II

Introduction to Probability

D. 2. 1. (***Absolute and Relative Frequency***)
The *absolute frequency of an event* E in n trials, denoted by $F(E)$, is the number of cases in which it occurs.
The *relative frequency of an event* E in n trials is defined as:

$$f(E) := \frac{F(E)}{n}.$$

Ex. 2. 1.
A symmetrical coin will be tossed. We denote the two possible outcomes of the experiment H (for head) and T (for tail).
If we repeat the experiment 35 times and the coin lands 21 times heads-up, then we have

$$F(H) = 21, \quad f(H) = \frac{21}{35} = 0.6.$$

Ex. 2. 2. (***A historical example)***
The following table shows the results of an experiment with tossing a coin and observing the event H (for head):

Experimented by	n	$F(H)$	$f(H)$
Buffon	4040	2048	0.5069
K. Pearson	12000	6019	0.5016
K. Pearson	24000	12012	0.5005

It can be seen that with the increasing number of trial the relative frequencies will be stabilised about the number 0.5.

D. 2. 2. (***Statistical or Empirical Definition of Probability***)
Let the probability of the event E be denoted by $P(E)$. Then

$$P(E) := \lim_{n\to\infty} \frac{F(E)}{n}.$$

E. 2. 3.
On February 1, 2003, the Space shuttle Columbia exploded. This was the second disaster in 113 space missions for NASA.
On the basis of this information, what is the probability that a future mission is successfully completed?

Solution:

Probability of successful flights $\approx \frac{111}{113} \approx 0.98$.

D. 2. 3. (***Classical Definition of Probability***)

$$P(E) := \frac{k}{n}.$$

Here are:

k : number of favourable outcomes,
n : total number of possible outcomes.

Assumption: All outcomes are equally likely.

Ex. 2. 4.
A die will be thrown. Calculate the probabilities for the following events:

A : „An even umber will be thrown".
B : „At least a 3 will be thrown".
C : „At most a 5 will be thrown".

Solution:

$$P(A) = \frac{3}{6} = 0.5, \qquad P(B) = \frac{4}{6} \approx 0.67, \qquad P(C) = \frac{5}{6} \approx 0.83.$$

D. 2. 4. (***Subjective Probability***)
A probability that is derived from an individual's personal judgement about how likely is an event to occur is called *subjective probability*.

Ex. 2. 5.
The probability that a certain country will win the next football championship.

D. 2. 5. (***Axiomatic Definition of Probability- Kolmogorov***)
Axiom 1:
Let $P(E)$ be a unique mapping of $E \in F$ into $[0,\ 1]$ called *probability*.

Axiom 2:

$$P(\Omega) = 1.$$

Axiom 3:

$$\left\langle E_i \cap E_k = \varnothing \quad , \quad i \neq k,\ i,k = 1,\ 2, \ldots, n \right\rangle \quad \Rightarrow \quad \left\langle P\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} P\left(E_j\right) \right\rangle .$$

T. 2. 1.

$$\langle E_1 = E_2 \rangle \quad \Rightarrow \quad \langle P(E_1) = P(E_2) \rangle$$

T. 2. 2.

$$P(\bar{E}) = 1 - P(\bar{E}) .$$

Proof:

$$P(E \cup \bar{E}) = P(E) + P(\bar{E}) \qquad (\because\ E \cap \bar{E} = \varnothing,\ \text{axiom 3})$$

$$1 = P(E) + P(\bar{E}) \qquad (\because\ E \cup \bar{E} = \Omega,\ \text{axiom 2})$$

T. 2. 3.

$$P(\varnothing) = 0 .$$

Proof:

$$P(\varnothing) = 1 - P(\Omega) \qquad (\because\ T.2.2 \text{ with } E = \Omega)$$

$$= 0 \qquad (\because\ \text{axiom 2})$$

T. 2. 4.

$$\langle E_1 \subseteq E_2 \rangle \quad \Rightarrow \quad \langle P(E_1) \leq P(E_2) \rangle .$$

T. 2. 5.
Let $E_j,\ j = 1,\ 2, \ldots, n$ be mutually exclusive and exhaustive events. Then

$$\sum_{j=1}^{n} P(E_j) = 1 .$$

Proof:

$$P\left(\bigcup_{j=1}^{n} E_j \right) = \sum_{j=1}^{n} P\left(E_j\right) \qquad (\because\ \text{axiom 3})$$

$$P(\Omega) = \sum_{j=1}^{n} P\left(E_j\right)$$

$$1 = \sum_{j=1}^{n} P\left(E_j\right) \qquad (\because\ \text{axiom 2})$$

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