_1}(x_1,x_2;\lambda)=0.60x_1^{-0.40}\cdot x_2^{0.25}-8\lambda=0$$
$$L_{x_2}(x_1,x_2;\lambda)=0.25x_1^{0.60}\cdot x_2^{-0.75}-5\lambda=0$$
$$L_\lambda(x_1,x_2;\lambda)=8x_1+5x_2-680=0$$

$$\frac{0.6x_1^{-0.4}x_2^{0.25}}{0.25x_1^{0.6}x_2^{-0.75}} = \frac{8\lambda}{5\lambda}$$

$$2.4x_1^{-1}x_2 = \frac{8}{5} \qquad \Rightarrow \qquad x_2 = \frac{2}{3}x_1$$

Substitute in $L_\lambda(x_1, x_2; \lambda)$: $x_1 = 60, \quad x_2 = 40$, and $L_{x_1}(x_1 = 60, x_2 = 40) = 0 \Rightarrow \lambda \approx 0.037$.

Because of $\lambda \approx 0.037$ an increase of the existing budget by 1 unit will lead to an approximate increase of utility by 0.037.

19.

1.

$$L(x_1, x_2; \lambda) = x_1^2 \cdot x_2^2 - \lambda(x_1 + 2x_2 - 20)$$

$$L_{x_1}(x_1, x_2; \lambda) = 2x_1 \cdot x_2^2 - \lambda = 0$$
$$L(x_1, x_2; \lambda) = 2x_1^2 \cdot x_2 - 2\lambda = 0$$
$$L_\lambda(x_1, x_2; \lambda) = x_1 + 2x_2 - 20 = 0$$

$$\begin{cases} 2x_1 \cdot x_2^2 = -\lambda \\ 2x_1^2 \cdot x_2 = -2\lambda \end{cases}$$

$$\frac{2x_1^2 \cdot x_2}{2x_1 \cdot x_2^2} = \frac{-2\lambda}{-\lambda}$$

$$\frac{x_1^2 \cdot x_2}{x_1 \cdot x_2^2} = 2 \quad \Rightarrow \quad x_1 = 2x_2$$

$$2x_2 + 2x_2 = 20, \qquad x_2 = 5, \quad x_1 = 10, \quad \lambda = 500$$

Therefore

$$x_1 = 10, \quad x_2 = 5, \quad \lambda = 500.$$

2.

An increase of the right-hand side of the constraint from 20 to 21 will lead to an approximate increase of the value of the objective function by 500 units.

20.

$$U(x_1, x_2^{0.5}) = x_1^{0.5} \cdot x_2^{0.5} \to \max$$

subject to

$$2x_1 + 3x_2 = 100.$$

$$L(x_1, x_2, \lambda) = x_1^{0.5} x_2^{0.5} - \lambda\left(2x_1 + 3x_2 - 100\right) \to \max$$

$$L_{x_1}(x_1, x_2, \lambda) = 0.5x_1^{-0.5} x_2^{0.5} - 2\lambda$$
$$L_{x_2}(x_1, x_2, \lambda) = 0.5x_1^{0.5} x_2^{-0.5} - 3\lambda$$
$$L_{\lambda}(x_1, x_2, \lambda) = -2x_1 - 3x_2 + 100$$

$$\begin{cases} 0.5x_1^{-0.5} x_2^{0.5} - 2\lambda = 0 \\ 0.5x_1^{0.5} x_2^{-0.5} - 3\lambda = 0 \\ -2x_1 - 3x_2 + 100 = 0 \end{cases}$$

$$\frac{0.5x_1^{-0.5} x_2^{0.5}}{0.5x_1^{0.5} x_2^{-0.5}} = \frac{2\lambda}{3\lambda} \quad \Leftrightarrow \quad \frac{x_1^{-0.5} x_2^{0.5}}{x_1^{0.5} x_2^{-0.5}} = \frac{2}{3} \quad \Leftrightarrow \quad \frac{x_2}{x_1} = \frac{2}{3} \quad \Leftrightarrow \quad x_2 = \frac{2}{3} x_1$$

Substituting the last relation into the budget constraint, we obtain:

$$2x_1 + 3 \cdot \frac{2}{3} x_1 = 100 \quad \Leftrightarrow \quad x_1 = 25, \quad x_2 = \frac{50}{3}, \qquad \lambda = \frac{1}{2\sqrt{6}} = 0.2041241452.$$

$$L_{x_1 x_1}(x_1, x_2, \lambda) = -0.25x_1^{-1.5} x_2^{0.5}$$

$$L_{x_1 x_2}(x_1, x_2, \lambda) = 0.25x_1^{-0.5} x_2^{-0.5} = L_{x_2 x_1}(x_1, x_2, \lambda)$$

$$L_{x_2 x_2}(x_1, x_2, \lambda) = -0.25x_1^{0.5} x_2^{-1.5}$$

$$\bar{H}(x_1, x_2, \lambda) := \begin{pmatrix} 0 & 2 & 3 \\ 2 & -0.25x_1^{-1.5} x_2^{0.5} & 0.25x_1^{-0.5} x_2^{-0.5} \\ 3 & 0.25x_2^{-0.5} x_1^{-0.5} & -0.25x_2^{0.5} x_2^{-1.5} \end{pmatrix}$$

$$\bar{H}\left(x_1 = 25, x_2 = \frac{50}{3} \right) \approx \begin{pmatrix} 0 & 2 & 3 \\ 2 & -0.0082 & 0.0122 \\ 3 & 0.0122 & -0.0184 \end{pmatrix}$$

$$\det \bar{H} = 0.2938 > 0.$$

Therefore, the Lagrange function assumes its relative maximum at $\begin{pmatrix} 25 & \frac{50}{3} & 0.2041 \end{pmatrix}^T$.

Thus, purchasing 25 units of G_1 and $\frac{50}{3}$ units of G_2 will maximise the consumer's utility. Increasing the budget of \$100 units by \$1 dollar will lead to an increase of utility by about 0.2041241452 units.

21

1.

a) $$P(p_1, p_2) = \left(21 - \frac{1}{10}p_1\right)p_1 + \left(50 - \frac{2}{5}p_2\right)p_2 - \left[2000 + 10\left(21 - \frac{1}{10}p_1 + 50 - \frac{2}{5}p_2\right)\right]$$

$$P(p_1, p_2) = -\frac{1}{10}p_1^2 - \frac{2}{5}p_2^2 + 22p_1 + 54p_2 - 2710$$

$$P_{p_1}(p_1, p_2) = -\frac{1}{5}p_1 + 22 := 0 \quad \Rightarrow \quad p_1 = 110$$

$$P(p_1, p_2) = -\frac{4}{5}p_2 + 54 := 0 \quad \Rightarrow \quad p_2 = 67.50$$

$$\left\langle \det H = \det\begin{pmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{4}{5} \end{pmatrix} = \frac{4}{25} > 0 \ \wedge \ P_{p_1p_1}(p_1, p_2) = -\frac{1}{5} < 0 \right\rangle$$

$$\Rightarrow$$

The profit will be maximised for $p_1 = 110$ and $p_2 = 67.50$. $P(p_1 = 110, p_2 = 67.5) = 322.50$

b)

If the producer does not discriminate, $p_1 = p_2$. Thus,

$$\tilde{P}(p_1) = \left(21 - \frac{1}{10}p_1\right)p_1 + \left(50 - \frac{2}{5}p_1\right)p_1 - \left[2000 + 10\left(21 - \frac{1}{10}p_1 + 50 - \frac{2}{5}p_1\right)\right]$$

$$\tilde{P}(p_1) = -\frac{1}{2}p_1^2 + 76p_1 - 2710$$

$$\frac{d\,\tilde{P}'(p_1)}{dp_1} = -p_1 + 76 := 0, \quad p_1 = 76, \quad \frac{d^2\,\tilde{P}'(p_1)}{d^2 p_1} = -1 < 0.$$

Therefore, the profit will be maximised for $p_1 = p_2 = 76 =: p$, $\tilde{P}(76) = 178$.

2.

Profit differential = 322.5 – 178 = 144.5

(Last updated: 07.02.2014)