Chapter 7

Linear Optimization

Transportation Problem

D. 7. 1. (*Hitchcock-Koopmans's Transportation Problem*)
The *(classical) Hitchcock-Koopmans's transportation problem* is defined as follows:

$$\text{Minimise } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to

$$\sum_{i=1}^{m} x_{ij} = a_i, \quad i = 1,2,...,m$$
$$\sum_{j=1}^{n} x_{ij} = b_j, \quad j = 1,2,...,n$$

$$x_{ij} \geq 0, \quad i = 1,2,...,m;\ j = 1,2,...,n.$$

Here are:

x_{ij} : the amount of goods moved from origin i to destination j
c_{ij} : the cost of moving a unit amount of goods from origin i to destination j
a_i : the supply available at origin i
b_j : the demand for goods at destination j
m : total number of origins (sources)
n : total number of destinations (sinks).

The classical transportation is called *closed* (*or balanced*)if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j .$$

It is called *open* (or *unbalanced*)if

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j .$$

Ex. 7. 1.
A certain commodity is to be transported from the sources S_1, S_2 and S_3 to the destinations D_1, D_2, D_3 and D_4. The following table shows the availabilities of the commodity, the requirements of the destinations and the costs per unit for the transportation of the commodity from the three sources to the four destinations:

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities
S_1	5	2	4	3	30
S_2	6	4	9	5	40
S_3	2	3	8	1	55
Requirements →	15	20	40	50	

Determine a transportation plan that minimises the total transportation cost.

Th. 7. 1.
The classical transportation problem has always a feasible solution.

Proof:
Let

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j := a.$$

We show that

$$x_{ij} = \frac{a_i b_j}{a}, \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n,$$

is a feasible solution of the classical transportation problem:

$$x_{ij} = \frac{a_i b_j}{a} \geq 0, \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n.$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{a} = \frac{a_i \sum_{j=1}^{n} b_j}{a} = a_i, \quad i = 1, 2, \ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{a} = \frac{b_j \sum_{j=1}^{n} a_i}{a} = b_j, \quad j = 1, 2, \ldots, n.$$

Th. 7. 2
The set of solution of the classical transportation problem is unbounded.

Proof:

$$\begin{cases} \sum_{j=1}^{n} x_{ij} = a_i < \infty \\ \sum_{i=1}^{m} x_{ij} = b_j < \infty \quad \Rightarrow \quad 0 \le x_{ij} < \infty \\ x_{ij} \ge 0, \ \forall \text{i,j} \end{cases}$$

C. 7. 1.
The classical transportation problem has always (at least) an optimal solution.

Th. 7. 3.
In the system of constraints of the classical transportation problem at least one equation is dependent on the remaining constraints.

Proof (Only for $m = 2, \ n = 3$*)*

x_{11}	x_{12}	x_{13}	a_1
x_{21}	x_{22}	x_{23}	a_2
b_1	b_2	b_3	

(i) $x_{11} + x_{12} + x_{13} = a_1$

(ii) $x_{21} + x_{22} + x_{23} = a_2$

(iii) $x_{11} + x_{21} = b_1$

(iv) $x_{12} + x_{22} = b_2$

(v) $x_{13} + x_{23} = b_3$

Subtracting equation (ii) from the sum of the equation3 (iii), (iv) and (v), we obtain equation (i).

C. 7. 2.

$$r(A) \le m + n - 1,$$

$r(A)$: Rank of the coefficient matrix A.

R. 7. 1. (*Illustration of C. 2. 7.*)

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$r(A) \leq 2 + 3 - 1 = 4.$$

C. 7. 3.
A feasible basic solution of the classical transportation has at most $m + n - 1$ positive components.

D. 7. 3. (*Degenerate Solution*)
A feasible basic solution (of the classical transportation problem) with less than $m + n - 1$ positive components is called a *degenerate solution*.

Th. 7. 4.
Assuming $a_i \geq 0,\ i = 1, 2, ..., m;\ b_j \geq 0,\ j = 1, 2, ..., n,$ to be integer, then every feasible basic solution of the classical transportation problem will also be integer.

R. 7. 2.
Because of the above results the transportation problem is considered as a *special case* of linear optimization:

R. 7. 3.
The transportation algorithm comprises two phases:
In *Phase 1* a basic feasible solution will be found. In *Phase 2,* the basic feasible solution will be successively improved until an optimal solution has been found.

Alg. 7. 1. (*Transportation Algorithm*)

Phase 1:

Use one of the following three methods:

- Northwest corner method
- The minimum cell cost method
- Vogel's approximation method (VAM)

A. Northwest Corner

S1: Assign largest possible allocation to the cell in the upper-left-hand corner of the tableau.
S2: Repeat Step 1 until all allocations have been assigned.
S3: Stop. A basic feasible solution has been reached.

B. *Minimum Cell Cost*

S1: Find a cell that has the least cost
S2: Allocate as much as possible to this cell.
S3: Block those cells that cannot be allocated to
S4: Repeat above stels until all allocations have been assigned.

C. *Vogel's Approximation Method (VAM)*

S1: For each column and row, determine its penalty cost by subtraction their two of least cost
S2: Select row/column that has the highest penalty cost in Step 1.
S3: Allocate as much as possible to the selected row/column that has the least cost.
S4: Block those cells that cannot be further allocated to.
S5: Repeat above steps until all allocations have been assigned.

<u>Phase 2</u>:

S1: Set up the equation

$$u_i + v_j = c_{ij}$$

for basic variables (occupied cells).

S2: Solve the equation by assigning a value (preferably 0) to one of the variables u_i and v_j

S3: Calculate

$$c_{ij}^{'} = u_i + v_j$$

for nonbasic variables (empty cells).

S4: Compute the opportunity cost for each empty cell:

$$d_{ij} = c_{ij} - c_{ij}^{'} \ .$$

S5: If

$d_{ij} \geq 0$ for all empty sets, the given solution is optimal; otherwise continue.

S6: Select a cell with the smallest negative opportunity cost as the cell to be included in the next solution.

S7: Draw a *closed loop* for the unoccupied cell selected in *S6*. Note that the right angle turn in the loop is permitted only at occupied cells and at the original unoccupied cell. (A closed loop is a sequence of cells in the transportation problem such that

1. each pair of consecutive cells lies in either the same row or the same column

2. no three consecutive cells lie in the same row or column
3. the first and last cells of a sequence lie in the same row or column
4. no cell appears more than once in the sequence
5. The first cell is unoccupied and all the others are occupied.

S8: Assign alternate plus and minus signs at the unoccupied cells on the corner points of the loop aith the plus sign at the cell being evaluated.

S9: Determine the maximum number of units that can be shipped to this unocupied cell. The smallest valte with a negative position can be shipped to the entering cell. Now, add this quantity to the cells on the corner points of the loop marked with plus signs and subtract it from those cells marked with minus sign.
In this way an unoccupeis cell becomes a occupied cell.

S10: Repeat the whole procedure until an optimal solution is obtained.

<u>Ex. 7. 2.</u>
Consider the transportation problem *in Ex. 7. 1.:*

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities
S_1	5	2	4	3	30
S_2	6	4	9	5	40
S_3	2	3	8	1	55
Requirements →	15	20	40	50	

Find an initial solution by applying the following methods:

1. Northwest corner
2. Minimum cell cost
3. VAM

Solution:

1.

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities
S_1	5 **15**	2 **15**	4	3	30
S_2	6	4 **5**	9 **35**	5	40
S_3	2	3	8 **5**	1 **50**	55
Requirements →	15	20	40	50	125

$z_0 = 530$

2.

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities
S_1	5	2 **20**	4 **10**	3	30
S_2	6 **10**	4	9 **30**	5	40
S_3	2 **5**	3	8	1 **50**	55
Requirements →	15	20	40	50	125

$z_0 = 470$

3.

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities	d_i^r
S_1	5	2	4 **30**	3	30	1
S_2	6	4 **20**	9 **10**	5 **10**	40	1
S_3	2 **15**	3	8	1 **40**	55	1, 2
Requirements →	15	20	40	50	125	
d_j^c	3, 4	1	4, 1	2, 4		

$z_0 = 410$

Ex. 7. 3.
A transportation company ships truckloads of grain from 4 silos $S_i,\ i = 1,\ldots,4$, to 5 mills $M_j,\ j = 1,\ldots,5$. The supply and the demand together with the transportation costs per truckload on the different routes are summarised in the following table:

	M_1	M_2	M_3	M_4	M_5	Supply
S_1	6	2	8	7	5	40
S_2	4	3	7	5	9	70
S_3	2	1	3	6	4	60
S_4	5	6	4	8	3	30
Demand	30	60	50	40	20	200

The company seeks the minimum-cost shipping schedule between the silos and the mills.

Solution:
We use VAM to obtain an initial solution:

0. Iteration

	M_1	M_2	M_3	M_4	M_5	Supply	d_i^r
S_1	6	2 **40**	8	7	5	40	3
S_2	4	3 **20**	7	5 **40**	9 **10**	70	1, 2
S_3	2 **30**	1	3 **20**	6	4 **10**	60	1, 2, 1
S_4	5	6	4 **30**	8	3	30	1
Demand	30	60	50	40	20	200	
d_j^c	2	1, 2	1	1	1		

$z_0 = 700$

	M_1	M_2	M_3	M_4	M_5	Supply	u_i
S_1	6 6	- 2 **40**	8 7	7 4	+ 5 8	40	-1
S_2	4 7	+ 3 **20**	7 8	5 **40**	- 9 **10**	70	0
S_3	2 **30**	1 -2	3 **20**	6 0	4 **10**	60	-5
S_4	5 3	6 -1	4 **30**	8 1	3 5	30	-4
Demand	30	60	50	40	20	200	
v_j	7	3	8	5	9		

	M_1	M_2	M_3	M_4	M_5	Supply	u_i
S_1	6 3	2 **30**	8 4	7 4	5 **10**	40	1
S_2	4 4	3 **30**	7 5	5 **40**	9 6	70	2
S_3	2 **30**	1 1	+ 3 **20**	6 3	- 4 **10**	60	0
S_4	5 3	6 2	- 4 **30**	8 4	+ 3 5	30	1
Demand	30	60	50	40	20	200	
v_j	1	3	3	4	9		

$z_1 = 680$

	M_1	M_2	M_3	M_4	M_5	Supply	u_i
S_1	6 5	+ 2 **30**	8 6	7 4	- 5 **10**	40	0
S_2	+ 4 6	- 3 **30**	7 7	5 **40**	9 6	70	1
S_3	+ 2 **30**	1 -1	+ 3 **30**	6 1	4 2	60	-3
S_4	5 3	6 0	- 4 **20**	8 2	+ 3 **10**	30	-2
Demand	30	60	50	40	20	200	
v_j	5	2	6	4	5		

$z_2 = 660$

	M_1	M_2	M_3	M_4	M_5	Supply	u_i
S_1	6 5	2 **40**	8 4	7 4	5 3	40	-1
S_2	4 **10**	3 **20**	7 5	5 **40**	9 4	70	0
S_3	2 **20**	1 3	3 **40**	6 3	4 2	60	-2
S_4	5 3	6 4	4 **10**	8 4	3 **20**	30	-1
Demand	30	60	50	40	20	200	
v_j	4	3	5	5	4		

$z_2 = 640$

We have now reached the optimal solution:

$x_{12}^* = 40,\ x_{21}^* = 10,\ x_{22}^* = 20,\ x_{24}^* = 40,\ x_{31}^* = 20,\ x_{32}^* = 40,\ x_{43}^* = 10, x_{45}^* = 20$

All other components are equal to zero. $z^* = 640$.

<u>Ex. 7. 4.</u> (*A Degenerate Problem)*

Three factories F_1, F_2 and F_3 supply four dealers D_1, D_2, D_3 and D_4 with a certain product. The available informations are given in the following table:

Factory	Dealer				Supply
	D_1	D_2	D_3	D_4	
F_1	2	2	2	4	1000
F_2	4	6	4	3	700
F_3	3	2	1	0	900
Demand	900	800	500	400	2600

1. Use the minimum cell cost method as initial solution.
2. Find an optimal transportation plan.

Solution:

1.

Dealer → Factory ↓	D_1	D_2	D_3	D_4	Supply
F_1	2 **900**	2 **100**	2	4	1000
F_2	4	6 **700**	4	3	700
F_3	3	2	1 **500**	0 **400**	900
Demand	900	800	500	400	2600

$z_0 = 6700$

2.

The number of positive basic variables is $5 < m + n - 1 = 3 + 4 - 1$. Therefore, the initial basic feasible solution is degenerate.

To resolve degeneracy, we make use of an arbitrarily small quantity $\varepsilon > 0$ (practically $\varepsilon = 0$).

Dealer → Factory ↓	D_1	D_2	D_3	D_4	Supply	u_i
F_1	2 - **900**	2 + **100**	2 1	4 0	1000	0
F_2	+ 4 6	- 6 **700**	4 5	3 4	700	4
F_3	3 2	2 **0**	1 **500**	0 **400**	900	0
Demand	900	800	500	400	2600	
v_j	2	2	1	0		

Dealer → Factory ↓	D_1	D_2	D_3	D_4	Supply	u_i
F_1	2 **200**	2 **800**	2 1	4 0	1000	0
F_2	4 **700**	6 4	4 3	3 2	700	2
F_3	3 2	2 **0**	1 **500**	0 **400**	900	0
Demand	900	800	500	400	2600	
v_j	2	2	1	0		

$z_1 = z^* = 5300.$

<u>Ex. 7. 5.</u> (*Multiple Solutions*)

A concrete company transports concrete from three plants P_1, P_2 and P_3 to four construction sites S_1, S_2, S_3 and S_4. The following table shows the necessary data:

Plant	Construction Sites				Supply
	S_1	S_2	S_3	S_4	
P_1	5	3	6	2	19
P_2	4	7	9	1	37
P_3	3	4	7	5	34
Demand	16	18	31	25	90

Find an optimal transportation programme.

Solution:

We use Vogel's approximation method to find an initial solution:

Site → Plant ↓	S_1	S_2	S_3	S_4	Supply	d_i^r
P_1	[5]	[3] **18**	[6] **1**	[2]	19	1, 2, 1
P_2	[4] **12**	[7]	[9]	[1] **25**	37	3, 5
P_3	[3] **4**	[4]	[7] **30**	[5]	34	1, 4
Demand	16	18	31	25	90	
d_c^c	1	1	1	1		

Site → Plant ↓	S_1	S_2	S_3	S_4	Supply	u_i
P_1	[5] 2	- [3] **18**	+ [6] **1**	[2] -1	19	0
P_2	[4] **12**	[7] 5	[9] 8	[1] **25**	37	2
P_3	[3] **4**	[4] +4	[7] - **30**	[5] 0	34	1
Demand	16	18	31	25	90	
v_j	2	3	6	-1		

$z_0 = 355 = z^*$.

Because of $c_{32} = c'_{32}$ we have a multiple optimal solution

Site→ Plant ↓	S_1	S_2	S_3	S_4	Supply	u_i
P_1	[5] 2	[3] 3	[6] **19**	[2] -2	19	-1
P_2	[4] **12**	[7] 3	[9] 8	[1] **25**	37	1
P_3	[3] **4**	[4] **18**	[7] **12**	[5] -1	34	0
Demand	16	18	31	25	90	
v_j	3	4	7	-1		

$z_1 = 355 = z^*$

$$X^* = \alpha \begin{pmatrix} 0 & 18 & 1 & 0 \\ 12 & 0 & 0 & 25 \\ 4 & 0 & 30 & 0 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0 & 0 & 19 & 0 \\ 12 & 0 & 0 & 25 \\ 4 & 18 & 12 & 0 \end{pmatrix}$$

$$0 \le \alpha \le 1$$

R. 7. 3. (***Prohibited Routes***)

Sometimes one or more of the routes in a transportation problem are *prohibited*. That is, units cannot be transported from a particular source to s particular destination. When this situation occurs, we must make sure that no units in the optimal solution are allocated to the cell representing this route. For that purpose a value of $M > 0$, arbitrarily large, is assigned as transportation cost for such a cell.

<u>Ex. 7. 6.</u>

The following table shows the supply of the canneries C_1, C_2, C_3 and C_4 to the factories $F_1, F_2, ..., F_6$ together with the corresponding unit transportation costs. The route from C_2 to F_5 is for the time being due to construction works closed:

	F_1	F_2	F_3	F_4	F_5	F_5	Supply
C_1	7	5	7	7	5	3	60
C_2	9	11	6	11		5	20
C_3	11	10	6	2	2	8	90
C_4	9	10	9	6	9	12	50
Demand	60	20	40	20	40	40	220

Determine an optimal transportation plan.

Solution:

We find an initial solution by applying Vogel's approximation method:

	F_1	F_2	F_3	F_4	F_5	F_6	Supply	d_i^r
C_1	7 **20**	5	7	7	5	3 **40**	60	2, 4,0
C_2	9 **10**	11	6 **10**	11	**M**	5	20	1, 3
C_3	11	10	6 **30**	2 **20**	2 **40**	8	90	0, 4, 2, 5
S_4	9 **30**	10 **20**	9	6	9	12	50	3, 0, 3,0
Demand	60	20	40	20	40	40	220	
d_j^c	2	5	0, 1	4, 5	3, 4	2, 7		

$z_0 = 1180$

	F_1	F_2	F_3	F_4	F_5	F_6	Supply	u_i
C_1	- 7 / **20**	+ 5 / 8	7 / 4	7 / 0	5 / 0	3 / **40**	60	-2
C_2	9 / **10**	11 / 10	6 / **10**	11 / 2	**M** / 2	5 / 5	20	0
C_3	11 / 9	10 / 10	6 / **30**	2 / **20**	2 / **40**	8 / 5	90	0
S_4	+ 9 / **30**	- 10 / **20**	9 / 6	6 / 2	9 / 2	12 / 5	50	0
Demand	60	20	40	20	40	40	220	
v_j	9	10	6	2	2	5		

	F_1	F_2	F_3	F_4	F_5	F_6	Supply	u_i
C_1	7 / **0**	5 / **20**	7 / 4	7 / 0	5 / 0	3 / **40**	60	-2
C_2	9 / **10**	11 / 7	6 / **10**	11 / 2	**M** / 2	5 / 5	20	0
C_3	11 / 9	10 / 7	6 / **30**	2 / **20**	2 / **40**	8 / 5	90	0
S_4	9 / **50**	10 / 7	9 / 6	6 / 2	9 / 2	12 / 5	50	0
Demand	60	20	40	20	40	40	220	
v_j	9	7	6	2	2	5		

$z_1 = 1120$

Because of $c_{26} = c'_{26} = 5$ there are multiple solutions:

	F_1	F_2	F_3	F_4	F_5	F_6	Supply	u_i
C_1	+ 7 / **0**	5 / **20**	7 / 4	7 / 0	5 / 0	- 3 / **40**	60	-2
C_2	- 9 / **10**	11 / 7	6 / **10**	11 / 2	**M** / 2	+ 5 / 5	20	0
C_3	11 / 9	10 / 7	6 / **30**	2 / **20**	2 / **40**	8 / 5	90	0
S_4	9 / **50**	10 / 7	9 / 6	6 / 2	9 / 2	12 / 5	50	0
Demand	60	20	40	20	40	40	220	
v_j	9	7	6	2	2	5		

	F_1	F_2	F_3	F_4	F_5	F_6	Supply
C_1	7 **10**	5 **20**	7	7	5	3 **30**	60
C_2	9	11	6 **10**	11	**M**	5 **10**	20
C_3	11	10	6 **30**	2 **20**	2 **40**	8	90
S_4	9 **50**	10	9	6	9	12	50
Demand	60	20	40	20	40	40	220

$$X^* = \alpha \begin{pmatrix} 0 & 20 & 0 & 0 & 0 & 40 \\ 10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 30 & 20 & 40 & 0 \\ 50 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 10 & 20 & 0 & 0 & 0 & 30 \\ 0 & 0 & 10 & 0 & 0 & 10 \\ 0 & 0 & 30 & 20 & 40 & 0 \\ 50 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$z^* = 1120$$

$$0 \le \alpha \le 1$$

<u>Ex. 7. 7.</u> (*Unbalanced Transportation Problem*)
A company has three factories F_1, F_2 and F_3 which supply warehouses W_1, W_2, W_3 and W_4 with a certain product. Monthly factory capacities, monthly warehouse demands and unit transportation cost are given in the following table:

Factory	W_1	W_2	W_3	Supply
F_1	28	17	26	500
F_2	19	12	16	300
Demand	250	250	500	

Find an optimal solution to the problem.

Solution:
Since the demand is higher than the supply, a dummy factory F_d will be introduced:

Factory	W_1	W_2	W_3	Supply
F_1	28	17	26	500
F_2	19	12	16	300
F_d	0	0	0	200
Demand	250	250	500	1000

We use VAM to obtain an initial solution:

0. Iteration

	W_1	W_2	W_3	Supply	d_i^r
F_1	28 **50**	17 **250**	26 **200**	500	9, 2
F_2	19	12	16 **300**	300	4, 3
F_d	0 **200**	0	0	200	0
Demand	250	250	500		
d_j^c	19, 9	12, 5	16, 8, 10		

$z_0 = 15650$

	W_1	W_2	W_3	Supply	u_i
F_1	28 **50**	17 **250**	26 **200**	500	0
F_2	19 18	12 **7**	16 **300**	300	-10
F_d	0 **200**	0 -9	0 -2	200	-28
Demand	250	250	500		
v_j	28	17	26		

$z_0 = 15650 = z^*$.

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