Chapter 11

Decision Theory

R. 11. 1.

Being a manager means primarily making decisions. For example:

- Should a company expand its existing facility or build a new facility?
- How many employees should he hire?
- What piece of equipment should be purchased?
- Should his company introduce the new product locally first or proceed to introduce it nationally or even internationally?

R. 11. 2.

Decision Theory is an interdisciplinary area of study concerned with how a decision-maker makes or should make decisions.
The decision theory is classified into

1. Descriptive
2. Prescriptive

theory.

Descriptive decision theory deals with how people actually make decisions.
Prescriptive decision theory formulates how decisions should be made from a finite set of possible alternatives in order to accommodate a set of axioms believed to ensure "optimality".

It must be remembered that in the decision theory there is no "optimum" as such. An "optimal" decision is always to be conceived within a set of rules.

D. 11. 1 (***Decision Table***)

The following table is called a *decision table* (*matrix*):

	s_1	.	.	.	s_j	.	.	.	s_n
a_1	x_{11}	.	.	.	x_{1j}	.	.	.	x_{1n}
.	.				.				.
.	.				.				.
.	.								
a_i	x_{i1}	.	.	.	x_{ij}	.	.	.	x_{in}
.	.				.				.
.	.				.				.
.	.				.				.
a_m	x_{m1}	.	.	.	x_{mj}	.	.	.	x_{mn}

Here are:

a_i: action (or alternative) $i = 1, 2, ..., m$;

s_j: state (of nature) $j = 1, 2, ..., n$;

x_{ij}: consequence of choosing the action a_i in the state of s_j.

If the decision problem involves monetary outcomes the x_{ij} may be single numbers. Otherwise we assume that the decision maker can value them numerically, i.e. we shall assume that he can measure the value of x_{ij} to him through some real-valued function v.
By 'measure the value' we mean that

$$v\left(x_{ij}\right) > v\left(x_{kl}\right)$$

if and only if the decision maker would prefer the consequence x_{ij} to the consequence x_{kl}.
Letting $v_{ij} = v(x_{ij})$, the general form of a decision table becomes:

	s_1	.	.	.	s_j	.	.	.	s_n
a_1	v_{11}	.	.	.	v_{1j}	.	.	.	v_{1n}
.	.				.				.
.	.				.				.
.	.				.				.
a_i	v_{i1}	.	.	.	v_{ij}	.	.	.	v_{in}
.	.				.				.
.	.				.				.
.	.				.				.
a_m	v_{m1}	.	.	.	v_{mj}	.	.	.	v_{mn}

The above table is also called the *payoff table*.

<u>D. 11. 2.</u> (*Dominance*)

1. Outcome dominance
2. Event dominance
3. Probabilistic dominance

1. Outcome dominance

In *outcome dominance* one alternative dominates some second alternative, if the worst payoff for this alternative is at least as good as the best payoff for the second alternative.

2. Event dominance

Event dominance occurs if one alternative has a payoff equal to or better than that of a second alternative for each state of nature (regardless of what event occurs, the first alternative is better than the second).

3. Probabilistic dominance

If, for any amount of money, one alternative has a uniformly equal or better chance of obtaining that amount or more, then that alternative dominates another alternative by probabilistic dominance.

<u>Ex. 11. 1.</u>
We illustrate the first two types of dominance by the following example:

Alternatives \ States	s_1	s_2	s_3
a_1	5	-1	2
a_2	2	1	0
a_3	4	2	5
a_4	-1	1	4
$P(s_j)$	0.1	0.4	0.5

The worst payoff outcome for alternative a_3 is 2 (when s_2 occurs), the best payoff for a_2 is also 2 (when s_1 occurs). Hence, alternative a_3 dominates a_2 by *outcome dominance*.

For each state of nature, the payoff for alternative a_3 is greater than that for a_2 and a_4. Hence, a_3 dominates a_2 and a_4 by *event dominance*.

<u>D. 11. 3.</u> (*Types of Decision Problems*)
We distinguish three different decision problems:

(1) Decision under certainty
The true state is known to the decision maker. Therefore, he can predict the consequence of his action with certainty

(2) Decision under uncertainty
The decision maker does not know which of the states will occur and has no way of estimating the probabilities of their occurrence.

(3) Decision under risk
The decision maker does not know the true state of nature by certain, but he can qualify his uncertainty through a probability distribution.

D. 11. 4. (***Decision under Certainty***)
A decision problem under certainty occurs if $n = 1$, i.e. we have the following decision table:

Actions	State s
a_1	v_1
.	.
.	.
.	.
a_i	v_i
.	.
.	.
.	.
a_m	v_m

If the outcomes are known and the values of the outcomes are certain, the task of the decision maker is to compute the optimal alternative or outcome with some optimization criterion in mind.
As an example: if the optimization criterion is cost minimisation and you are considering two different brands of a product, which appear to be equal in value to you, one costing 20% less than the other, then, all other things being equal, you will choose the less expensive brand.
Linear optimisation is an example of a technique for making decisions under the assumption of certainty.

D. 11. 5. (***Decision under Uncertainty***)
Here the decision maker feels that he can say nothing at all about the true state of nature. Not only is he unaware of the true state, but he cannot even quantify his uncertainty in any way. He is only prepared to say that each s_j describes a possible state of the world.

Ex. 11. 2. (***A Newsvendor Problem***)
The demand for a certain newspaper in a small town is 0, 1, 2, or 3 units. A newsvendor purchases the newspaper for 0.10 € and sells each paper for 0.25 €.
Therefore, we shall have the following profit table (matrix):

		s_1	s_2	s_3	s_4
		0	1	2	3
a_1	0	0	0	0	0
a_2	1	-10	15	15	15
a_3	2	-20	5	30	30
a_4	3	-30	-5	20	45

How many newspapers should the newsvendor order?

R. 11. 4. (***Decision Rule under Uncertainty***)
How should a decision maker choose in such a situation of strict uncertainty?
We shall discuss the following rules:

(1) The Laplace (equal likelihood) rule,
(2) The maximin rule,
(3) The minimax rule,
(4) The maximax rule,
(5) The minimax regret rule
(6) The pessimism-optimism rule.

R. 11. 5. (***Laplace Rule***)
Choose the action a_k such that

$$\sum_{j=1}^{n} (1/n)\, v_{kj} = \max_{i=1,2,\ldots,m} \sum_{j=1}^{n} (1/n)\, v_{ij} .$$

Ex. 11. 3.
We illustrate the Laplace rule by applying it to the newsvendor problem:

		s_1	s_2	s_3	s_4	
		0	1	2	3	$\sum_{j=1}^{n}(1/n)v_{ij}$
a_1	0	0	0	0	0	0
a_2	1	-10	15	15	15	8.75
a_3	2	-20	5	30	30	**11.25**
a_4	3	-30	-5	20	45	7.50

$$\max(0,\ 8.75,\ 11.25,\ 7.50) = 11.25.$$

Therefore, the newsvendor chooses the action a_3.

R. 11. 6.
The Laplace rule is, in fact, a rule for decisions with risk for the special case that all states of nature occur with the same probability. It is based on the so-called *"principle of insufficient reason*". Laplace argued that "knowing nothing about the true state of nature" is equivalent to "all states having equal probability".

R. 11. 7. (***Maximin Rule/Wald***)
Choose the action a_k such that

$$S_k = \max_{i=1,2,\ldots,m} S_i := \max_i \min_j v_{ij} .$$

We call S_i the *security level* of a_i, i. e. a_i guarantees the decision maker a return of at least S_i.

We note that this rule is a very pessimistic criterion of choice; for its general philosophy is to assume that the worst will happen. It can also mean “pure fear”.

Ex. 11. 4.

We illustrate the minmax rule by applying it to the newsvendor problem:

		s_1 0	s_2 1	s_3 2	s_4 3	$\min_j v_{ij}$
a_1	0	0	0	0	0	**0**
a_2	1	-10	15	15	15	-10
a_3	2	-20	5	30	30	-20
a_4	3	-30	-5	20	45	-30

$$\max(0,\ -10,\ -20,\ -30) = 0 .$$

Therefore, the newsvendor chooses the action a_1.

R. 11. 8. (***Minimax Rule***)

Choose the action a_k such that

$$w_{kj} = \min_i \max_j w_{ij} ,$$

where

$$w_{ij} := \max_{i,j} v_{ij} - v_{ij} .$$

Ex. 11. 5.

We illustrate the minimax rule by applying it to the newsvendor problem:

w_{ij}		s_1 0	s_2 1	s_3 2	s_4 3	$\max_j w_{ij}$
a_1	0	45	45	45	45	**45**
a_2	1	55	30	30	30	55
a_3	2	65	40	15	15	65
a_4	3	75	50	25	0	75

$$\min\left(45,\ 55,\ 60,\ 75\right) = 45 .$$

Therefore, the newsvendor chooses the action a_1.

R. 11. 9.
The maximin and the minimax rules will always lead to the same choice of action.

R. 11. 10. (***Maximax Rule***)
Choose the action a_k such that

$$v_{kj} = \max_i \max_j v_{ij}.$$

We note that this rule is a very optimistic criterion of choice; for its general philosophy is to assume that the best case will happen. It can also mean "pure greed" ("Go for the gold!")

Ex. 11. 6.
We illustrate the maximax rule by applying it to the newsvendor problem:

		s_1 0	s_2 1	s_3 2	s_4 3	$\max_j v_{ij}$
a_1	0	0	0	0	0	0
a_2	1	-10	15	15	15	15
a_3	2	-20	5	30	30	30
a_4	3	-30	-5	20	45	**45**

$$\max(0,\ 15,\ 30,\ 45) = 45.$$

Therefore, the newsvendor chooses the action a_4.

R. 11. 11. (***Minimax Regret Rule/Savage-Niehans***)
Choose the action a_k such that

$$v_{kj} = \min_i \max_j r_{ij},$$

where

$$r_{ij} = \max_i v_{ij} - v_{ij}.$$

The matrix $R := \left(r_{ij}\right),\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n,$ is the so-called *"regret matrix"*.
This rule might suggest "fear of guilt".

<u>Ex. 11. 7.</u>
We illustrate the *maximax* regret rule by applying it to the newsvendor problem:

r_{ij}		s_1 0	s_2 1	s_3 2	s_4 3	$\max_j r_{ij}$
a_1	0	0	15	30	45	45
a_2	1	10	0	15	30	30
a_3	2	20	10	0	15	**20**
a_4	3	30	20	10	0	30

$$\min(45,\ 30,\ 20,\ 30) = 20.$$

Therefore, the newsvendor chooses the action a_3.

<u>R. 11. 12.</u> (*Pessimism-Optimism Rule/Hurwicz*)
Choose the action a_k such that

$$v_{kj} = \max_i \left\{ \alpha \cdot \max_j v_{ij} + (1-\alpha) \min_j v_{ij} \right\},$$
$$0 \le \alpha \le 1,$$

where

α: optimism index,
$1-\alpha$: pessimism index.

This rule might mean “combining greed and fear”.

This approach attempts to strike a balance between the *maximax* and *maximin* criteria
A cautious decision maker will set $\alpha = 1$ which reduces the Hurwicz criterion to the *maximin* criterion. An adventurous decision maker will set $\alpha = 0$ which reduces the Hurwicz criterion to the *maximax* criterion.

Ex. 11. 8.
We illustrate the pessimism-optimism rule by applying it to the newsvendor problem:
Let $\alpha := 0.7$

		s_1 0	s_2 1	s_3 2	s_4 3	$\alpha \cdot \max_j v_{ij} + (1-\alpha) \min_j v_{ij}$
a_1	0	0	0	0	0	$0.3 \cdot 0 + 0.7 \cdot 0 = 0$
a_2	1	-10	15	15	15	$0.3 \cdot (-10) + 0.7 \cdot 15 = 7.5$
a_3	2	-20	5	30	30	$0.3 \cdot (-20) + 0.7 \cdot 30 = 15$
a_4	3	-30	-5	20	45	$0.3 \cdot (-30) + 0.7 \cdot 45 = \mathbf{22.5}$

$$\max_i (0;\ 7.5;\ 15;\ 22.5\ ) = 22.5 .$$

Therefore, the newsvendor chooses the action a_4.

R. 11. 13.
The above rules do not take into account the probability associated with the outcomes for each alternative; they merely focus on the value of the outcomes. The criticism of decision criteria under uncertainty is aimed at their failure to include important information about the chance of each state occurring.

D. 11. 6. (***Decision under Risk***)
A decision problem under risk occurs when the decision maker does not know the true state of nature by certain, but he can qualify his uncertainty through a probability distribution.

R. 11. 14.
How should a decision maker choose in a situation of risk? We shall discuss the following rules:

(1) The $\mu-$Rule,
(2) The "Bernoulli"-Rule,
(3) The $\mu\sigma-$Rule.

R. 11. 15. (***The $\mu-$Rule***)
Choose the action a_k such that

$$E_k = \max_i \sum_{j=1}^{n} v_{ij} \cdot p(s_j) ,$$

where

$p(s_j)$: the probability for the occurrence of s_j,
E_k : the expected value of a_k.

<u>D. 11. 7.</u> (*Expected Monetary Value*)
We call

$$EMV = \max_i \sum_{j=1}^{n} v_{ij} \cdot p(s_j)$$

the *expected monetary value (EMV).*

<u>Ex. 11. 9.</u>
We illustrate the $\mu-$rule by applying it to the newsvendor problem:

		s_1	s_2	s_3	s_4	
		$p(s_1)=0.10$	$p(s_2)=0.35$	$p(s_3)=0.40$	$p(s_4)=0.15$	
		0	1	2	3	E_i
a_1	0	0	0	0	0	0.00
a_2	1	-10	15	15	15	12.50
a_3	2	-20	5	30	30	**16.25**
a_4	3	-30	-5	20	45	10.00

$$\max(0.00,\ 12.50,\ 16.25,\ 10.00) = 16.25.$$

Therefore, the newsvendor chooses the action a_3. The expected monetary value for our problem is equal to 18.25.

<u>D. 11. 8.</u> (*Expected Value under Certainty*)
The expected value under certainty is defined to be:

$$EVUC = \sum_{j=1}^{n} \max_i v_{ij} \cdot p(s_j).$$

<u>Ex. 11. 10.</u>
The expected value under certainty for our newsvendor problem is

$$EVUC = 0 \cdot 0.10 + 15 \cdot 0.35 + 30 \cdot 0.40 + 45 \cdot 0.15 = 24.$$

<u>D. 11. 9.</u> (*Expected Value of Perfect Information*)

$$EVPI := EVUC - EMV.$$

<u>Ex. 11. 11.</u>
The expected value of perfect information for our newsvendor problem is

$$EVPI = 24 - 16.25 = 7.75.$$

<u>R. 11. 16.</u> (*The "Bernoulli"-Rule*)
Choose the action a_k such that

$$E_k = \max_i \sum_{j=1}^{n} u_{ij}\left(v_{ij}\right) \cdot p(s_j)\,,$$

where

$u_{ij}\left(v_{ij}\right),\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n$: the utility of v_{ij}.

<u>R. 11. 17.</u> (*Risk attitude*)
The risk attitude is determined as follows:

$$u''(v)\begin{cases} < 0 & \text{risk-averse} \\ = 0 & \text{risk-neutral} \\ > 0 & \text{risk-seeking} \end{cases}$$

<u>Ex. 11. 12</u>
We illustrate the Bernoulli rule by applying it to the newsvendor problem using the utility function

$$u_{ij} = -0.02v_{ij}^2 + 3v_{ij},\quad i, j = 1,2,3,4:$$

u_{ij}		s_1	s_2	s_3	s_4	
		$p(s_1) = 0.10$	$p(s_2) = 0.35$	$p(s_3) = 0.40$	$p(s_4) = 0.15$	
		0	1	2	3	E_i
a_1	0	0.0	0.0	0.0	0.0	0.000
a_2	1	-32.0	40.5	40.5	40.5	33.250
a_3	2	-68.0	14.5	72.0	72.0	37.875
a_4	3	-108.0	-15.5	52.0	94.5	18.750

$$\max(0.000,\ 32.250,\ 37.875,\ 18.750) = 37.875.$$

Therefore, the newsvendor chooses the action a_3.

<u>R. 11. 18.</u> (*The $\mu - \sigma$-Rule*)
Choose the action a_k such that

$$\Phi_k = \max_i \Phi\left(\mu_i, \sigma_i\right),$$

where

$\Phi\left(\mu, \sigma\right)$: *preference function.*

<u>R. 11. 19.</u> (*Risk attitude*)
The risk attitude is determined as follows:

$$\frac{\partial \Phi}{\partial \sigma}\begin{cases} < 0 & \text{risk-averse} \\ = 0 & \text{risk-neutral} \\ > 0 & \text{risk-seeking} \end{cases}$$

<u>Ex. 11. 13.</u>
We illustrate the Bernoulli rule by applying it to the newsvendor problem using the preference function

$$\Phi(\mu,\sigma) = 5\mu - 0.5\sigma.$$

$\mu_1 = 0.00, \quad \mu_2 = 12.50, \quad \mu_3 = 16.25, \quad \mu_4 = 10.00.$

$\sigma_1 = 0.00, \quad \sigma_2 = 7.50, \quad \sigma_3 = 16.7238602, \quad \sigma_4 = 21.50581317.$

		s_1	s_2	s_3	s_4			
		$p(s_1) = 0.10$	$p(s_2) = 0.35$	$p(s_3) = 0.40$	$p(s_4) = 0.15$			
		0	1	2	3	μ_i	σ_i	Φ_i
a_1	0	0	0	0	0	0.00	0.00	0.00
a_2	1	-10	15	15	15	12.50	7.50	58.75
a_3	2	-20	5	30	30	16.25	16.72	**72.89**
a_4	3	-30	-5	20	45	10.00	21.51	39.25

$$\max(0.00,\ 78.75.25,\ 72.89,\ 39.25) = 72.89.$$

Therefore, the newsvendor chooses the action a_3.

<u>R. 11. 20.</u> (*Decision Tree*)
A *decision tree* is a pictorial representation of a decision situation, normally found in discussions of decision-making under uncertainty or risk. It shows decision alternatives, states of nature, probabilities attached to the state of nature, and conditional benefits and losses. The tree approach is most useful in a sequential decision situation.
The decision tree is read from left to right. The leftmost node in a decision tree is called the *root node* (or the *decision* node) and is represented by a small square.
The branches emanating to the right from a decision node represent the set of decision alternatives that are available. One, and only one, of these alternatives can be selected. The small circles in the tree are called *chance nodes*. The number shown in parentheses on each branch of a chance node is the probability that the outcome shown on that branch will occur at the chance node.
The right end of each path through the tree is called an *endpoint*, and each point represents the final outcome of following a path from the root node of the decision tree to that endpoint.

R. 11. 21. (***Drawing a Decision Tree Step-by-Step***)

Step 1**:** Grow the decision tree.

Step 2**:** Assign probabilities to the event outcomes on the tree.
Step 3**:** Assign the cash flows to the tree.

Step 4**:** Fold back the decision tree and compute the expected values for each decision.

Ex. 11. 14. (***Bighorn Oil Company***)[1]
Bighorn Oil Company has leased the drilling rights on a large parcel of land in Wyoming that may or may not contain an oil reserve. A competitor has offered to lease the land for $200000 cash in return for drilling rights and all rights to any oil that might be found.
The offer will expire in three days. If Bighorn does not take the deal, it will be faced with the decision of whether to drill for oil on its own. Drilling costs are projected to be $400000. The company feels that there are four possible outcomes from drilling:

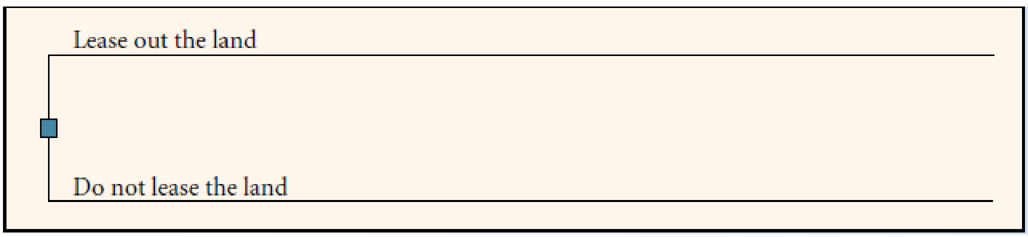
1. dry hole (no oil or natural gas)
2. natural gas
3. natural gas and some oil
4. oil only

If drilling yields a dry hole, the land will be basically worthless, because it is located in the badlands of Wyoming. If natural gas is discovered, Bighorn will recover only its drilling costs. If natural gas and some oil is discovered, revenue is projected to be $800,000. Finally, if only oil is discovered, revenues will be $1,600,000.

Draw a suitable decision tree

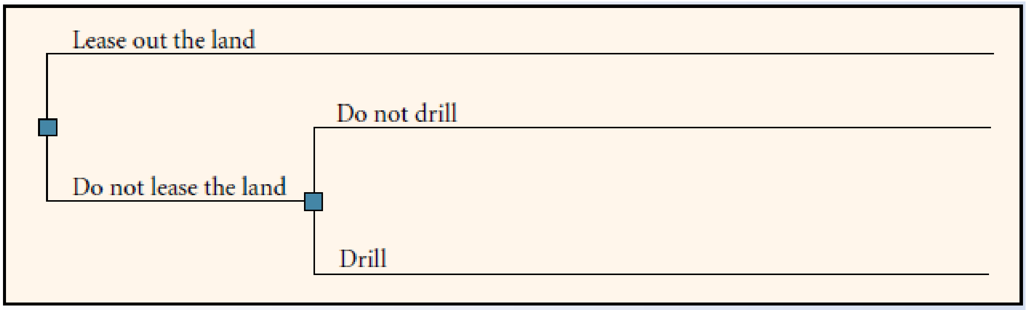
Solution:

Step 1**:** Grow the decision tree.
The initial decision to be made is whether to accept the lease:

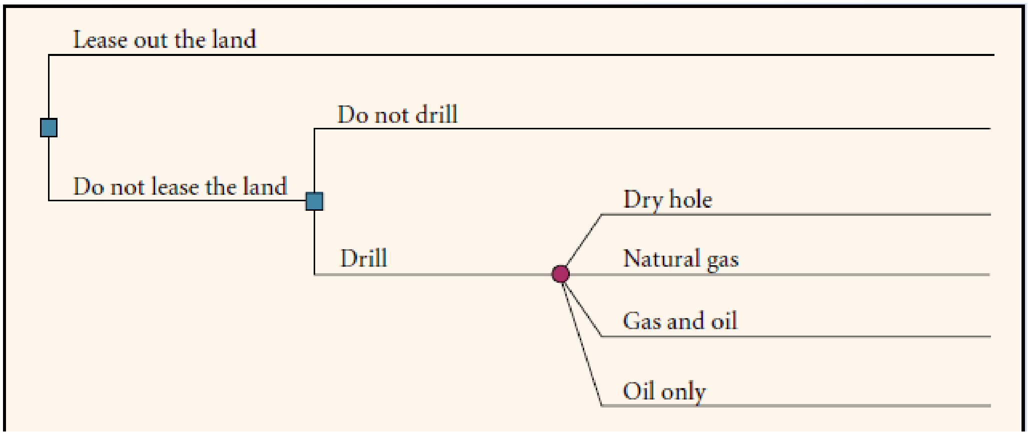


If the land is leased, no further decisions are required. However, if the land is not leased, Bighorn faces the decision of whether to drill on the property. The tree then grows to:

[1] Groebner, D. F.; Shannon, Patrick. W.; Frey, Phillip. C.; Smith, Kent. D.: Business Statistics , 2012



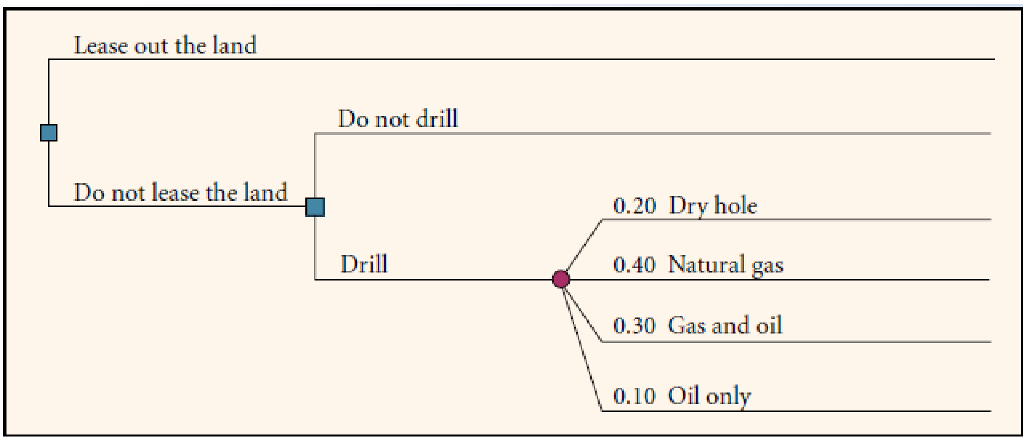
Now, if Bighorn decides to drill, there are four possible events that could occur:



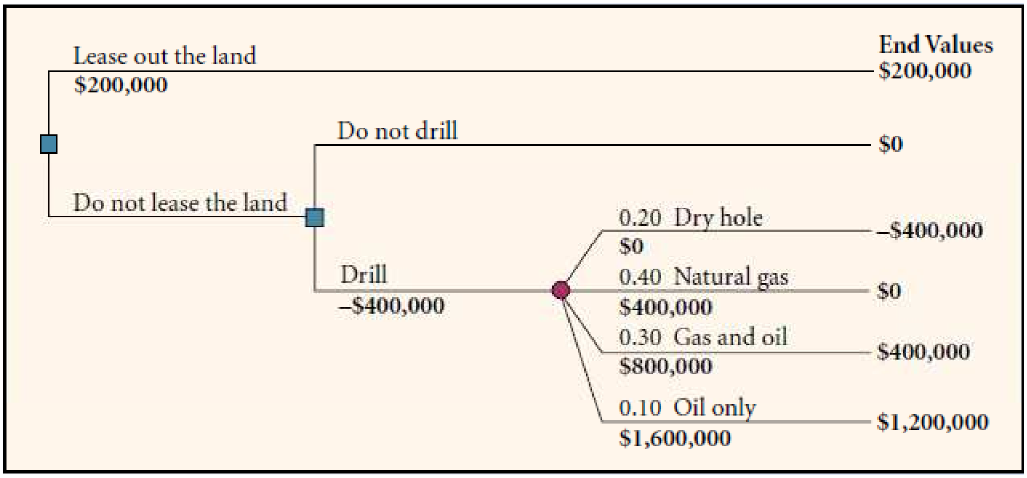
Step 2**:** Assign probabilities to the event outcomes on the tree.
In this example, the only event deals with the production result if Bighorn decides to drill. The company has subjectively assessed the probability of each of the four possible outcomes as follows:

Outcome	Probability
Dry hole	0.20
Natural gas	0.40
Gas and Oil	0.30
Oil only	0.10

The revised decision tree reflects these probabilities and becomes:



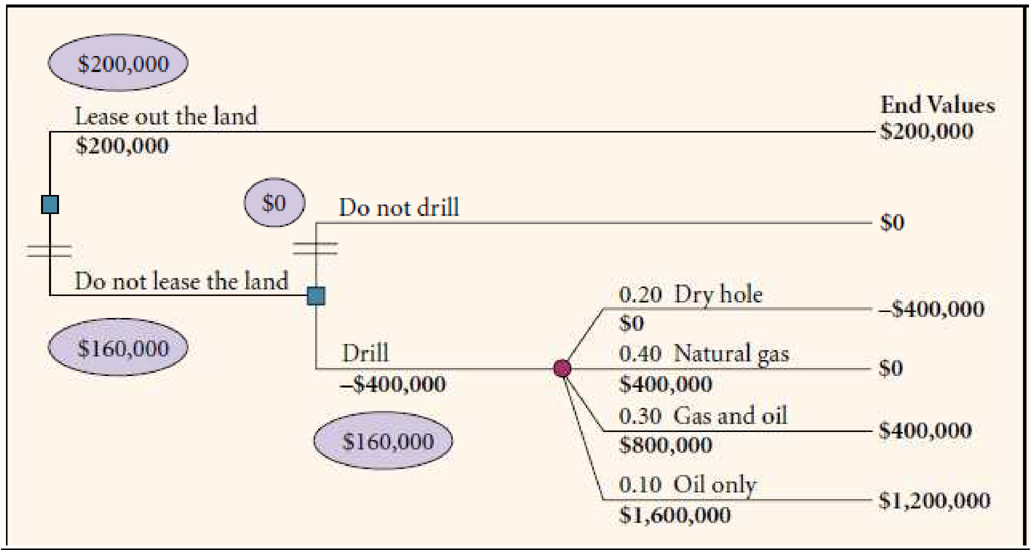
***Step 3*:** Assign the cash flows to the tree.
At each branch of the tree at which a revenue or cost occurs, show the dollar value. These revenues and costs are then totaled across the tree, and the end values for each branch are determined. These cash flows are palace on the tree as follows:



***Step 4*:** Fold back the decision tree and compute the expected values for each decision.
We need to compute the expected value for each decision alternative. This is done from the right side of the tree and working back to the left. We first determine:

$$E(\text{"Drill"}) = -400000 \cdot 0.20 + 0 \cdot 0.40 + 400000 \cdot 0.30 + 1200000 \cdot 0.10 = 160000$$

As we fold back the tree, we block all decision alternatives that do not have the highest expected value. This is shown in the decision tree as follows:



Note that we always select the decision with the highest expected payoff. In this example, the *best decision* is to lease the land and accept $200000 payment because it exceeds the $160000 expected value of the non-lease option.

(Last updated: 05.09.2014)