Chapter V

Discrete and Continuous Random Variables

<u>D. 5. 1.</u> (*Probability Function*)
If X is a discrete random variable, then the function

$$p(x) = P(X = x)$$

defined on the outcomes of X is called the *probability function* of the discrete random variable X .
If X has the outcomes x_i, $i = 1, 2, ..., n$, then we can write:

x_i	x_1	x_2	$\cdots$	x_n
$p_i = P(X = x_i) = f(x_i)$	p_1	p_2	$\cdots$	p_n

<u>Ex. 5. 1.</u> (see Ex. 4. 1.)

x_i	0	1	2
$p_i = P(X = x_i)$	$\frac{7}{20}$	$\frac{10}{20}$	$\frac{3}{20}$

<u>T. 5. 1.</u>

$$\sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} p_i = 1 .$$

<u>T. 5. 2.</u>

$$F(x) = P(X < x) = \begin{cases} 0 & \text{for} \quad x \le x_1 \\ \sum_{i=1}^{k} p_i & \text{for} \quad x_k < x \le x_{k+1}, \quad k = 1,\ 2,\ ...,n-1 \\ 1 & \text{for} \quad x > x_n \end{cases}$$

<u>Ex. 5. 2.</u>
A die will be thrown. Let

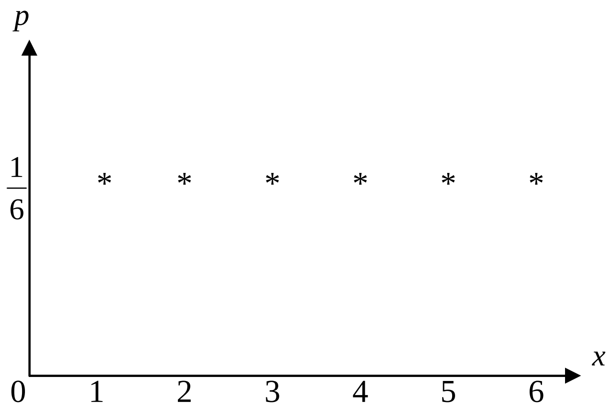
X : „the number appearing above“.

Probability function:

1) in tabular form:

x_i	1	2	3	4	5	6
p_i	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

2) in graphical form:



Distribution function:

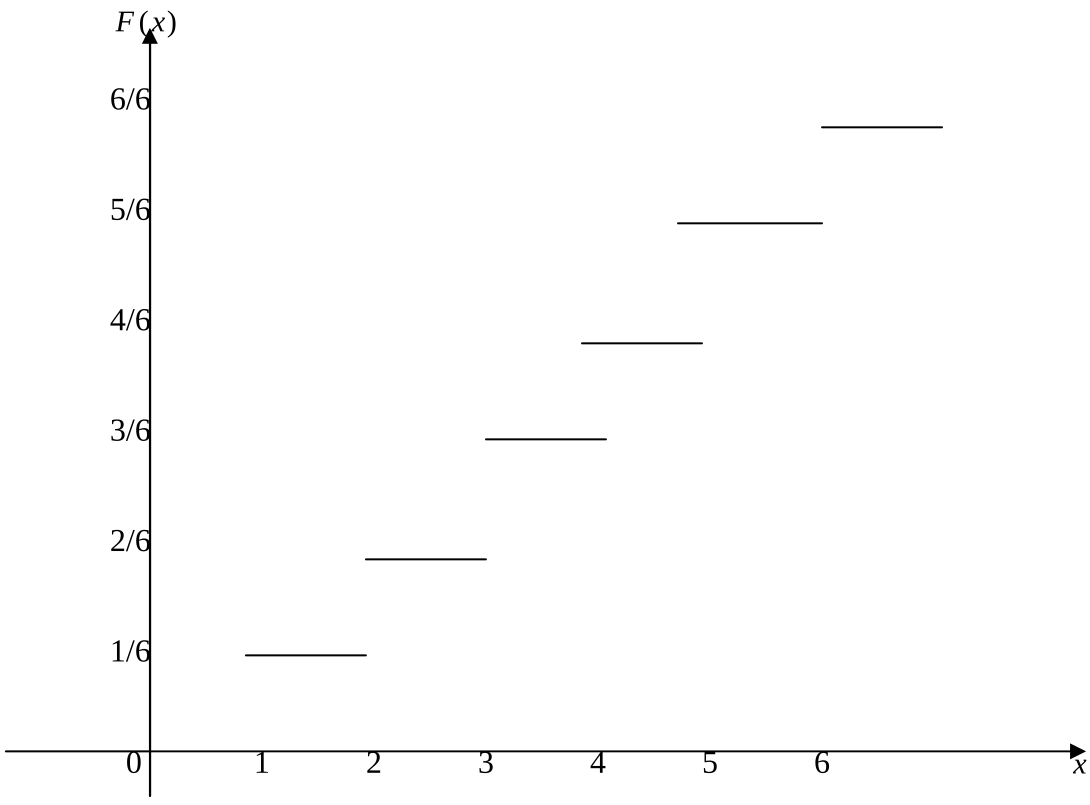
1) in analytic form:

$$F(x) = \begin{cases} 0 & \text{when} \quad -\infty < x \le 1 \\ \frac{1}{6} & \text{when} \quad 1 < x \le 2 \\ \frac{2}{6} & \text{when} \quad 2 < x \le 3 \\ \frac{3}{6} & \text{when} \quad 3 < x \le 4 \\ \frac{4}{6} & \text{when} \quad 4 < x \le 5 \\ \frac{5}{6} & \text{when} \quad 5 < x \le 6 \\ 1 & \text{when} \quad 6 < x < +\infty \end{cases}$$

2) in tabular form:

x	$F(x)$
$]-\infty,\ 1]$	0
$]1,\ 2]$	$\frac{1}{6}$
$]2,\ 3]$	$\frac{2}{6}$
$]3,\ 4]$	$\frac{3}{6}$
$]4,\ 5]$	$\frac{4}{6}$
$]5,\ 6]$	$\frac{5}{6}$
$]6,\ +\infty[$	$\frac{6}{6}=1$

3) in graphical form:



<u>D. 5. 3.</u> (*Density Function*)
Let $F(x)$ be a differentiable distribution function of a continuous random variable X. The function

$$f(x) := F'(x)$$

is called the (*probability) density function of* X.

<u>T. 5. 3.</u> (*Important Properties of a Density Function*)

1.

$$F(x) = P(X < x)$$

$$= P(-\infty < X < x)$$

$$= \int_{-\infty}^{x} f(t)dt .$$

2.

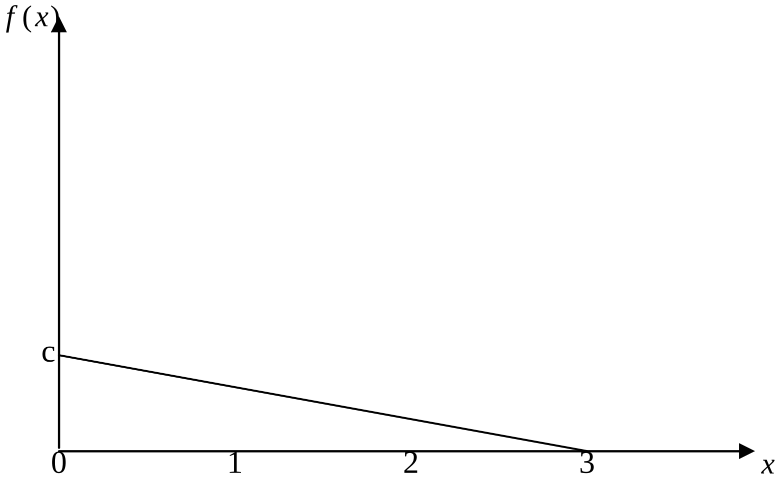
$$P(a \leq X < b) = F(b) - F(a)$$

$$= \int_{a}^{b} f(x)dx .$$

3.

$$P(-\infty < X < +\infty) = \int_{-\infty}^{+\infty} f(x)dx = 1 .$$

<u>Ex. 5. 3.</u>
The following graph represents the probability density function of a random variable X :



1. Determine $f(0) = c$.
2. Find the analytical representation of $f(x)$.
3. Find $F(x)$.
4. Calculate $P(1.5 < X < 3)$.
5. Interpret the above results geometrically.

Solution:

$$y = ax + b$$

$$x = 0 \quad \Rightarrow \quad y = b = c$$

$$y = 0 \quad \Rightarrow \quad x = -\frac{b}{a} = -\frac{c}{a} = 3$$

$$a = -\frac{c}{3} .$$

1.

$$y = f(x) = -\frac{c}{3}x + c$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^3 f(x)dx = 1$$

$$\int_0^3 \left(-\frac{c}{3}x + c \right) dx = \left[-\frac{c}{3} \cdot \frac{x^2}{2} + cx \right]_0^3$$

$$= -c \div \frac{9}{6} + 3c = -c \cdot \left(\frac{3}{2} - 3 \right) = 1$$

$$c = \frac{2}{3} .$$

2.

$$f(x) = -\frac{\frac{2}{3}}{3} \cdot x + \frac{2}{3}$$

$$f(x) = -\frac{2}{9} \cdot x + \frac{2}{3}, \qquad x \in [0, \quad 3] .$$

3.

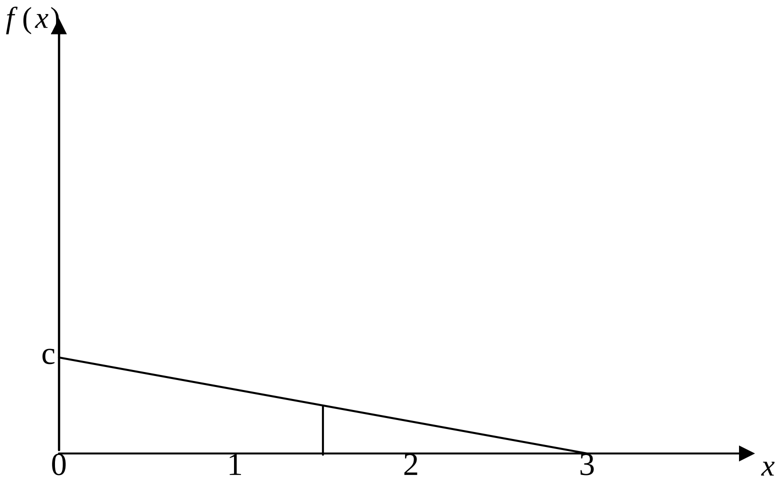
$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_0^3 \left(-\frac{2}{9}t + \frac{2}{3} \right) dt = \left[-\frac{2}{9} \cdot \frac{t^2}{2} + \frac{2}{3}t \right]_0^x$$

$$F(x) = -\frac{x^2}{9} + \frac{2}{3}x, \quad x \in [0, \ 3].$$

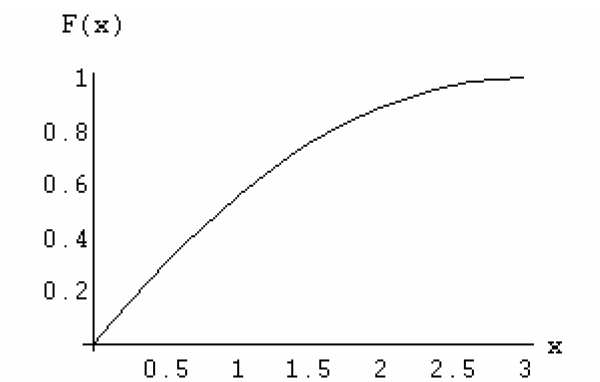
4.

$$P\big(1.5 < X < 3)\big) \approx P(1.5 \le X < 3) = \int_{1.5}^{3} f(x)dx = F(3) - F(1.5) = \frac{1}{4}$$

5.



$f(x)$



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