Chapter III

Probability Algebra

<u>T. 3. 1.</u> (*Addition Rule*)
Let A and B be two *arbitrary* events. Then, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

<u>Ex. 3. 1.</u>
At a certain university, men engage in various sports in the following proportions:

Soccer (A): 60%
Basketball (B): 50%
Both soccer and basketball: 30%

If a man is selected at random for an interview, what is the chance that he will

1. play soccer or basketball?
2. play neither sport?

Solution:

1.
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.60 + 0.50 - 0.30 \\ &= 0.80 \end{aligned}$$
2.
$$\begin{aligned} 1 - P(A \cup B) &= 1 - 0.80 \\ &= 0.20 \end{aligned}$$

<u>R. 3. 1.</u>
Axiom III of the axiomatic definition of probability is a special case of the addition theorem for two mutually exclusive events.

<u>D. 3. 1.</u> (*Conditional Probability*)
Let $A, B \in S$. The *conditional probability* of A given B is:

$$P(A/B) := \begin{cases} \dfrac{P(A \cap B)}{P(B)} & if \quad P(B) > 0 \\ 0 & if \quad P(B) = 0 \end{cases}$$

<u>Ex. 3. 2.</u>
A math lecturer gave his group two tests. 25% of the group passed both tests and 42% passed the first test.
What percent of those who passed the first test passed also the second test?

Solution:

Let

$A:$ „The group passed the second test. "

$B:$ „The group passed the first test. "

We have:

$$P(B) = 0.42\,, \qquad P(A \cap B) = 0.25\,.$$

Therefore

$$P(A/B) = \frac{0.25}{0.42} \approx 0.60$$

R. 3. 2.
Similarly, we define the conditional probability of B given A**:**

$$P(B/A) := \begin{cases} \dfrac{P(B \cap A)}{P(A)} & \textit{if} \quad P(A) > 0 \\ 0 & \textit{if} \quad P(A) = 0 \end{cases}$$

R. 3. 3.
Since $P(A \cap B) = P(B \cap A)$, we have

$$P(A) \cdot P(A/B) = P(B) \cdot P(B/A)\,.$$

T. 3. 2. (***Multiplication Rule for Two Events***)
Let $A, B \in S$ be two *arbitrary* events. Then

$$P(A \cap B) = P(B) \cdot P(A/B)$$

$$= P(A) \cdot P(B/A)$$

Ex. 3. 3.
Three defective light bulbs inadvertently got mixed with 6 good ones. If two bulbs are chosen at random for a ceiling lamp, what is the probability that they both are good?

Solution:

$A:$ „Second bulb is good. "

$B:$ „First bulb is good. "

$$P(A \cap B) = \frac{6}{9} \cdot \frac{5}{8} \approx 0.42$$

R. 3. 3.
The multiplication rule can also be extended to n *arbitrary events* $E_i, i = 1, 2, ..., n$:

$$P(\bigcap_{i=1}^{n} E_i) = P(E_1) \cdot P(E_2 / E_1) \cdot P(E_3 / E_1 \cap E_2)...P(E_n / \bigcap_{i=1}^{n-1} E_i).$$

D. 3. 2. **(*Independent Events*)**
The event A is said to be *independent* of B if

$$P(A) = P(A / B).$$

Otherwise, A is said to be *dependent* of B .

T. 3. 2.

$$\langle A \text{ is independen t of } B \rangle \Rightarrow \langle B \text{ is independen t of } A \rangle.$$

Proof:

We have

$$P(B / A) = \frac{P(A / B) \cdot P(B)}{P(A)}$$
$$= \frac{P(A) \cdot P(B)}{P(A)}$$
$$= P(B)$$

R. 3. 3.
Based on the last result, we usually speak of two *independent events*.

Ex. 3. 4.
Consider choosing a card from a well-shuffled Skat deck of 32 playing cards. Let:

A : „The chosen card is an ace. "

B : „The chosen card is a diamond card. "

C : „The chosen card is ace of diamonds. "

Determine if the following pairs of events are independent:

1. (A, B)
2. (C, A)

Solution:
1.

$$P(A) = \frac{1}{8} = P(A/B)$$

Therefore, the events A, B are independent.

2.

$$P(C) = \frac{1}{32} \neq \frac{1}{4} P(C/A)$$

Therefore, the events A, C are dependent.

<u>T. 3. 3.</u> (*Multiplication Rule for Two Independent Events*)
Let $A, B \in S$ be two *independent* events. Then

$$P(A \cap B) = P(A) \cdot P(B)$$

Proof:

$$\begin{aligned} P(A \cap B) &= P(B) \cdot P(A/B) \\ &= P(A) \cdot P(B) \end{aligned}$$

<u>R. 3. 4.</u>
The notion independence can also be extended to $n > 2$ events.

<u>T. 3. 4.</u> (*Multiplication Rule for Independent Events*)
Let $E_i, i = 1, 2, ..., n$, be *independent* events. Then

$$P\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} P(E_i).$$

<u>Ex. 3. 5.</u>
Consider three machines $M_i, i = 1, 2, 3$, with the reliabilities

M_1: 0.9, M_2: 0.8, M_3: 0.85.

Assuming that the three machines work independently of one another, calculate the probability that

1. no machine will fall out.
2. all machines will fall out.

Solution:
Let

E_i : „Machine M_i will <u>not</u> fall out”, $i = 1, 2, 3$.

1.

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)\cdot P(E_2)\cdot P(E_3)$$

$$= 0.9 \cdot 0.8 \cdot 0.85 = 0.612$$

2.

$$P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3) = P(\bar{E}_1)\cdot P(\bar{E}_2)\cdot P(\bar{E}_3)$$

$$= (1-0.9)\cdot(1-0.8)\cdot(1-0.85) = 0.003$$

T. 3. 5.

Let $E_i, i = 1, 2, ..., n$, be *independent* events. Then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = 1 - \prod_{i=1}^{n}\left(1 - P(E_i)\right)$$

Ex. 3. 5. (*continued*)

3. at least one of the machines does not fall out.
4. at least one of the machines falls out.

Solution:

3. $P(E_1 \cup E_2 \cup E_3) = 1-(1-P(E_1))\cdot(1-P(E_2))\cdot(1-P(E_3))$

$$= 1-(1-0.9)\cdot(1-0.8)\cdot(1-0.85)$$

$$= 1-0.003 = 0.997$$

4. $P(\bar{E}_1 \cup \bar{E}_2 \cup \bar{E}_3) = 1-(1-P(\bar{E}_1))\cdot(1-P(\bar{E}_2))\cdot(1-P(\bar{E}_3))$

$$= 1-(1-0.1)\cdot(1-0.2)\cdot(1-0.15)$$

$$= 1-0.612 = 0.388$$

Ex. 3. 6.

The probability to hit a plane by a single shot is equal to 0.004. What is the chance of bringing down a plane by 250 simultaneous shots. It will be assumed that the plane will be brought down if it is hit at least once.

Solution:

Let

$E_i,\ \ i = 1, 2, ..., 250$: “hit the plane by the $i-$th shot“.

$$P\left(\bigcup_{i=1}^{250} E_i\right) = 1 - \prod_{i=1}^{250}(1-P(E_i))$$

$$= 1-(1-0.004)^{250}$$

$$= 1-0.367142 = 0.632858$$

T. 3. 6. (***Total probability***)

Let $B_i, i = 1, 2, ..., n$, form a group of mutually exclusive and exhaustive events. For an arbitrary event *A* we have

$$P(A) = \sum_{i=1}^{n} P(A/B_i) \cdot P(B_i).$$

Proof:

$$A = \Omega \cap A$$

$$= \left(\bigcup_{i=}^{n} B_i \right) \cap A$$

$$= \bigcup_{i=1}^{n} (B_i \cap A)$$

$$P(A) = P\left(\bigcup_{i=1}^{n} (B_i \cap A) \right)$$

$$= \sum_{i=1}^{n} P(B_i \cap A)$$

$$= \sum_{i=1}^{n} P(A/B_i) \cdot P(B_i).$$

Ex. 3. 7.

Consider three groups of jars.

The first group consists of three jars each containing 2 white and 3 red marbles.

The second group consists of 2 jars each containing 4 white and 1 red marbles

The third group consists of only 1 jar containing 0 white and 8 red marbles.

A jar will be chosen at random and a marble taken out.

1. What is the chance of having chosen a white marble?
2. What is the chance of having chosen a red marble?

Solution:

Let

A: „a white marble will be chosen",

B_i, $i = 1, 2, 3$: „the jar chosen belongs to the group i ".

1.

$$P(A) = \sum_{i=1}^{3} P(A/B_i) \cdot P(B_i)$$

$$= \frac{3}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{4}{5} + \frac{1}{6} \cdot 0 = \frac{7}{15}$$

2.

$$P(\bar{A}) = \sum_{i=1}^{3} P(\bar{A}/B_i) \cdot P(B_i)$$
$$= \frac{3}{6} \cdot \frac{3}{5} + \frac{2}{6} \cdot \frac{1}{5} + \frac{1}{6} \cdot 1 = \frac{8}{15}$$

<u>T. 3. 7.</u> (*Bayes*)

Let B_i, $i = 1, 2, ..., n$, form a group of mutually exclusive and exhaustive events. For an arbitrary event $A \neq \emptyset$ we have:

$$P(B_i / A) = \frac{P(A/B_i) \cdot P(B_i)}{\sum_{i=1}^{n} P(A/B_i) \cdot P(B_i)}, \qquad i = 1, 2, ..., n.$$

Proof:

$$P(B_i \cap A) = P(A) \cdot P(B_i / A) = P(B_i) \cdot P(A/B_i)$$

$$P(B_i / A) = \frac{P(B_i) \cdot P(A/B_i)}{P(A)}$$
$$= \frac{P(A/B_i) \cdot P(B_i)}{\sum_{i=1}^{n} P(A/B_i) \cdot P(B_i)}, \qquad i = 1, 2, ..., n.$$

<u>Ex. 3. 8.</u>

A factory produces a product on three machines. The first machine produces 25%, the second 35% and the third 40% of the total production. Experience shows that 5% of the products produced on the first, 4% on the second and 2% on the third machine are defective. Determine the probability that a randomly selected defective product has been produced on

1. the first
2. the second
3. the third

machine.

Solution:

Let

A: „a product is defective.”,

B_i: „a product has been produced on the machine i" .

We have:

$P(B_1) = 0.25$, $P(B_2) = 0.35$, $P(B_3) = 0.40$,

$P(A/B_1) = 0.05$, $P(A/B_2) = 0.04$, $P(A/B_3) = 0.02$

1.

$$P(B_1 / A) = \frac{0.05 \cdot 0.25}{0.05 \cdot 0.25 + 0.04 \cdot 0.35 \cdot 0.02 \cdot 0.40} = \frac{25}{69},$$

2.

$$P(B_2 / A) = \frac{0.04 \cdot 0.35}{0.05 \cdot 0.25 + 0.04 \cdot 0.35 \cdot 0.02 \cdot 0.40} = \frac{28}{69},$$

3.

$$P(B_3 / A) = \frac{0.02 \cdot 0.35}{0.05 \cdot 0.25 + 0.04 \cdot 0.35 \cdot 0.02 \cdot 0.40} = \frac{16}{69}.$$

R. 3. 5.
In Bayes Theorem, $P(A/B_i)$ are called prior *probabilities* and posterior $P(B_i / A)$ *probabilities*.

T. 3. 8.
Consider a finite set of N elements, $M \le N$ elements of which have a certain property. Let us choose a sample of $n \le N$ elements.
The probability that the sample contains $m \le n$ elements with the above-mentioned property is in case of

1. *nonreplacement:*

$$P_{noplacemen\,t} = \frac{\binom{M}{m} \cdot \binom{N-M}{n-m}}{\binom{N}{n}}$$

2. *replacement*

$$P_{nreplacet} = \binom{n}{m} \cdot p^m \cdot q^{n-m},$$

$$\frac{M}{N} =: p, \qquad q := 1 - p$$

Ex. 3. 9.
A box contains 25 items, 10 of which are defective. A sample of two items will be taken.

1. without replacement
2. with replacement.

Determine the probability that

i) both items are good
ii) exactly one item is defective
iii) both items are defective.

Solution:

$N = 25$, $M = 10$, $n = 2$, $p = \frac{10}{25} = \frac{2}{5}$

		1.	2.
i)	$m = 0$	$P = \frac{7}{20}$	$P = \frac{9}{25}$
ii)	$m = 1$	$P = \frac{1}{2}$	$P = \frac{12}{25}$
iii)	$m = 2$	$P = \frac{3}{20}$	$P = \frac{4}{25}$

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