Kapitel X
Das uneigentliche Integral

(Lösungen)

10. 1.

a)

$$\int_0^{\infty} \frac{dx}{a^2 + x^2} = \frac{1}{a^2} \int_0^{\infty} \frac{dx}{1 + \left(\frac{x}{a}\right)^2},$$

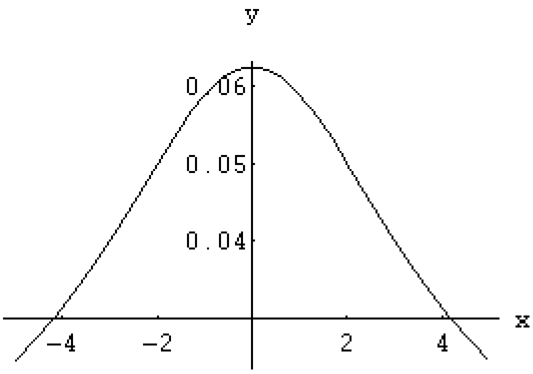
$$u = \frac{x}{a}, \qquad \frac{du}{dx} = \frac{1}{a}, \qquad dx = a \cdot du$$

$$\int_0^{\infty} \frac{dx}{a^2 + x^2} = \frac{1}{a^2} \int_0^{\infty} \frac{a \cdot du}{1 + u^2} = \left[\frac{1}{a} \arctan u\right]_0^{\infty}$$

$$= \left[\frac{1}{a} \cdot \arctan \frac{x}{a}\right]_0^{\infty} = \frac{1}{a}\left[\lim_{x \to +\infty} \arctan \frac{x}{a} - \lim_{x \to 0} \arctan \frac{x}{a}\right]$$

$$= \frac{1}{a} \cdot \frac{\pi}{2}, \quad a > 0$$

Skizze für $a = 4$:

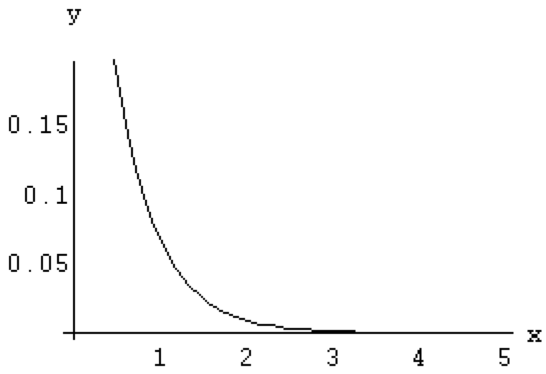


b)

$$u = -2x, \qquad \frac{du}{dx} = -2, \qquad dx = -\frac{1}{2} du.$$

$$\int_0^{\infty} \frac{1}{2} e^{-2x} dx = \frac{1}{2} \cdot \int_0^{+\infty} e^u \cdot \left(\frac{1}{2}\right) \cdot du = \left[-\frac{1}{4} \cdot e^u\right]_0^{+\infty}$$

$$= -\frac{1}{4}\left(\lim_{x \to +\infty} e^{-2x} - \lim_{x \to 0} e^{2x}\right) = \frac{1}{4}.$$



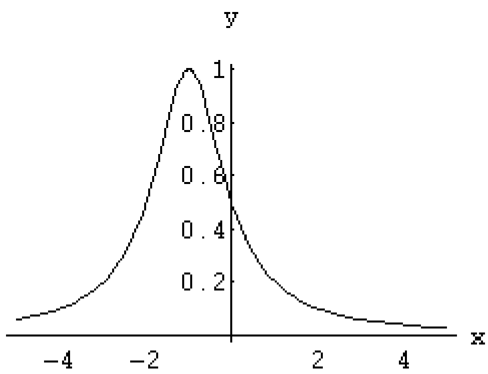
c)

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{+\infty} \frac{dx}{(x+1)^2+1},$$

$u = x+1, \qquad du = dx;$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{+\infty} \frac{du}{1+u^2} = [\arctan u]_{-\infty}^{+\infty} = [\arctan(x+1)]_{-\infty}^{+\infty}$$

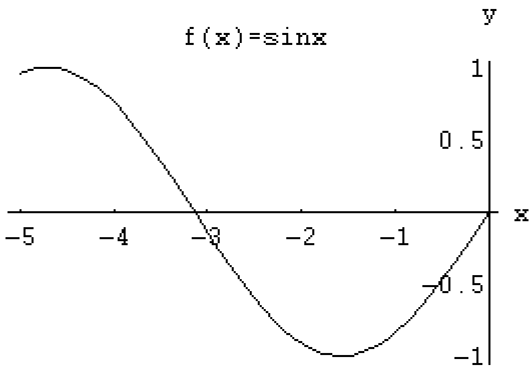
$$= \lim_{x\to+\infty} \arctan(x+1) - \lim_{x\to-\infty} \arctan(x+1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$



d)

$$\int_{-\infty}^{0} \sin x dx = [-\cos x]_{-\infty}^{0} = -\left(\lim_{x\to 0} \cos x - \lim_{x\to-\infty} \cos x\right) = -1 + \lim_{x\to-\infty} \cos x.$$

Da $\lim_{x\to-\infty} \cos x$ nicht existiert, existiert auch das Integral $\int_{-\infty}^{0} \sin x dx$ nicht.



10. 2.

$$\int_2^{\infty}\frac{dx}{x^2-1}=\lim_{t\to+\infty}\int_2^{t}\frac{dx}{(x-1)\cdot(x+1)}$$

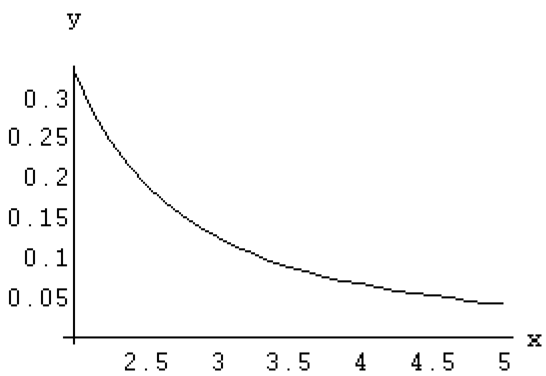
$$=\lim_{t\to+\infty}\int_2^{t}\left[\frac{1}{2\cdot(x-1)}-\frac{1}{2\cdot(x+1)}\right]\cdot dx$$ (Partialbruchzerlegung)

$$=\lim_{t\to+\infty}\left[\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|\right]_2^t$$

$$=\frac{1}{2}\lim_{t\to+\infty}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_2^t=\frac{1}{2}\lim\left[\ln\left|\frac{t-1}{t+1}\right|-\ln\frac{1}{3}\right]$$

$$=\frac{1}{2}\lim_{t\to+\infty}\ln\left|\frac{1-\frac{1}{t}}{1+\frac{1}{t}}\right|-\frac{1}{2}\lim_{t\to+\infty}\ln\frac{1}{3}$$

$$=\frac{1}{2}\cdot(0)+\frac{1}{2}\ln\left(\frac{1}{3}\right)^{-1}=\frac{1}{2}\ln 3=\ln\sqrt{3}\,.$$



10. 3.

$$CH\int_{-\infty}^{+\infty} x^3 dx = \lim_{\omega\to\infty}\left[\frac{x^4}{4}\right]_{-\omega}^{+\omega} = \lim_{\omega\to\infty}\left(\frac{\omega^4}{4}-\frac{\omega^4}{4}\right) = 0\,.$$

Der Cauchysche Hauptwert des uneigentlichen Integrals existiert und ist gleich Null. Das uneigentliche Integral selbst existiert nicht, denn es gilt

$$\int_{-\infty}^{+\infty} x^3 dx = \int_{-\infty}^{0} x^3 dx + \int_{0}^{+\infty} x^3 dx = \left[\frac{x^4}{4}\right]_{-\infty}^{0} + \left[\frac{x^4}{4}\right]_{0}^{+\infty}$$

$$\left(\lim_{x\to 0}\frac{x^4}{4} - \lim_{x\to-\infty}\frac{x^4}{4}\right) + \left(\lim_{x\to+\infty}\frac{x^4}{4} - \lim_{x\to 0}\frac{x^4}{4}\right).$$

10. 4.

Wegen $e^x > 1, \forall x > 0$, gilt $x^2 \cdot e^x > x^2, \forall x > 0$. Hieraus folgt

$$f(x) := \frac{1}{x^2 \cdot e^x} < \frac{1}{x^2} =: g(x)\,,$$

$$\int_{1}^{+\infty} g(x)dx = \int_{1}^{+\infty} x^{-2} \cdot dx = \left[-x^{-1}\right]_{1}^{+\infty} = -\left(\lim_{x\to+\infty}\frac{1}{x} - \lim_{x\to 1}\frac{1}{x}\right) = 1,$$

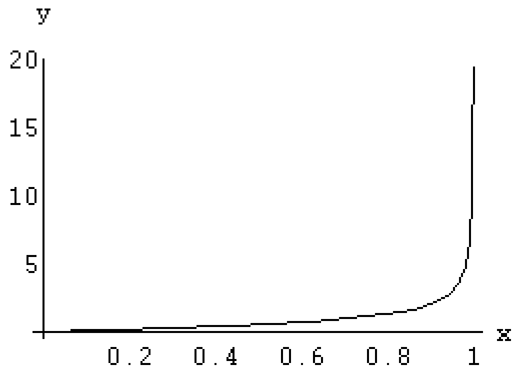
d. h. das uneigentliche Integral existiert.

10. 5.

a)

$$\int_{0}^{1}\frac{xdx}{\sqrt{1-x^2}} = \lim_{\varepsilon\to 0^+}\int_{0}^{1-\varepsilon}\frac{x}{\sqrt{1-x^2}}dx = \lim_{\varepsilon\to 0^+}\left\{\left[-\sqrt{1-x^2}\right]_{0}^{1-\varepsilon}\right\}$$

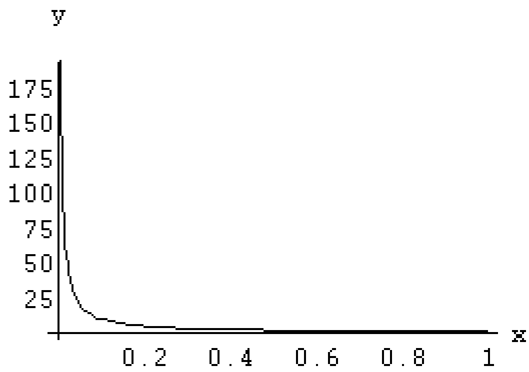
$$= \lim_{\varepsilon\to 0^+}\left\{-\sqrt{1-(1-\varepsilon)^2}+1\right\} = 1\,.$$



b)

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dx}{x} = \lim_{\varepsilon \to 0^+} \left\{ [\ln x]_\varepsilon^1 \right\} = \lim_{\varepsilon \to 0^+} (\ln 1 - \ln \varepsilon) = +\infty \,,$$

das Integral ist also divergent.



c)

$$\int_0^1 \ln x dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln x dx = \lim_{\varepsilon \to 0_+} \left\{ [x \cdot \ln x - x]_\varepsilon^1 \right\}$$

$$= \lim_{\varepsilon \to 0^+} \{-1 - \varepsilon \cdot \ln \varepsilon + \varepsilon\} = \lim_{\varepsilon \to 0^+} \left\{ -1 - \frac{\ln \varepsilon}{\frac{1}{\varepsilon}} + \varepsilon \right\} = -1 \qquad (\because \text{ L'Hospital})$$

d)

$$\int_0^2 \frac{dx}{(x-1)^4} = \lim_{\varepsilon_1 \to 0^+} \int_0^{1-\varepsilon_1} \frac{dx}{(x-1)^4} . + \lim_{\varepsilon_2 \to 0^+} \int_{1+\varepsilon_2}^2 \frac{dx}{(x-1)^4} = +\infty \,,$$

das Integral ist also divergent.

10. 6.

a)

$$\int_0^1 x \cdot \ln x dx,$$

$$u = \ln x \,, \qquad u' = \frac{1}{x};$$

$$v' = x \,, \qquad v = \frac{x^2}{2}.$$

$$\int x \ln x dx = \frac{x^2}{2} \cdot \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \cdot \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C = \frac{x^2}{2} \cdot \left(\ln x - \frac{1}{2} \right) + C \,.$$

$$\int_0^1 x \cdot \ln x dx = \lim_{\varepsilon \to o^+} \left\{ \frac{x^2}{2} \cdot \left(\ln x - \frac{1}{2} \right) \right\}_{\varepsilon}^{1} = \frac{1}{2} \cdot \left(\ln 1 - \frac{1}{2} \right) = -\frac{1}{4} .$$

b)

$$\int_1^2 \frac{dx}{x \ln x}.$$

$$\int \frac{dx}{x \ln x}. = \int \frac{\frac{1}{x}}{\ln x} \cdot dx = \ln(\ln(x)) + C)$$

$$\int_1^2 \frac{dx}{x \ln x}. = \lim_{\varepsilon \to 1^+} [\ln(\ln x)]_{\varepsilon}^{2} = \lim_{\varepsilon \to 1^+} (\ln(\ln(2) - \ln(\ln \varepsilon)))$$

$$= \lim_{\varepsilon \to 1^+} \left(\ln \frac{\ln 2}{\ln \varepsilon} \right) = \lim_{\varepsilon \to 1^+} \left(\ln 2 \cdot \ln \frac{1}{\ln \varepsilon} \right) = +\infty .$$

Das Integral existiert also nicht.

(Letzte Aktualisierung: 22.02.05)