Chapter V

Continuous Functions

D. 5. 1. (Continuity at One Point)
Let f be a function defined on a neighbourhood of x_0, $N_\varepsilon(x_0)$. f is *continuous at* x_0 if $\lim_{x \to x_0} f(x)$ exists and

$$\lim_{x \to x_0} f(x) = f(x_0)$$

If f is not continuous at x_0, it is said to be *discontinuous at* x_0.

T. 5. 1.

$$f \text{ and } g \text{ are continuous at } x_0 \;\Rightarrow\; f + g \text{ is continuous at } x_0$$

(The converse is not true!)

T. 5. 2.

$$f \text{ and } g \text{ are continuous at } x_0 \;\Rightarrow\; f \cdot g \text{ is continuous at } x_0$$

(The converse is not true!)

T. 5. 3.

$$f \text{ and } g \text{ are continuous at } x_0 \text{ and } g(x_0) \neq 0$$
$$\Rightarrow$$
$$\frac{f}{g} \text{ is continuous at } x_0$$

(The converse is not true!)

T. 5. 4.

$$f \text{ and } g \text{ are continuous at } x_0 \text{ and } g \text{ is continuous at } y_0 = f(x_0)$$
$$\Rightarrow$$
$$g \circ f \text{ is continuous at } x_0$$

(The converse is not true!)

D. 5. 2. (Removable Discontinuity)
Take a function f defined on a pointed neighborhood N of. x_0. Suppose that f has a finite limit a at x_0. Then we say that f has a *removable discontinuity* at x_0.

D. 5. 3. (One-Sided Discontinuity)
The function is *left* (resp. *right*) *continuous* at x_0 if it has a left limit (resp. right limit) at x_0 and

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

$$\text{(resp. } \lim_{x \to x_0^+} f(x) = f(x_0)\text{)}$$

D. 5. 4. (Continuity on an Interval)
Let f be a function defined on an open interval I. f is *continuous on* I if it is continuous at every point of I.

R. 5. 1.

f is continuous on $I \Rightarrow f(I)$ is an interval

R. 5. 2.

f is continuous on the closed interval $I \Rightarrow f(I)$ is a closed interval

T. 5. 5.(Intermediate Value Theorem – First Version)
Let f be a function, defined on the interval $[a, b]$. If f is continuous on $[a, b]$, then f achieves any value between $f(a)$ and $f(b)$.

T. 5. 6.(Intermediate Value Theorem – Second Version)
Let f be a function, defined on the interval $[a, b]$. If f is continuous on $[a, b]$ and if $f(a) \cdot f(b) < 0$, then there exists a real number $c \in]a, b[$ such that $f(c) = 0$.

T. 5. 7.
Let f be a function defined on an interval I. Suppose that f is continuous on I and that f is strictly monotonous on I. Then f is a bijection from I onto $f(I)$. The inverse function f^{-1} is continuous on $f(I)$.

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