Chapter VIII

Some Special Continuous Distributions

<u>D. 8. 1.</u> (*Normal Distribution*)
A continuous variable X has *normal* (or *Gaussian*) distribution if its probability density function is of the form

$$f(x;\mu,\sigma) = \frac{1}{\sigma}\cdot\varphi\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x\in R^1$$

<u>T. 8. 1.</u>
The *distribution function* of a normally distributed variable X is given by

$$F(x;\mu,\sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{t^2}{2}}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt, \qquad x\in R^1.$$

Proof: (see: D. 5. 3.)

<u>R. 8. 1.</u>
The old German money 10 Mark notes had Karl F. Gauss printed on the back, and a small bell shaped curve and its formula in the background:

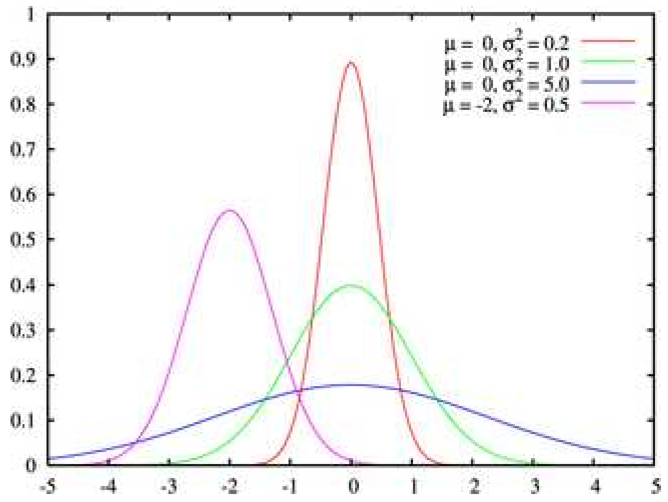
T. 8. 2.
Let X be a normally distributed variable. Then

$$E(X) = \mu,$$

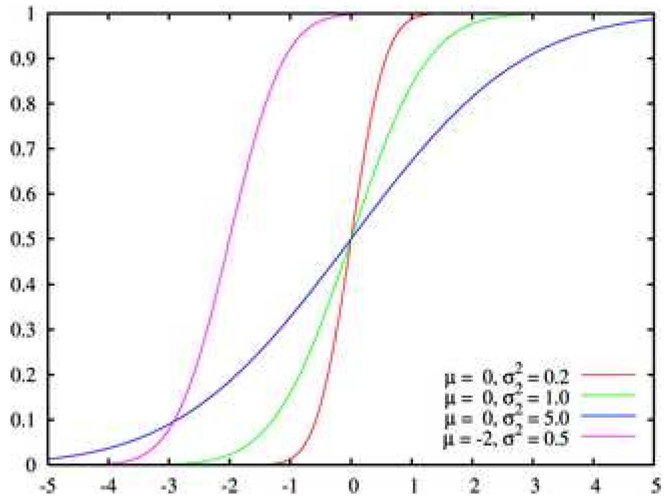
$$D^2(X) = \sigma^2.$$

R. 8. 2.
The following charts show the probability density and the distribution functions of a normally distributed variable X for different values of μ and σ^2:

Probability Density Function



Distribution Function



R. 8. 3.(*Some Important Properties of the Normal Density Function)*
The probability density function of a normally distributed random variable has the following properties:

1. The function assumes its maximum at μ .

2. Increasing the mean shifts the distribution to the right without changing the shape, decreasing the mean shifts it to the left.

3. Decreasing the standard deviation "shrinks" it while making the peak higher, increasing the standard deviation makes it flatter.

4. The function is symmetric and centred about the mean. Hence, the area under the curve to the left of the mean equals the area under the curve to the right of the mean. Both of these areas are equal to 0.5.

5. The function has inflection points at $x = \mu \pm \sigma$.

6. Mean = median = mode.

7. $x \to \pm\infty \quad \Rightarrow \quad f(x;\mu,\sigma) \to 0$.

R. 8. 4. (*Standardised Normal Distribution*)
Using the standardisation

$$Z = \frac{X - \mu}{\sigma}$$

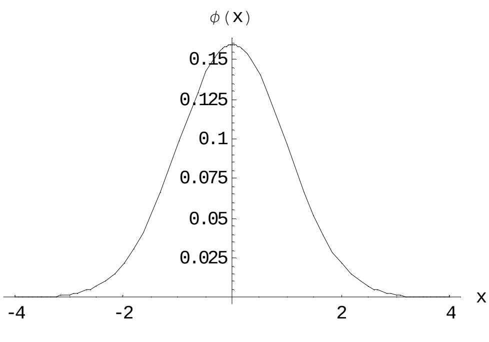
with

$$E(Z) = 0, \qquad D^2(Z) = 1$$

(See D. 6. 3.), we obtain:

1. The *probability density function of the standardised normal distribution*:

$$f(x) = \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad x \in R^1 .$$



2. The *standardised normal distribution function*:

$$F(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \qquad x \in R^1.$$

R. 8. 5.

$$\varphi(-x;\ 0,\ 1) = \varphi(x;\ 0,\ 1),$$

$$\Phi(-x;\ 0,\ 1) = 1 - \Phi(x;\ 0,\ 1).$$

T. 8. 3.

$$P(|X-\mu| < c) = 2\cdot\Phi\left(\frac{c}{\sigma}\right) - 1, \quad c \in R^1.$$

Ex. 8. 1.

Consider a normally distributed random variable X with the mean 1 and the variance 9.

1. Find the probability that

   a. X lies in the interval $]2,\ 5[$,
   b. X is at least equal to 2.7,
   c. X deviates from the mean by less than 1.5.

2. Determine a number z for which the probability $P(X \geq z)$ is at least 0.1.

Solution:

a.

$$\begin{aligned} P(2 < X < 5) \approx P(2 \leq X < 5) &= F(5) - F(2) \\ &= \Phi\left(\frac{5-1}{\sqrt{9}}\right) - \Phi\left(\frac{2-1}{\sqrt{9}}\right) \\ &= \Phi\left(\frac{4}{3}\right) - \Phi\left(\frac{1}{3}\right) \\ &= 0.908241 - 0.629300 = 0.278941. \end{aligned}$$

b.

$$\begin{aligned} P(X \geq 2.7) &= 1 - P(X < 2.7) \\ &= 1 - F(2.7) \\ &= 1 - \Phi\left(\frac{2.7-1}{3}\right) \\ &= 1 - \Phi(0.56666) \\ &= 1 - 0.715661 = 0.274339. \end{aligned}$$

c.

$$P(|X-1|<1.5) = 2\cdot\Phi\left(\frac{1.5}{3}\right)-1$$
$$= 2\cdot\Phi(0.5)-1$$
$$= 2\cdot 0.691462 - 1 = 0.382929\,.$$

2.

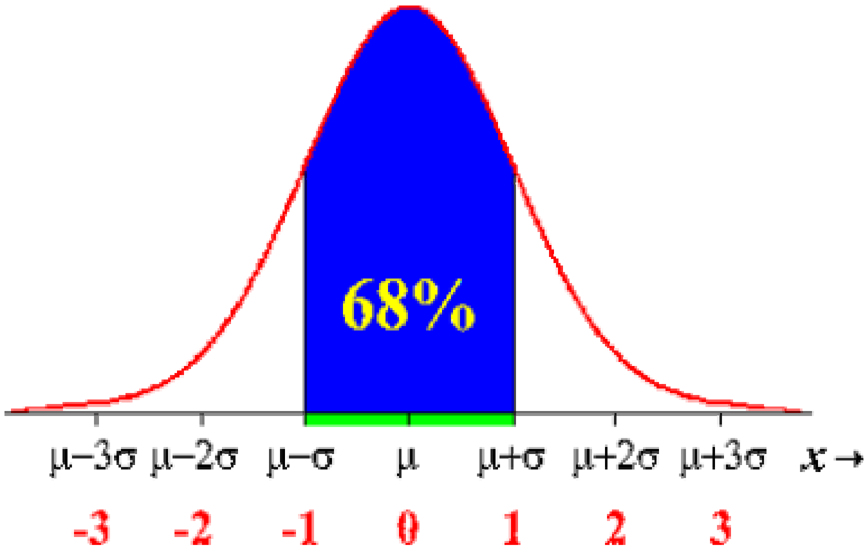
$$P(X \ge z) \ge 0.1$$
$$P(X < z) \le 0.9$$
$$P(X < z) = F(z) = \phi\left(\frac{z-1}{3}\right) \le 0.9 = \Phi(1.28)$$
$$\Phi\left(\frac{z-1}{3}\right) \le \Phi(1.28)$$
$$\frac{z-1}{3} \le 1.28$$
$$z \le 4.84\,.$$

<u>R. 8. 6.</u>(*“68-95-99.7 Rule” or “Empirical Rule”*)
All normal density curves satisfy the following property which follows from T. 8. 3.:

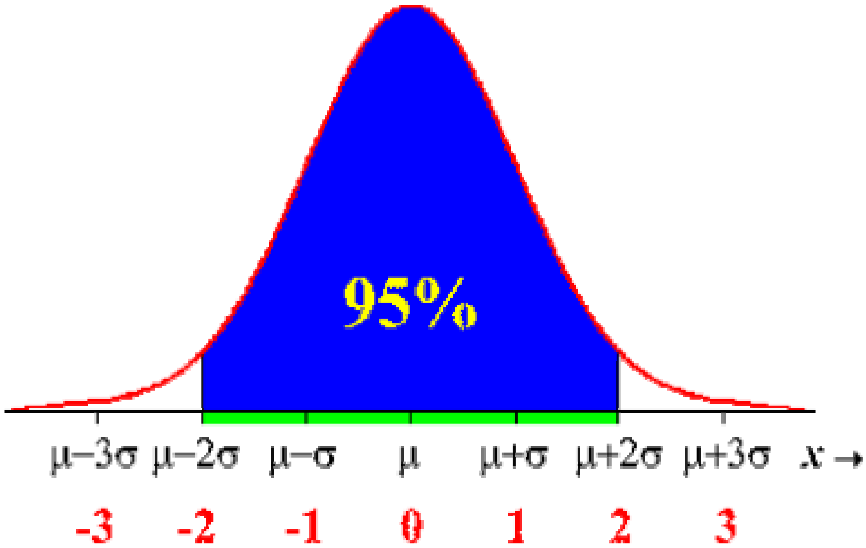
$$P(|X-\mu|<\sigma) = 0.6828$$
$$P(|X-\mu|<2\sigma) = 0.9544$$
$$P(|X-\mu|<3\sigma) = 0.9972\,.$$

This means:

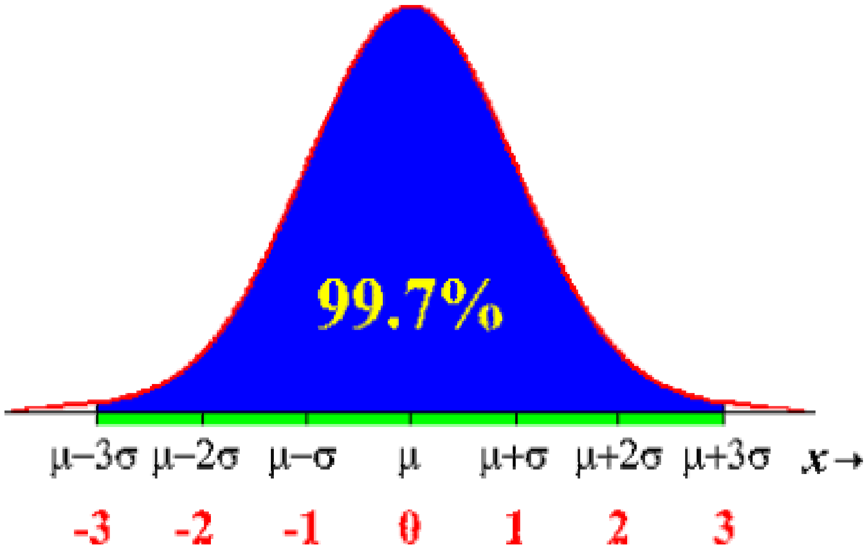
Approximately 68% of the observations fall within 1 standard deviation of the mean:



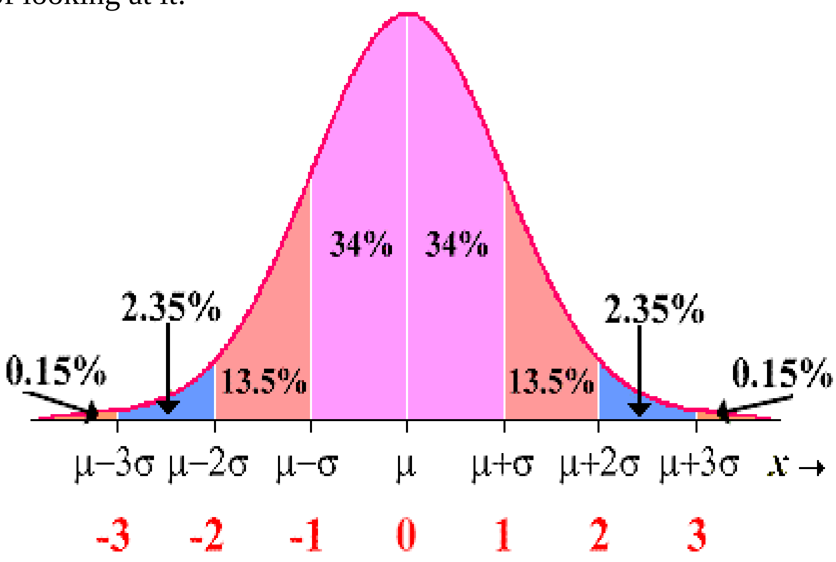
Approximately 95% of the observations fall within 2 standard deviations of the mean:



Approximately 99.7% of the observations fall within 3 standard deviations of the mean:



Another way of looking at it:



Ex. 8. 1. (*Continued*)
Approximately

68% of the observations fall within the interval $[-2,\ 4]$,
95% of the observations fall within the interval $[-5,\ 7]$,
99.7% of the observations fall within the interval $[-8,\ 10]$.

T. 8. 4.
Let $X_i,\ i=1,\ 2,...,$ be binomially distributed random variables with parameters n and p .
Then the sequence of the corresponding standardised random variables

$$\begin{aligned} X_i &:= \frac{X_i - \mu}{\sigma} \\ &= \frac{X_i - n\cdot p}{\sqrt{n\cdot p\cdot q}}, \qquad i=1,\ 2,... \end{aligned}$$

converges against a *standardised normally* distributed random variable.
In particular, we have

$$P\left(X_i < x\right) = \Phi\left(\frac{x - n\cdot p}{\sqrt{n\cdot p\cdot q}}\right).$$

R. 8. 5. (*Normal Approximation to Binomial*)
The above theorem can be used to approximate binomial distribution by normal distribution.
It will be recommended to use the following rule of thumb:

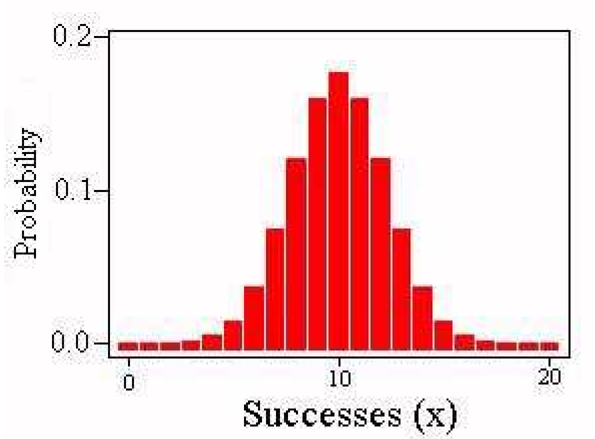
"If

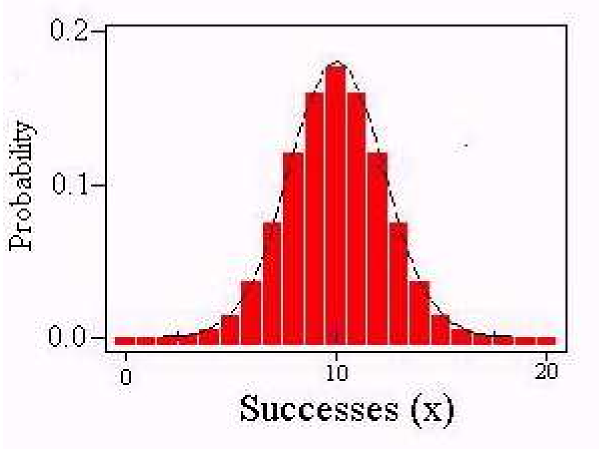
$n\cdot p > 5$ and $n\cdot q > 5$,

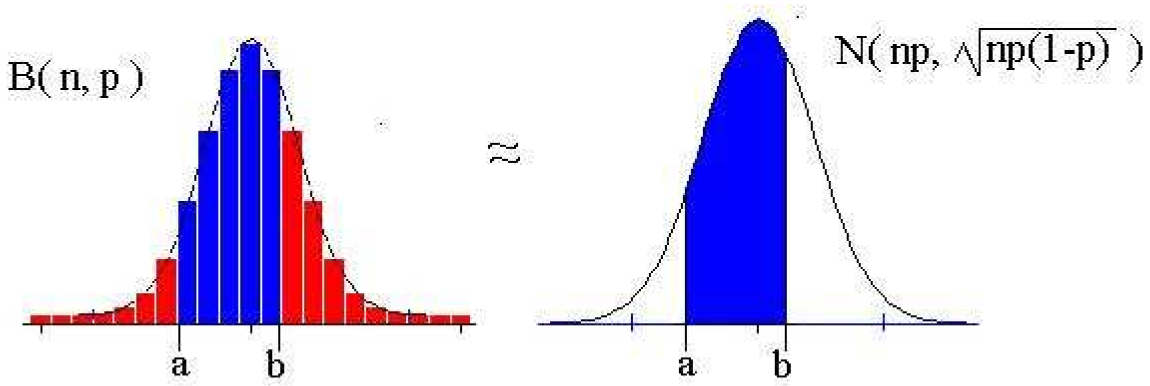
then the binomial distribution can be approximated by normal distribution."

Ex. 8. 2.
Shown below is the probability distribution of a binomial random variable X with $n = 20$ and $q = 0.5$:

Successes(x)	Probability
0	0.000001
1	0.000019
2	0.000181
3	0.001087
4	0.004621
5	0.014786
6	0.036964
7	0.073929
8	0.120134
9	0.160179
10	0.176197
11	0.160179
12	0.120134
13	0.073929
14	0.036964
15	0.014786
16	0.004621
17	0.001087
18	0.000181
19	0.000019
20	0.000001







Ex. 8. 3.
Experience shows that 90% of the products of a firm are of highest quality. Find the probability of finding at least 950 such products in a sample of 1000.

Solution:
Let X denote the number of products of highest quality. Then, we have

$$n = 1000, \qquad p = 0.9 .$$

Using the binomial distribution, we find the „ugly“ expression:

$$P(X \geq 950) = \sum_{x=950}^{1000} \binom{1000}{x} \cdot 0.9^{x} \bullet 0.1^{1000-x} .$$

Let us, therefore, approximate it by the normal distribution (Obviously the conditions for the application of the above rule of thumb are fulfilled!)

$$E(X) = \mu = n \cdot p = 1000 \cdot 0.9 = 900$$
$$D^2(X) = \sigma^2 = n \cdot p \cdot q = 900 \cdot 0.1 = 90$$

$$P(X \geq 950) = 1 - P(X < 950) = 1 - \Phi\left(\frac{950 - 900}{\sqrt{90}}\right) = 1 - \Phi(5.3) = 0.0001 .$$

(Last revised: 09.06.08)