Part I
Analysis in Economics

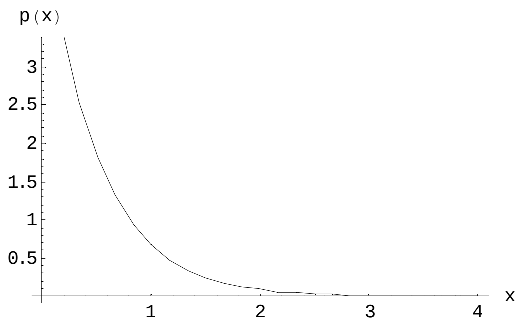
<u>D. 1. 1.</u> (*Function*)
A *function* f from a set A into a set B , denoted by $f: A \to B$, is a correspondence that assigns to each element $x \in A$ exactly one element $y \in B$. We call y the *image* of x under f and denote it by $f(x)$. The *domain* of f is the set $f(A) = \{ f(x) \mid x \in A \} \subseteq B$

<u>R. 1. 1.</u>
In this part, we only work with functions whose domains and ranges are sets of real numbers. We call such functions *real functions of one real variable*.

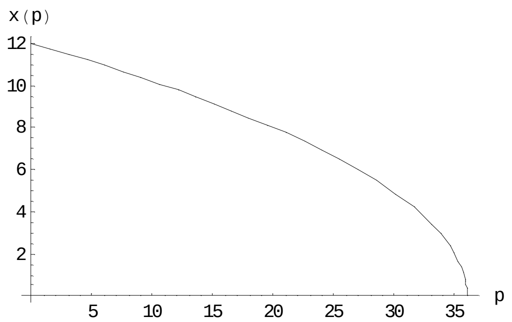
<u>Ex. 1. 1.</u>
Some important economic functions:

1. (Demand functions)

i) $p(x) = 5e^{-0.2x}$



ii) $x(p) = 2\sqrt{36 - p}$.

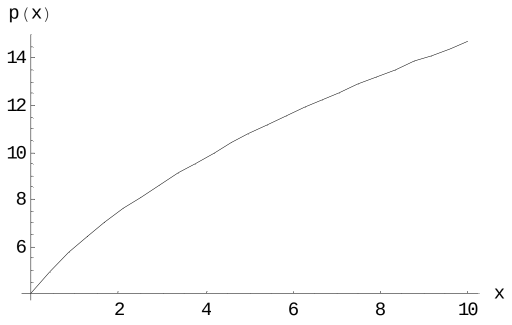


Here are:

p : the price per unit of product
x : the demand.

2. (Supply functions)

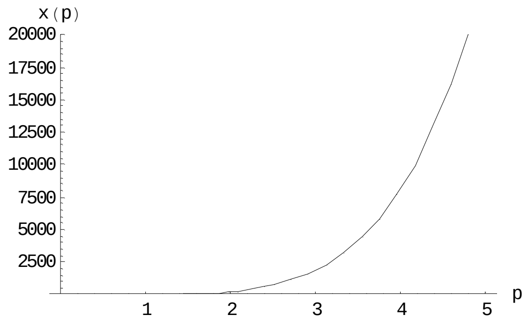
i) $p(x) = 2\sqrt{5x+4}$



ii) $x(p) = -50 + 8p^5$.

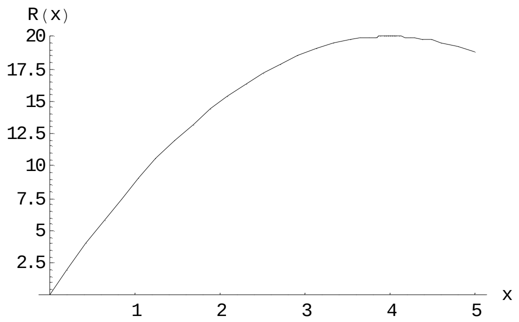
Here are:

p : the price per unit of product
x : the supply.



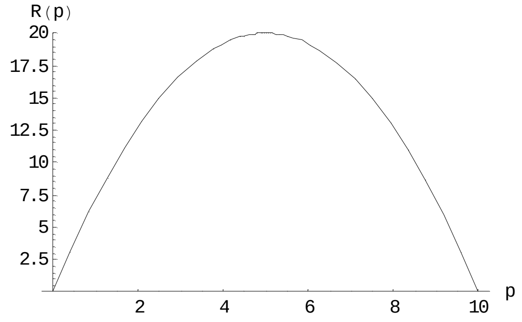
3. (Revenue functions)

i) $R(x) := x \cdot p(x) = x \cdot (10 - 1.25x)$
$= 10x - 1.25x^2$



ii) $R(p) = p \cdot x(p) = p \cdot (8 - 0.8p)$

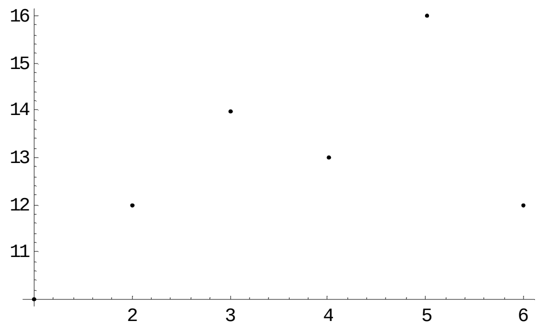
$= 8p - 0.8p^2$



iii) (A discrete revenue function)

The following table shows the revenue [thousand €] of a firm in the first six month of a year:

Month	1	2	3	4	5	6
Revenue	10	12	14	13	16	12



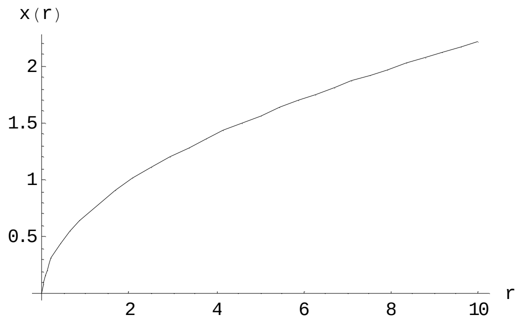
4. (Production functions)

Let

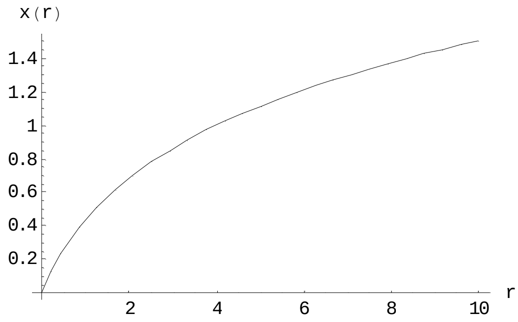
r: Factor input

x: Output.

i) (Cobb-Douglas production function): $x(r) = 0.7r^{0.5}$



ii) (CES production function): $x(r) = \left(r^{-0.5} + 0.5\right)^{-2}$



iii) (Limitational production function):

$$x(r) = \begin{cases} 0.75r & \text{für} \quad r \le 20 \\ 15 & \text{für} \quad r > 20 \end{cases}$$

5. (Total cost functions)
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x: Production
C: Total costs
C_f: Fixed costs
C_v: Variable costs
$C(x) = C_f + C_v(x)$

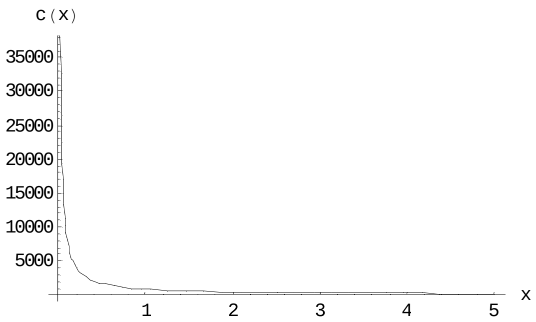
i) (Neoclassical production function): $C(x) = 2001 + 36 \cdot e^{0.01x}$

ii) (Piecewise defined cost function):

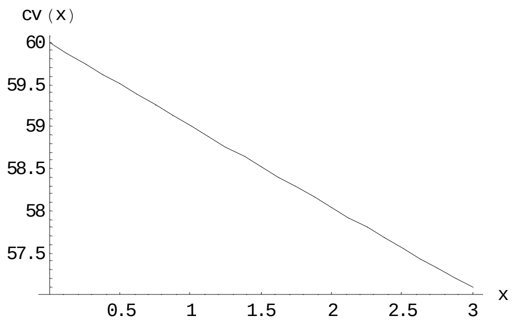
$$C(x) = \begin{cases} 0.25x + 3 & \text{für} \quad 0 < x \le 4 \\ 0.2x + 5 & \text{für} \quad 4 < x \le 8 \\ 0.5x + 3 & \text{für} \quad 8 < x \le 12 \\ 0.12x^2 - 2.5x + 21 & \text{für} \quad 12 < x \le 16 \end{cases}$$

6. (Average, average variable and average cost functions)

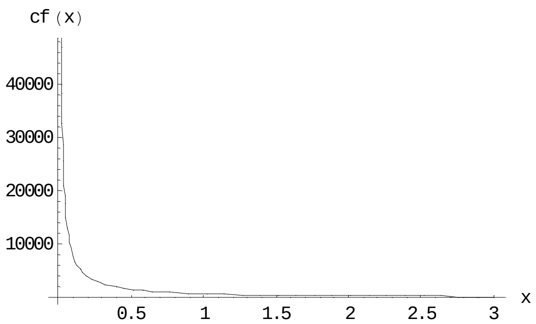
$$c(x) := \frac{C(x)}{x}, \quad x > 0, \qquad c(x) = \frac{800 + 0.01x^3 - x^2 + 60x}{x} = \frac{800}{x} + 0.01x^2 - x + 60$$



$$c_v(x) := \frac{C_v(x)}{x}, \quad x > 0, \qquad c_v(x) = \frac{0.01x^3 - x^2 + 60x}{x} = 0.01x^2 - x + 60$$

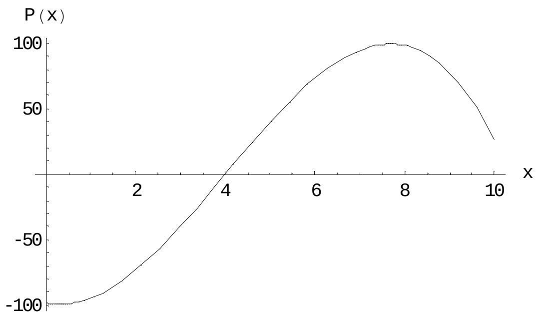


$$c_f := \frac{C_f}{x}, \quad x > 0, \qquad c_f(x) = \frac{800}{x}, \quad x > 0$$



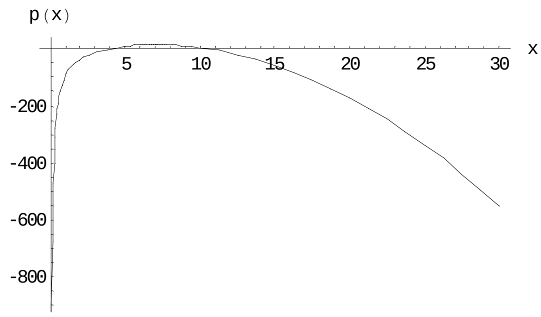
7. (Profit functions)

$$P(x) := R(x) - C(x), \qquad P(x) = 52.50x - \left(x^3 - 12x^2 + 60x + 98\right)$$
$$= -x^3 + 12x^2 - 7.5x - 98$$



8. (Average profit function)

$$p(x) := \frac{P(x)}{x}, \quad x > 0, \qquad p(x) = \frac{-x^3 + 12x^2 - 7.5x - 98}{x} = -x^2 + 12x - 7.5 - \frac{98}{x}$$



Ex. 1. 2.
Find the domain of the

1. production function

$$x(r) = \sqrt{2r - 200}$$

2. demand function

$$p(x) = \frac{100}{\sqrt{x}} - 4\sqrt{x} + 20$$

3. function

$$E(Y) = 200 \cdot \ln(Y + 100) - 750,$$

(E : monthly expenditure on energy; Y : monthly family income)

Solution:
1. $r \geq 100$
2. $x > 0$
3. $Y > 0$.

D. 1. 2. (***Limit of a Function***)
Let $f: A \to B$ be a real function and $x_0,\ l \in R$. Then

$$\left\langle \lim_{x \to a} f(x) = b \right\rangle \quad \Leftrightarrow \quad \left\langle \forall \varepsilon > 0\ \exists \delta > 0 : \ \left(0 < |x - a| < \partial\right) \ \Rightarrow \ |f(x) - b| < \varepsilon \right\rangle.$$

D. 1. 3. (***One-sided Limits***)
1. Let $f: A \to B$ be a real function defined in $]x_0,\ x_0 + \delta[,\ \delta > 0$.
   A number l_r is called the *limit of* $f(x)$ as x *approaches* x_0 *from the right* (or simply called the *right-hand limit of* f *at* x_0, symbolically

$$\left\langle \lim_{x\to x_0^+} f(x) = l_r \right\rangle, \text{ if } \left\langle \forall \varepsilon > 0 \;\; \exists \delta > 0: \;\; (x_0 < x < x_0 + \delta) \;\Rightarrow\; |f(x) - l_r| < \varepsilon \right\rangle.$$

2. Let $f: A \to B$ be a real function defined in $]x_0 - \delta, x_0[,\ \delta > 0$.
   A number l_l is called the *limit of* $f(x)$ as x *approaches* x_0 *from the left* (or simply called the *left-hand limit of* f *at* x_0, symbolically

$$\left\langle \lim_{x\to x_0^-} f(x) = l_l \right\rangle, \text{ if } \left\langle \forall \varepsilon > 0 \;\; \exists \delta > 0: \;\; (x_0 - \delta < x < x_0) \;\Rightarrow\; |f(x) - l_l| < \varepsilon \right\rangle.$$

T. 1. 1.
A function $f: A \to B$ has a limit as x approaches x_0 if and only if the left-hand and right-hand limits at x_0 exist and are equal. In symbols, we write

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x).$$

T. 1. 2. (***Uniqueness of Limits***)

$$\left\langle \lim_{x\to x_0} f(x) = a \wedge \lim_{x\to x_0} f(x) = b \right\rangle \;\Rightarrow\; a = b$$

D. 1. 4. (*Infinite Limits*)

1.
$$\left\langle \lim_{x\to x_0^-} f(x) = \infty \right\rangle, \text{ if } \left\langle \forall n \in N \;\; \exists \delta > 0: \;\; (x_0 - \delta < x < x_0) \;\Rightarrow\; f(x) > n \right\rangle.$$

2.
$$\left\langle \lim_{x\to x_0^-} f(x) = -\infty \right\rangle, \text{ if } \left\langle \forall n \in Z \setminus (N \cup \{0\}) \;\; \exists \delta > 0: \;\; (x_0 - \delta < x < x_0) \;\Rightarrow\; f(x) < n \right\rangle.$$

3.
$$\left\langle \lim_{x\to x_0^+} f(x) = \infty \right\rangle, \text{ if } \left\langle \forall n \in N \;\; \exists \delta > 0: \;\; (x_0 < x < x_0 + \delta) \;\Rightarrow\; f(x) > n \right\rangle.$$

4.
$$\left\langle \lim_{x\to x_0^+} f(x) = -\infty \right\rangle, \text{ if } \left\langle \forall n \in Z \setminus (N \cup \{0\}) \;\; \exists \delta > 0: \;\; (x_0 < x < x_0 + \delta) \;\Rightarrow\; f(x) < n \right\rangle.$$

5.
$$\left\langle \lim_{x\to x_0} f(x) = \infty \right\rangle, \text{ if } \left\langle \forall n \in N \;\; \exists \delta > 0: \;\; 0 < |x - x_0| < \delta \;\Rightarrow\; f(x) > n \right\rangle.$$

6.
$$\left\langle \lim_{x\to x_0} f(x) = -\infty \right\rangle, \text{ if } \left\langle \forall n \in Z \setminus (N \cup \{0\}) \;\; \exists \delta > 0: \;\; (0 < |x - x_0| < \delta) \;\Rightarrow\; f(x) < n \right\rangle.$$

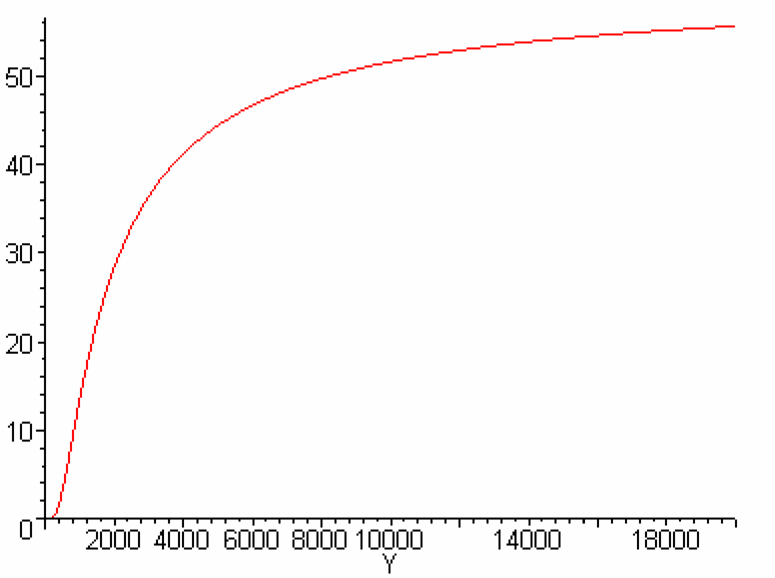
Ex. 1. 3.
The following function expresses the dependency of the butter consumption B [€/month] of a family on its monthly income Y [€/month}:

$$B(Y) = 60 \cdot e^{-\frac{1500}{Y}}, \quad Y > 0.$$

Investigate the change in butter consumption when the family income approaches zero?

Solution:

$$\lim_{Y \to 0^+} B(Y) = \lim_{Y \to 0^+} 60 \cdot e^{-\frac{1500}{Y}} = 60 \cdot \lim_{Y \to 0^+} \frac{1}{e^{\frac{1500}{Y}}} = 60 \cdot 0 = 0.$$



D. 1. 5. (*Limit at Infinity*)
1.

$$\left\langle \lim_{x \to \infty} f(x) = l \right\rangle, \text{ if } \left\langle \forall \varepsilon > 0 \ \exists n \in N: \ x > n \ \Rightarrow \ |f(x) - l| < \varepsilon \right\rangle.$$

2.

$$\left\langle \lim_{x \to -\infty} f(x) = l \right\rangle, \text{ if } \left\langle \forall \varepsilon > 0 \ \exists n \in Z \setminus (Z \cup \{0\}) \ \exists \delta > 0: \ x < n \Rightarrow \ |f(x) - l| < \varepsilon \right\rangle.$$

Ex. 1. 4.
Given the function described in Ex. 1.3., find its point of satiation for the income approaching infinity.

Solution:

$$\lim_{Y \to \infty} B(Y) = \lim_{Y \to \infty} 60 \cdot e^{-\frac{1500}{Y}} = 60 \cdot e^0 = 60 \cdot 1 = 60$$

T. 1. 3.
Let $a, b, x_0 \in R$.

1. $\lim_{x \to x_0} b = b$.
2. $\lim_{x \to x_0} x = x_0$.
3. $\lim_{x \to x_0} (ax + b) = ax_0 + b$.
4. $\lim_{x \to x_0} |x| = x_0$.
5. $\lim_{x \to x_0} \sin x = \sin x_0$.
6. $\lim_{x \to x_0} \cos x = \cos x_0$.

T. 1. 4. (***The Algebra of Limits***)
Let f, g be two real functions. If $\lim_{x \to x_0} f(x) = a$ and $\lim_{x \to x_0} g(x) = b$, then the following rules hold:

1. $\lim_{x \to x_0} (f(x) + g(x)) = a + b$.
2. $\lim_{x \to x_0} (f(x) - g(x)) = a - b$.
3. $\lim_{x \to x_0} k \cdot f(x) = k \cdot a$ (k is any constant).
4. $\lim_{x \to x_0} f(x) \cdot g(x) = a \cdot b$.
5. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{a}{b}$ if $b \neq 0$.

T. 1. 5. (***The Sandwich Theorem***)
Let $A \subseteq R$ and $f, g, h : A \to R$ be three real functions.

$$\left\langle g(x) \leq f(x) \leq h(x),\ \forall x \in A \quad \wedge \quad \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = a \right\rangle \quad \Leftrightarrow \quad \lim_{x \to x_0} f(x) = a \quad .$$

T. 1. 6. (***Elementary Limits of Circular Functions***)
1. $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.
2. $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$.
3. $\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$.

T. 1. 7. (***Limit Theorem for Composites***)
Let f and g be two real functions.

$$\left\langle \lim_{x \to a} f(x) = b \ \wedge \ \lim_{a \to b} g(u) = g(b) \right\rangle \quad \Leftrightarrow \quad \left\langle \lim_{x \to a} g(f(x)) = g(b) \right\rangle .$$

or, equivalently,

$$\lim_{x\to a} g \circ f(x) = g\left(\lim_{x\to a} f(x)\right).$$

D. 1. 6. (***Continuity***)
Let $f : A \to R$ be a real function. f is said to be continuous at $x = x_0$ if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

or, equivalently

$$\forall \varepsilon > 0,\ \exists \delta > 0: \ |x - x_0| < \delta \ \Rightarrow \ |f(x) - f(x_0)| < \varepsilon .$$

R. 1. 2.
Here and throughout, the symbol $f \in C(x_0)$ means „ f is continuous at x_0 “. Thus, we obtain a rule to test for continuity at a given Point:

R. 1. 3. (***Continuity Test***)

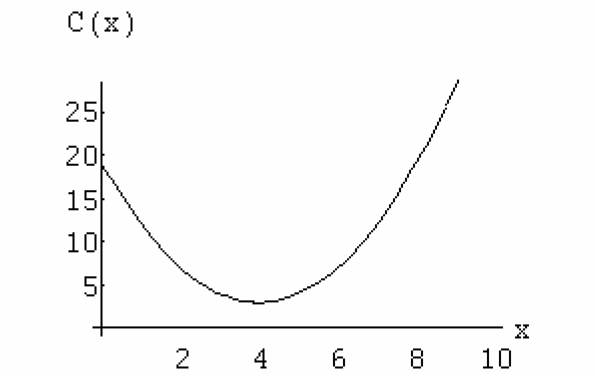
$$f \in C(x_0) \iff (i)\ x_0 \in A, \quad (ii)\ \lim_{x\to x_0} f(x) \text{ exists, and} \quad (iii)\ \lim_{x\to x_0} f(x) = f(x_0).$$

Ex. 1. 5.
A firm that produces some amount of output x has the following cost function:

$$C(x) = \begin{cases} (x-4)^2 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

1. What is the right-hand limit of this function as output approaches zero?
2. What is the function actually equal to when output is zero?
3. Is the function continuous? If not, explain why not.

Solution:
1.
$$\lim_{x\to 0^+} C(x) = 19 .$$
2.
$$C(0) = 0 .$$
3.
No, since $\lim_{x\to 0} C(0) \neq C(0)$.



T. 1. 8.
If f is continuous at $x = x_0$, then the following combinations are also continuous at $x = x_0$:

$(i)\ f + g$,
$(ii)\ f - g$,
$(iii)\ k \cdot f \quad (k \in R)$,
$(iv)\ f \cdot g$,
$(v)\ \dfrac{f}{g}$, provided $g(x_0) \neq 0$.

T. 1. 9.
If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composite $g \circ f$ is continuous at x_0.

D. 1. 7. (One-sided Continuity)
1. A function f is called *continuous from the left at* x_0, if

$$\lim_{x \to x_0^-} f(x) = f(x_0).$$

2. A function f is called *continuous from the right at* x_0, if

$$\lim_{x \to x_0^+} f(x) = f(x_0).$$

3. If f is defined on $[a,\ b]$, continuous on $]a,\ b[$ and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$, then f is said to be *continuous on* $[a,\ b]$.

T. 1. 10.

$$f \in C(x_0) \iff \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

<u>T. 1. 11.</u> (*Fundamental Theorem on Continuous Real Functions*)
If $f:\ [a,\ b]\to R$ is a continuous function, then $f([a,\ b])=[c,\ d]$ for some suitable $c,d\in R$. In addition, if a real function is continuous on $[a,\ b]$, then f attains an absolute maximum value M and an absolute minimum value m somewhere on this interval.

<u>T. 1. 12.</u> (*Min-Max Theorem for Continuous Functions*)
If $f:\ [a,\ b]\to R$ is a continuous function, then there exist $x_1, x_2\in[a,\ b]$ such that

$$f(x_1)\le f(x)\le f(x_2),\quad \forall x\in[a,\ b].$$

<u>T. 1. 13.</u> (*Intermediate Value Theorem for Continuous Functions*)
If $f:\ [a,\ b]\to R$ is a continuous function and if $f(a)\neq f(b)$, then for any $k\in[f(a),\ f(b)]$ (or $[f(b),\ f(a)]$), there exists a number $c\in[a,\ b]$ such that $f(c)=k$.

<u>T. 1. 14.</u> (*Intermediate Zero Theorem*)
If $f:\ [a,\ b]\to R$ is a continuous function and $f(a)\cdot f(b)<0$, then there exists $c\in[a,\ b]$ such that $f(c)=0$

<u>T. 1. 15.</u> (*Boundedness Theorem*)
If $f:\ [a,\ b]\to R$ is a continuous function, then there exists a positive number M such that

$$|f(x)|\le M,\quad \forall x\in[a,\ b].$$

<u>D. 1. 8.</u> (*Uniform Continuity*)
If $f:\ [a,\ b]\to R$ be a real function. f is said to be *uniformly continuous on* f if

$$\forall\varepsilon>0,\ \exists\delta>0:\ \ \forall x,y\in A,\ |x-y|<\delta\ \Rightarrow\ |f(x)-f(y)|<\varepsilon.$$

<u>R. 1. 4.</u>

Uniform continuity $\Rightarrow$ continuity, but not the converse

<u>D. 1. 9.</u> (*Derivative*)
Let $f:\ A\to R$ and $x_0\in\ ]a,\ b[\ \subseteq A$. The *derivative* of f is a function f' whose value at x_0 is the number

$$f'(x_0):=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

or, equivalently

$$f'(x_0):=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x_0)}{\Delta x}$$

with $x=x_0+\Delta x$.

Generally speaking, the derivative of f at any $x_0 \in \left]a,\ b\right[\subseteq A$ is given by

$$f\,'(x) := \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

If the derivative $f\,'(x_0)$ exists, we say that f *has a derivative* (or, *is differentiable*) *at* x_0 . If f has a derivative at every point of its domain, then f is said to be *differentiable*.

<u>Ex. 1. 6.</u>
A firm has the following revenue function

$$E(x) = 150x - 0.5x^2.$$

Find its derivative at $x_0 = 2$.

Solution:

$$E\,'(2) = \lim_{h\to 0}\frac{E(2+h)-E(2)}{h}$$

$$= \lim_{h\to 0}\frac{\left[150\cdot(2+h)-0.5\cdot(2+h)^2\right]-\left[150\cdot 2-0.5\cdot 2^2\right]}{h}$$

$$= \lim_{h\to 0}\frac{148h-h^2}{h}$$

$$= \lim_{h\to 0}(148-h) = 148.$$

<u>D. 1. 10.</u> (*One-sided Derivative*)
Let $f:\ A \to R$ be a real function. Then f is said to be *differentiable on* $[a,\ b]$ if $f\,'$ exists for all $x \in \left]a,\ b\right[$ and the limits

(i) $$f_+'(a) := \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$$ (*right-hand derivative at a*)

or, equivalently,

$$f_+'(a) := \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

and

(ii) $$f_-'(a) := \lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h}$$ (*left-hand derivative at b*)

or, equivalently,

$$f_-^{'}(a) := \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

exists at the endpoints a and b.

T. 1. 16.

A real function $f: A \to R$ is differentiable at $x = x_0$ if and only if

$$f_+^{'}(x_0) = f_-^{'}(x_0) = f'(x_0).$$

T. 1. 17. (***Differentiability-Continuity Theorem***)

If a real function f is differentiable at x_0, then f is continuous at x_0.

T. 1. 18. (***Algebra of Derivatives***)

If f and g are real functions that are differentiable at x, then $f \pm g$, $f \cdot g$ and f / g are differentiable at x (in the case of f / g provided that $g(x) \neq 0$). Moreover,

(i) $\left(f \pm g\right)'(x) = f'(x) \pm g'(x)$ (*Sum-Difference Rule*)

(ii) $\left(f \cdot g\right)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$ (*Product Rule*)

(iii) $\left(f / g\right)'(x) = \dfrac{f(x) \cdot g'(x) - g(x) \cdot f'(x)}{\left(g(x)\right)^2}$ (*Quotient Rule*)

T. 1. 19. (***Chain Rule***)

Let the function F be defined as f composed with g, that is $F = f \circ g = f(g(x))$. Then F' is given by

$$F' = \frac{dF}{dx} = f'(g(x)) \cdot g'(x).$$

R. 1. 5.

Alternatively, in Leibniz notation, the chain rule can be expressed as

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Ex. 1. 7.

A firm's cost function is given by

$$C(x) = 200 \cdot e^{0.01x+400}, \quad x \geq 0.$$

Find the marginal cost function.

Solution:

$$z = 0.01x + 400, \qquad \frac{dz}{dx} = 0.01$$

$$C(x) = 200 \cdot e^z, \qquad \frac{dC}{dz} = 200 \cdot e^z$$

$$C'(x) = \frac{dC(x)}{dx} = \frac{dz}{dx} \cdot \frac{dC(x)}{dz} = 0.01 \cdot 200 \cdot e^z = 2e^{0.01x+400}.$$

D. 1. 11. (***Elasticity Function***)

Let f be a differentiable function on A. The function

$$\varepsilon_{f,x}(x) := \frac{x}{f(x)} \cdot f'(x)$$

will be called the *elasticity function of* f *on* A.

R. 1. 6.

1. The elasticity $\varepsilon_{f,x}$ can be interpreted as follows: A percentage change of x leads to an approximate change of f by $\varepsilon_{f,x}$.

2. $\varepsilon_{f,x} > 0$ means that both x and f change in the same direction; $\varepsilon_{f,x} < 0$ means that x and f change in opposite directions.

3. Following cases can be distinguished:

$\lvert\varepsilon_{f,x}\rvert < 1$:	f is *inelastic,*
$\lvert\varepsilon_{f,x}\rvert > 1$:	f is *elastic*
$\lvert\varepsilon_{f,x}\rvert = 1$:	f is *proportional elastic,*
$\lvert\varepsilon_{f,x}\rvert \to \infty$:	f is *completely elastic,*
$\varepsilon_{f,x} \equiv 0$	f is *completely inelastic.*

4. Let f be denoted by $y = f(x)$ und its inverse by $x = g(y)$. Then

$$\varepsilon_{y,x} \cdot \varepsilon_{x,y} = 1.$$

Ex. 1. 8.

A firm has the revenue function

$$R(x) = 300x - 2.5x^2.$$

Find the elasticity function $\varepsilon_{E,x}(x)$ at the point $x = 10$ and interpret your result

Solution:

$$\varepsilon_{R,x}(x) = \frac{x}{300x - 2.5x^2} \cdot (300 - 5x),$$

$$\varepsilon_{R,x}(10) \approx 0.91.$$

An increase of demand from 10 units by 1% will lead to an increase of the revenue of the firm by approximately 0.91%. The revenue function is inelastic at $x = 10$.

<u>D. 1. 12.</u> (*Differential*)
Let f be a differentiable function at x and $\Delta x \neq 0$. The difference between $f(x+\Delta x)$ and $f(x)$, denoted by Δf,

$$\Delta f := f(x+\Delta x) - f(x)$$

is called the *increment of* f *from* x *to* $x+\Delta x$.
The product $f'(x)\Delta x$ is called *differential of* f *at* x *with increment* Δx, and is denoted by df,

$$df := f'(x)\Delta x$$

or, equivalently,

$$dy := f'(x)\Delta x.$$

<u>R. 1. 7.</u>
In the above definition, Δx can be any nonzero value. However, in most applications of differentials, we choose $dx = \Delta x$. Thus, we also write

$$dy := f'(x)dx.$$

We have

$$\Delta f \approx df$$

or, equivalently

$$\Delta y \approx f'(x)\Delta x = f'(x)dx$$

<u>Ex. 1. 9.</u>
The total cost function of a firm is given by

$$C(x) = 0.06x^3 - 2x^2 + 60x + 200.$$

The firm would like to increase its production from 10 to 12 units. Find the approximate increase of costs using the differential of the cost function.

Solution:

$$dC(x) = C'(x)\cdot dx_{|x=10,\ dx=2} = \left(0.18x^2 - 4x + 60\right)\cdot dx_{|x=10,\ dx=2} = 76.$$

<u>D. 1. 13.</u> (*Maximum and Minimum of a Function*)
Let $f: A \to R$ be a function.
1. A number M is called the *maximum value of* f *over* A if

$$f(x) \leq M, \ \forall x \in A \ \wedge \ f(c) = M \text{ for some } c \in A.$$

2. A number m is called the *minimum value of* f *over* A if

$$f(x) \geq m, \ \forall x \in A \ \wedge \ f(c) = m \text{ for some } c \in A.$$

T. 1. 20.
Let $f: A \to R$ be a function and $c \in A$. If $f'(c)$ exists and $f'(c) \neq 0$, then $f(c)$ is neither a maximum nor a minimum value of f in any neighbourhood of c.

Proof:
If $f'(c)$ and $f'(c) > 0$, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0 .$$

The there exists an Interval $]c - \delta, \ c + \delta[$ such that

$$\frac{f(x) - f(c)}{x - c} > 0, \ \forall x \in \ ]c - \delta, \ c[\cup]c, \ c + \delta[.$$

This implies that $f(x) - f(c)$ and $x - c$ have the same sign in $]c - \delta, \ c[\cup]c, \ c + \delta[$, that is,

$$f(x) < f(c) \text{ if } x < c$$

and

$$f(x) > f(c) \text{ if } x > c .$$

Hence, $f(c)$ is neither a maximum nor a minimum value of f in any neighbourhood of c. A similar argument holds if $f'(c) < 0$.

T. 1. 21. (***Contrapositive of T. 1. 20***)
Let $f: A \to R$ be a function and $c \in A$. If $f(c)$ is either a maximum or a minimum value of f in some neighbourhood of c, then either $f'(c)$ does not exist or $f'(c) = 0$.

D. 1. 14. (***Critical Number***)
Let $f: A \to R$ be a function. A number $c \in A$ is called a *critical number of* f if either $f'(c)$ does not exist or $f'(c) = 0$.

T. 1. 22. (***Rolle***)
Let $[a, \ b] \subseteq A$ and $f: A \to R$ be a function. If
(i) f is continuous on $[a, \ b]$,
(ii) f is differentiable on $]a, \ b[$, and
(iii) $f(a) = f(b)$,

then there exists an element $c \in \ ]a, \ b[$ such that $f'(c) = 0$

T. 1. 23. (***Mean Value Theorem***)

Let $[a,\ b] \subseteq A$ and $f:\ A \to R$ be a function. If f is continuous on $[a,\ b]$ and differentiable on $]a,\ b[$, then there exists a number $c \in\ ]a,\ b[$ such that

$$\frac{f(b)-f(a)}{b-a} = f\,'(c)\,.$$

D. 1. 15. (***Monotonic Functions***)

Let $f:\ A \to R$ be a function.

(i) f is said to be *monotonic increasing* if $f(x_1) \le f(x_2),\ \ \forall x_1, x_2 \in A: x_1 < x_2$.

(ii) f is said to be *monotonic decreasing* if $f(x_1) \ge f(x_2),\ \ \forall x_1, x_2 \in A: x_1 < x_2$.

(iii) f is said to be *strictly increasing* if $f(x_1) < f(x_2),\ \ \forall x_1, x_2 \in A: x_1 < x_2$.

(iv) f is said to be *strictly decreasing* if $f(x_1) > f(x_2),\ \ \forall x_1, x_2 \in A: x_1 < x_2$.

(v) f is said to be *monotonic* if f is either increasing in A or decreasing in A.

(vi) f is said to be *strictly monotonic* if f is either strictly increasing in A or strictly decreasing in A.

T. 1. 24. (***Monotonicity Theorem***)

Let $f:\ A \to R$ be a function. If f is continuous on $[a,\ b] \subseteq A$ and differentiable on $]a,\ b[$, then

(i) $f\,'(x) > 0,\ \forall x \in\ ]a,\ b[\ \ \Rightarrow\ f$ is strictly increasing on $[a,\ b]$.

(ii) $f\,'(x) < 0,\ \forall x \in\ ]a,\ b[\ \ \Rightarrow\ f$ is strictly decreasing on $[a,\ b]$.

Ex. 1. 10.

Find the intervals in which the profit function

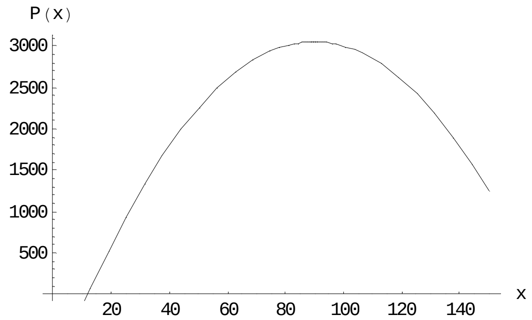
$$P(x) = -0.5x^2 + 90x - 1000$$

is strictly monotonic and sketch the graph of f.

Solution:

$$P\,'(x) = -x + 90\,,$$
$$P\,'(x) > 0\ \Rightarrow\ x < 90\,.$$

$\therefore\ f$ is strictly increasing in $]0,\ 90[$ and strictly decreasing for $x > 90$.



D. 1. 16. (***Extrema of a Function***)
Let $f: A \to R$ be a function and $x_0 \in A$. We say that

(i) $f(x_0)$ is a *relative maximum of f on A* if $f(x) \leq f(x_0), \ \forall x \in A \cap U_\varepsilon(x_0)$.

(ii) $f(x_0)$ is a *relative minimum of f on A* if $f(x) \geq f(x_0), \ \forall x \in A \cap U_\varepsilon(x_0)$.

(iii) $f(x_0)$ is an *absolute maximum of f on A* if $f(x) \leq f(x_0), \ \forall x \in A$.

(iii) $f(x_0)$ is an *absolute minimum of f on A* if $f(x) \geq f(x_0), \ \forall x \in A$.

T. 1. 25. (***A Necessary Condition for Relative Extrema***)
If a function f has a relative extremum at a number x_0, then $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

T. 1. 26.
If f is a differentiable function on a set A and has a relative extremum at x_0, then $f'(x_0) = 0$.

T. 1. 27. (***First Derivative Test for Extrema***)
Let x_0 be a critical number of f and f be continuous at x_0. If there exists an $\varepsilon > 0$ such that

(i)

$$\left\langle f'(x) < 0, \ \forall x \in \left]x_0 - \varepsilon, \ x_0\right[\ \wedge \ f'(x) > 0, \ \forall x \in \left]x_0, \ x_0 + \varepsilon\right[\right\rangle$$
then
$$\left\langle f(x_0) \text{ is a relative minimum} \right\rangle.$$

(ii)

$$\left\langle f'(x) > 0, \ \forall x \in \left]x_0 - \varepsilon, \ x_0\right[\ \wedge \ f'(x) < 0, \ \forall x \in \left]x_0, \ x_0 + \varepsilon\right[\right\rangle$$
then
$$\left\langle f(x_0) \text{ is a relative maximum} \right\rangle.$$

(iii)

$$\left\langle f'(x) \text{ keeps the same sign on } \left]x_0 - \varepsilon, \ x_0\right[\cup \left]x_0, \ x_0 + \varepsilon\right[\right\rangle$$
then
$$\left\langle f(x_0) \text{ is not an maximum} \right\rangle.$$

T. 1. 28. (***Second Derivative Test for Extrema***)
If x_0 is a critical number of a function f which is twice differentiable on an interval $\left]x_0 - \varepsilon, \ x_0 + \varepsilon\right[$ for some $\varepsilon > 0$, then

(i)

$$f''(x_0) < 0 \quad \Rightarrow \quad f(x_0) \text{ is a relative maximum of } f.$$

(ii)

$$f''(x_0) > 0 \quad \Rightarrow \quad f(x_0) \text{ is a relative minimum of } f.$$

R. 1. 8.
An algorithm for obtaining the extrema of a given continuous function f on the interval $[a,\ b]$ is given as follows:

1. Determine all critical numbers of f on $]a,\ b[$.
2. Calculate the value of f at all its critical numbers and also $f(a)$ and $f(b)$.
3. Compare all the value of f in 2. and the largest one is the maximum value of f on $[a,\ b]$ while the smallest value is the minimum value of f on $[a,\ b]$.

Ex. 1. 11.
The profit function of a firm depending on the price charged for its product is given by

$$P(p) = -p^3 + 4800p\text{-}119000, \qquad 0 \le p \le 45.$$

Find the price for which its profit will be maximised.

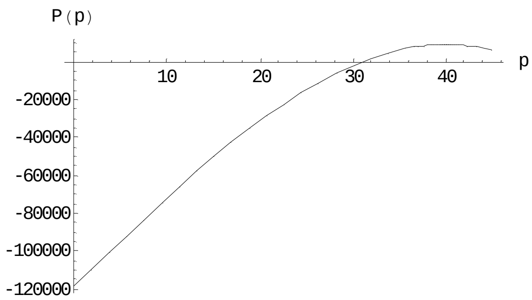
Solution:
1.
$$P'(p) = -3p^2 + 4800 := 0, \quad 0 < p < 45 \qquad \Rightarrow \qquad p = 40$$

$$P''(p) = -6p < 0.$$

Hence, the only critical numbers of P on $]0,\ 45[$ is $p = 40$ with $P(40) = 9000$.

2.
$$P(0) = -119000, \quad P(45) = 5875.$$

3.
Because of $-119000 < 5875 < 9000$ the firm's profit will be maximised for $p = 40$:



D. 1. 17. (***Convex and Concave Function***)
Let f be a function which is continuous on $[a,\ b]$ and is differentiable on $]a,\ b[$. We say that

(i) f is convex if the graph of f lies above the tangent lines to f throughout $]a,\ b[$.
(ii) f is concave if the graph of f lies below the tangent lines to f throughout $]a,\ b[$.

<u>T. 1. 29.</u> (*Test for Convexity and Concavity*)

Let $f: A \to R$ be a function whose second derivative exists on $]a,\ b[$. Then we have:

(i) $f''(x) > 0,\ \forall x \in\ ]a,\ b[\quad \Rightarrow \quad f$ is convex.

(ii) $f''(x) < 0,\ \forall x \in\ ]a,\ b[\quad \Rightarrow \quad f$ is concave.

<u>D. 1. 18.</u> (*Inflection Points*)

The point $(x_0,\ f(x_0))$ is called a *point of inflection* of a function f if there exists an interval $]x_0 - \varepsilon,\ x_0 + \varepsilon[$ such that $f''(x) > 0,\ \forall x \in\ ]x_0 - \in,\ x_0[$ and $f''(x) < 0,\ \forall x \in\ ]x_0,\ x_0 + \varepsilon[$.

<u>T. 1. 30.</u> (*Test for Inflection Points*)

If $(x_0,\ f(x_0))$ is an inflection point of f, then either $f''(x_0) = 0$ or $f''(x_0)$ does not exist.

<u>Ex. 1. 12.</u>

A firm has the total cost function

$$C(x) = 8400x + \left(1008000x^2 - 3060x^3 + 3x^4\right) \cdot 10^{-4}, \quad x \geq 100.$$

Discuss the most important properties of its average cost function $c(x) := \dfrac{C(x)}{x}$.

Solution:

$$c(x) = 8400 + 10^{-4}(1008000x - 3060x^2 + 3x^3)$$

1. *<u>Domain</u>*

$$D(c(x)) = \left[100,\ \bar{x}\right] \quad ; \qquad \bar{x} : \text{maximum production capacity.}$$

Let us assume that the firm has a maximum capacity of $\bar{x} = 800$.

1. *<u>Continuity</u>*

$c(x)$ ist continuous on D.

3. *<u>Points of Intersection with the Axes</u>*

a) *with the $x-$axis*

$$c(x) := 0 \quad \Rightarrow \quad 8400 + 10^{-4}(1008000x - 3060x^2 + 3x^3) = 0$$

It can be (numerically) shown that the graph of $c(x)$ does not cut the $x-$axis.

b) *With the $y-$ axis*

Because of $x \geq 100$ there is also no point of intersection with the $y-$ axis.

4. *<u>Monotonicity</u>*

$c'(x) = 10^{-4}(1008000 - 6120x + 9x^2)$

$c'(x) := 0 \quad \Rightarrow \quad x_1 = 280, \quad x_2 = 400$

$(x-280)\cdot(x-400) \geq 0 \quad \Rightarrow \quad c(x)$ is non-decreasing.

The solution of the above inequality leads to the following results:

$c(x)$ is non-decreasing for $\forall x \in]100, 280[\cup]400, 800[$
$c(x)$ is non-increasing for $\forall x \in]280, 400[$.

5. *Extrema*

$c''(x) = 10^{-4}(-6120x + 18x)$; $\quad c''(280) = -0.108 < 0$; $\quad c''(400) = 0.108 > 0$.

Thus, $c(x)$ assumes a relative maximum at $x = 280$ with $c(280) = 19219.20$ and a relative minimum at $x = 400$ with $c(400) = 18960.00$.
Because of $c(100) = 15720.00$ and $c(800) = 46800.00$ the absolute minimum and the absolute maximum of $c(x)$ are $(100,\ 15720)$ and $(800,\ 46800)$, respectively.

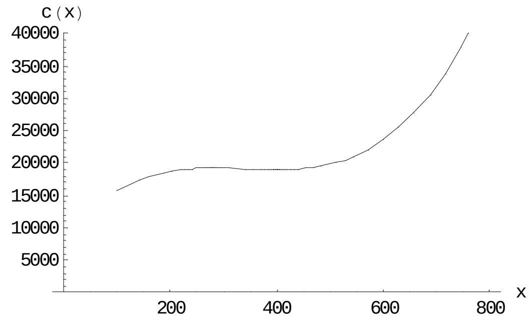
6. *Convexity and Concavity*

$$\left\langle c''(x) = 10^{-4}(-6120x + 18x) \geq 0 \right\rangle \quad \Rightarrow \quad \left\langle \begin{matrix} c(x) \text{ is convex}, \forall x \in]340,\ 800[\\ c(x) \text{ is cocave}, \forall x \in]100,\ 340[\end{matrix} \right\rangle, \text{ i.e. } x \geq 340$$

$\therefore$ $(340,\ 19089.60)$ is a point of inflection of $c(x)$.

The average costs increase progressively in $]340, 800[$ and regressively in $]100, 340[$.

7. *Graph of the Function:*



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