Chapter VI

Parameters of a Random Variable

D. 6. 1. (***Expected Value***)
The *expected value* of the random variable X , denoted by $E\left(X\right)$, is defined as

$$E(X) := \begin{cases} \sum_{i=1}^{\infty} x_i \cdot p_i & \text{when } X \text{ discrete} \\ \int_{-\infty}^{+\infty} x \cdot f(x)dx & \text{when } X \text{ continuous} \end{cases}$$

under the assumption

$$\sum_{i=1}^{\infty} |x_i| \cdot p_i < \infty$$

and

$$\int_{-\infty}^{+\infty} |x| \cdot f(x)dx < \infty .$$

R. 6. 1.
The expected value of a discrete random variable is the weighted mean of all outcomes x_i of X with the probabilities p_i acting as weights.

R. 6. 2.
The notion "expected value" in the probability theory has similarities with the notion "mean value", they are not, however, identical. The following example illustrates this fact:

Ex. 6. 1.
A die will be tossed with the following outcomes:

3, 5, 4, 3, 1

The average mean is equal to

$$\bar{x} = \frac{1}{5} \cdot \left(3+5+4+3++1\right) = 3.2$$

On the other hand, the expected value of the random variable

X : „number of dots facing uppermost“

will be

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

Whereas the mean value can vary from trial to trial, the expected value is an objective number independent of concrete outcomes of trials. The mean value approaches the expected value if the number of trials approaches infinity.

Ex. 6. 2.
Given the probability density function

$$f(x) = -\frac{2}{9}x + \frac{2}{3}, \quad x \in [0,\ 3] \ ,$$

find the expected value $E(X)$.

Solution:

$$E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \int_0^3 x \cdot \left(-\frac{2}{9}x + \frac{2}{3}\right)dx$$

$$= \int_0^3 \left(-\frac{2}{9}x^2 + \frac{2}{3}x\right)dx$$

$$= -\frac{2}{9}\cdot\frac{x^3}{3} + \frac{2}{3}\cdot\frac{x^2}{2}\Bigg|_0^3$$

$$= -\frac{2}{9}\cdot\frac{27}{3} + \frac{9}{3} = 1$$

D. 6. 2. (***Variance or Dispersion, Standard Deviation***)
The *dispersion* or *variance* of the random variable X, denoted by $D^2(X)$, is defined as

$$D^2(X) := E(X - E(X))^2$$

i. e.

$$D^2(X) := \begin{cases} \sum_{i=1}^{\infty} (x_i - E(X))^2 \cdot p_i & \text{when } X \text{ discrete} \\ \int_{-\infty}^{+\infty} (x - E(X))^2 \cdot f(x)dx & \text{when } X \text{ continuous} \end{cases}$$

under the assumption that the expected value exists.

The *standard deviation,* denoted by D, is defined as

$$D(X) := \sqrt{D^2(X)} \quad (>0)$$

R. 6. 3.
The following relations can be easily verified:

$$D^2(X) = E(X^2) - (E(X))^2,$$

i. e.

$$D^2(X) := \begin{cases} \sum_{i=1}^{\infty} x_i^2 \cdot p_i - \left(\sum_{i=1}^{\infty} x_i \cdot p_i \right)^2 & \text{when } X \text{ discrete} \\ \int_{.\infty}^{+\infty} x^2 \cdot f(x)dx - \left(\int_{-\infty}^{+\infty} x \cdot f(x) \right)^2 & \text{when } X \text{ continuous} \end{cases}$$

Ex. 6. 1. (*continued)*
Find the variance and the standard deviation of X .

Solution:

$$D^2(X) = (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + ... + (6-3.5)^2 \cdot \frac{1}{6} \approx 2.92,$$

or

$$D^2(X) = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + ... + 36 \cdot \frac{1}{6} - 3.5^2 \approx 2.92 .$$

$$D(X) \approx 1.71 .$$

Ex. 6. 2. (*continued)*
Find the variance and the standard deviation of X .

Solution:

$$D^2(X) = \int_0^3 (x-1)^2 \cdot \left(\frac{2}{9}x + \frac{2}{3} \right) dx$$

$$= \int_0^3 \left(-\frac{2}{9}x^3 + \frac{10}{9}x^2 - \frac{14}{9}x + \frac{2}{3} \right) dx$$

$$= -\frac{2}{9} \cdot \frac{x^4}{4} + \frac{10}{9} \cdot \frac{x^3}{3} - \frac{14}{9} \cdot \frac{x^2}{2} + \frac{2}{3}x \Big|_0^3$$

$$= \frac{1}{2}$$

or

$$D^2(X)=\int_0^3 x^2\cdot\left(-\frac{2}{9}x+\frac{2}{3}\right)dx-1$$

$$=\int_0^3\left(-\frac{2}{9}x^3+\frac{2}{3}x^2\right)dx-1$$

$$=-\frac{2}{9}\cdot\frac{x^4}{4}+\frac{2}{3}\cdot\frac{x^3}{3}\Bigg|_0^3-1$$

$$=\frac{1}{2}.$$

R. 6. 4.
Let X and Z be two random variables. Given the transformation

$$Z=g(X),$$

we would like to derive informations about Z assuming we know the distribution of X . This will be illustrated by the following example:

Ex. 6. 3.
Let us consider the function

$$Z=4X$$

with the following probability and distribution functions for the random variable X :

x_i	2	4
$P(X=x_i)$	$\frac{1}{4}$	$\frac{3}{4}$

$$F(x):=\begin{cases}0 & \text{when} \quad x\le 2\\ \frac{1}{4} & \text{when} \quad 2<x\le 4\\ 1 & \text{when} \quad x>4\end{cases}.$$

We are interested in finding the probability and distribution functions for the random variable Z :

$$P(X=x_i)=P(4X=4x_i)$$

$$=P(Z=4x_i)$$

$$=P(Z=z_i), \quad i=1,\ 2,$$

and

$$
\begin{aligned}
F(z) &= P(Z < z) \\
&= P(4X < z) \\
&= P\left(X < \frac{z}{4}\right) \\
&= P\left(\frac{z}{4}\right) = F(x).
\end{aligned}
$$

Thus, we have the probability function:

z_i	8	16
$P(Z = z_i)$	$\frac{1}{4}$	$\frac{3}{4}$

and the distribution function:

$$
F(z) := \begin{cases} 0 & \text{when} \quad z \le 8 \\ \frac{1}{4} & \text{when} \quad 8 < z \le 16 \\ 1 & \text{when} \quad z > 16 \end{cases}.
$$

Let us now calculate the expected value of Z :

$$
\begin{aligned}
E(Z) &= z_1 \cdot P(Z = z_1) + z_2 \cdot P(Z = z_2) \\
&= 8 \cdot P(Z = 8) + 16 \cdot P(Z = 16) \\
&= 8 \cdot P(4X = 8) + 16 \cdot P(4X = 16) \\
&= 8 \cdot P(X = 2) + 16 \cdot P(X = 4) \\
&= 8 \cdot \frac{1}{4} + 16 \cdot \frac{3}{4} \\
&= 14 \quad .
\end{aligned}
$$

<u>R. 6. 5.</u>
Generalising the results of the above example and assuming that the corresponding expected values exist, the following relations can be proved:

$$
E(Z) = E\left(g(x)\right) = \begin{cases} \sum_{i=1}^{\infty} g(x_i) \cdot P(X = x_i) & \text{when } X \text{ discrete} \\ \int_{-\infty}^{+\infty} g(x) \cdot f(x) dx & \text{when } X \text{ continuous} \end{cases}
$$

Ex. 6. 4.
Consider the random variable X with the probability function

x_i	x_1	x_2	$\dots$	x_n
$p_i = P(X = x_i) = f(x_i)$	p_1	p_2	$\dots$	p_n

Let

$$Z = aX + b, \quad a, b = const.$$

Then, we have:

$$\begin{aligned} E(Z) &= \sum_{i=1}^{\infty} \left(ax_i + b\right) \cdot p_i \\ &= a \cdot \sum_{i=1}^{\infty} x_i \cdot p_i + b \cdot \sum_{i=1}^{\infty} p_i \\ &= a \cdot \sum_{i=1}^{\infty} x_i \cdot p_i + b\,. \end{aligned}$$

Hence, assuming that $E(X)$ exists, we have

$$E\left(aX + b\right) = a \cdot E(X) + b\,.$$

Similarly, it can be proved:

$$D^2\left(aX + b\right) = a^2 \cdot D^2(X),$$

assuming that $D^2\left(X\right)$ exists.

Ex. 6. 5.
Consider the random variable X with

$$E\left(X\right) = \mu, \quad D^2\left(X\right) = \sigma^2 \ \ (\sigma^2 \neq 0)\,.$$

For the function

$$Z = g(X) := \frac{X - \mu}{\sigma} = \frac{1}{\sigma} \cdot X - \frac{\mu}{\sigma}\,,$$

we obtain

$$\begin{aligned} E\left(Z\right) &= \frac{1}{\sigma} \cdot E\left(X\right) - \frac{\mu}{\sigma} \\ &= \frac{1}{\sigma} \cdot \mu - \frac{\mu}{\sigma} = 0 \\ &= 0\,, \end{aligned}$$

$$D^2(Y) = \frac{1}{\sigma^2} \cdot D^2(X)$$

$$= \frac{\sigma^2(X)}{\sigma^2(X)}$$

$$= 1.$$

<u>**D. 6. 3.**</u> **(*Standardisation, Standardised Random Variable*)**
The random variable Z is called a *standardised random variable* if

$$E(Z) = 0, \quad D^2(Z) = 1.$$

The process

$$Z = g(X) := \frac{X - \mu}{\sigma} = \frac{1}{\sigma} \cdot X - \frac{\mu}{\sigma}$$

is called *standardisation*.

(Last revised: 03.06.08)