Kapitel VIII

Das unbestimmte Integral
(Lösungen)

8. 1.
a)

$$\int (x^3 + \frac{2}{x} - \frac{4}{x^3})dx = \int x^3 dx + \int \frac{2}{x}dx - \int \frac{4}{x^3}dx$$

$$= \int x^3 dx + 2 \cdot \int \frac{1}{x}dx - 4 \cdot \int x^{-3}dx = \frac{x^4}{4} + 2\ln|x| + \frac{2}{x^2} + C\,,\ x \neq 0\,.$$

b)

$$\int \sqrt{x^3}dx = \int x^{\frac{3}{2}}dx = \frac{2}{5}x^{\frac{5}{2}} + C = x^2 \cdot \sqrt{x} + C\,,\ x \geq 0$$

8. 2.
a)
Wegen

$$x^2 + 2x + 2 = (x+1)^2 + 1$$

gilt.

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{1 + (1+x)^2}\,.$$

$$u = 1 + x\,, \qquad \frac{du}{dx} = 1\,, \qquad dx = du\,.$$

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{du}{1+u^2} = \arctan u + C$$

$$= \arctan(1+x) + C\,.$$

b)

$$u = 4 - x^2\,, \qquad du = -2x \cdot dx\,.$$

$$\int \frac{x \cdot dx}{4 - x^2} = -\frac{1}{2}\int \frac{du}{u} + C$$

$$= -\frac{1}{2}\ln(4 - x^2) + C$$

$$= \ln \frac{1}{\sqrt{4-x^2}} + C$$

c)

$u = 1 + e^x$, $\quad du = e^x \cdot dx$, $\quad dx = \frac{du}{u-1}$;

$$\int \frac{dx}{1+e^x} = \int \frac{du}{u \cdot (u-1)} = \int \left(\frac{1}{u-1} - \frac{1}{u} \right) \cdot du$$

$$= \int \left(\frac{du}{u-1} \right) - \int \frac{du}{u} = \int \frac{d(u-1)}{u-1} - \int \frac{du}{u}$$

$$= \ln(u-1) - \ln u + C$$

$$= \ln e^x - \ln(1+e^x) + C$$

$$= \ln \frac{e^x}{1+e^x} + C = \ln \left(\frac{1+e^x}{e^x} \right)^{-1} + C$$

$$= -\ln(1+e^{-x}) + C .$$

d)

$u = \ln(x+1)$, $\quad du = \frac{dx}{x+1}$,

$$\int \frac{\sqrt[3]{\ln(x+1)}}{x+1} dx = \int u^{\frac{1}{3}} du = \frac{3}{4} u^{\frac{4}{3}} + C$$

$$= \frac{3}{4} [\ln(x+1)]^{\frac{4}{3}} + C$$

8. 3.

a)

$u = \ln x$, $\quad \frac{du}{dx} = \frac{1}{x}$, $\quad dx = x \cdot du$.

$$\int \frac{(\ln x)^2}{x} dx = \int \frac{u^2}{x} \cdot x du = \frac{u^3}{3} + C = \frac{1}{3} \cdot (\ln x)^3 + C .$$

b)

$$\int \frac{1}{9+2x^2}dx = \int \frac{dx}{9\cdot\left(1+\left(\frac{\sqrt{3}}{3}\right)^2\right)};$$

$$u = \frac{\sqrt{2}}{3}\cdot x, \qquad \frac{du}{dx} = \frac{\sqrt{2}}{3}, \qquad dx = \frac{3}{\sqrt{2}}\cdot du;$$

$$\int \frac{1}{9+2x^2}dx = \frac{3}{9\sqrt{2}}\int \frac{du}{1+u^2} = \frac{3}{9\sqrt{2}}\arctan u + C = \frac{1}{3\sqrt{2}}\arctan\frac{\sqrt{2}}{3}x + C$$

8. 4.

a)

$$u = \arctan x, \qquad \frac{du}{dx} = \frac{1}{1+x^2}, \qquad dx = \left(1+x^2\right)\cdot du.$$

$$\int \frac{\arctan x}{1+x^2}dx = \int \frac{u}{1+x^2}\cdot\left(1+x^2\right)du = \frac{u^2}{2} + C = \frac{1}{2}\cdot(\arctan x)^2 + C.$$

b)

$$u = 8x^3 - 1, \qquad \frac{du}{dx} = 24x^3, \qquad dx = \frac{1}{24x^2}\cdot du.$$

$$\int x^2\cdot\sqrt{8x^2-1}\cdot dx = \int x^2\cdot u^{\frac{1}{2}}\cdot\frac{1}{24x^2}du = \frac{1}{24}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}} + C = \frac{1}{36}\cdot\left(8x^3-1\right)\cdot\sqrt{8x^3-1} + C.$$

8. 5.

a)

$$u = 4x-2, \qquad \frac{du}{dx} = 4, \qquad dx = \frac{1}{4}\cdot du$$

$$\int \frac{dx}{(4x-2)^3} = \frac{1}{4}\cdot\int\frac{1}{u^3}\cdot du = \frac{1}{4}\int u^{-3}\cdot du = -\frac{1}{8}\cdot u^{-2} + C$$

$$= -\frac{\frac{1}{8}}{(4x-2)^2} + C, \qquad x \neq \frac{1}{2}.$$

b)

$$u(x) = x, \qquad u'(x) = 1;$$

$$v'(x) = e^{3x}, \qquad v(x) = \frac{1}{3}e^{3x}$$

$$\int x \cdot e^{3x} dx = \frac{1}{3} x \cdot e^{3x} - \frac{1}{3} \int e^{3x} dx = \frac{1}{3} x \cdot e^{3x} - \frac{1}{9} e^{3x} + C$$

$$= \frac{1}{3} e^{3x} \cdot \left(x - \frac{1}{3} \right) + C .$$

c)

$u(x) = x^2$, $\qquad u'(x) = 2x$;

$v'(x) = \sin 4x$, $\qquad v(x) = -\frac{1}{4} \cos 4x$

$$\int x^2 \sin 4x dx. = \left(-\frac{1}{4} \cos 4x \right) \cdot x^2 + \frac{1}{2} \cdot \int x \cdot \cos 4x dx .$$

$u(x) = x$, $\qquad u'(x) = 1$;

$v'(x) = \cos 4x$, $\qquad v(x) = \frac{1}{4} \sin 4x$

$$\int x \cdot \cos 4x dx = \frac{1}{4} x \cdot \sin 4x - \frac{1}{4} \int \sin 4x dx = \frac{1}{4} x \cdot \sin 4x + \frac{1}{16} \cdot \cos 4x + C_1$$

$$\int x^2 \sin 4x dx. = -\frac{1}{4} x^2 \cdot \cos 4x + \frac{1}{8} x \cdot \sin 4x + \frac{1}{32} \cos 4x + C .$$

8. 6.

a)

$$\left(2x^3 + 9x^2 + 8x + 5\right) : \left(x^2 + 4x + 3\right) = 2x + 1 + 2 \cdot \frac{-x + 1}{x^2 + 4x + 3}$$

$$\frac{-x + 1}{x^2 + 4x + 3} = \frac{A}{x + 1} + \frac{B}{x + 3} \quad \Rightarrow \quad A = 1, \ \ B = -2$$

$$\int \frac{2x^3 + 9x^2 + 8x + 5}{x^2 + 4x + 3} dx = \int (2x + 1) dx + 2 \cdot \left[\int \frac{dx}{x + 1} - 2 \cdot \int \frac{dx}{x + 3} \right]$$

$$= x^2 + x + 2 \ln \frac{|x + 1|}{(x + 3)^2} + C .$$

b)

$$\frac{4x^3 - 2x^2 + 9x - 18}{x^2 \cdot \left(x^2 + 9\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{CX + D}{x^2 + 9} \quad \Rightarrow A = 1, \ B = -2 , \ C = 3 , \ D = 0 .$$

$$\int \frac{4x^3 - 2x^2 + 9x - 18}{x^2 (x^2 + 9)} dx = \int \frac{dx}{x} - 2 \cdot \int \frac{dx}{x^2} + 3 \cdot \int \frac{x}{x^2 + 9} dx$$

$$= \ln|x| + \frac{2}{x} + \frac{3}{2}\ln\left|x^2 + 9\right| + C$$

8. 7.

$$t := \sqrt{2x-1}, \quad x = \frac{1+t^2}{2}, \qquad dx = t \cdot dt$$

$$\int \frac{dx}{x+\sqrt{2x-1}} = \int \frac{t \cdot dt}{\frac{1+t^2}{2}+t}. = \int \frac{2t}{(1+t)^2}.dt := I_1$$

$$u = 1+t;$$

$$I_1 = 2 \cdot \int \frac{u-1}{u^2} \cdot du = 2 \cdot \int \frac{du}{u} - 2\int \frac{du}{u^2} = 2\ln|u| + \frac{2}{u} + C$$

$$u = 1+t = 1+\sqrt{2x-1},$$

$$u = 1+t = 1+\sqrt{2x-1},$$

$$I = 2\ln\left(1+\sqrt{2x-1}\right) + \frac{2}{1+\sqrt{2x-1}} + C, \qquad 2x-1 > 0.$$

8. 8.

$$t := e^x, \qquad dt = t \cdot dx, \qquad dx = \frac{dt}{t};$$

$$\int \frac{dx}{e^{2x}-1}. = \int \frac{dt}{t \cdot (t^2-1)} = \int \frac{dt}{t \cdot (t+1) \cdot (t-1)} = \int \left(1 - \frac{1}{t} + \frac{1}{2 \cdot (t+1)} + \frac{1}{2 \cdot (t-1)}\right) \cdot dt$$

$$= -\ln|t| + \frac{1}{2}\ln|t+1| + \frac{1}{2}\ln|t-1| + C$$

$$= -\ln e^x + \frac{1}{2}\ln\left(e^x+1\right) + \frac{1}{2}\ln\left|e^x-1\right| + C \qquad x \neq 0$$

8. 9.

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$t = \tan\frac{x}{2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2dt}{1+t^2}$$

$$\int \frac{dx}{\cos x} = \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} \cdot dt = 2 \cdot \int \frac{dt}{1-t^2} = \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) \cdot dt$$

$$= \ln|1-t| + \ln|1+t| + C = \ln\left|1-t^2\right| + C$$

$$= \ln\left|1 - \tan^2 \frac{x}{2}\right| + C \, .$$

8. 10.

a)

$$x^2 - 6x + 12 = (x-3)^2 + 3 = t^2 + 2 \quad \text{mit } t := x - 3\,; \quad dt = dx$$

$$\int \frac{dx}{x^2 - 6x + 12} = \int \frac{dt}{t^2 + 3} = \frac{1}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}} \arctan \frac{x-3}{\sqrt{3}} + C \, .$$

b)

$$\int \frac{dx}{x^2 - 4x + 4} = \int \frac{dx}{(x-2)^2} = \left[\int \frac{dt}{t^2}\right]_{t=x-2} = \left[-\frac{1}{t} + C\right]_{t=x-2} = \frac{1}{2-x} + C \, .$$

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