Chapter 6

Linear Optimization

Duality

D. 6. 1. (***Symmetric Duality***)
The following problems are said to be *symmetrically dual* to one another:

(SP) $\max\left\{z = c^T x \mid Ax \le b,\ x \ge 0\right\}$

(SD) $\min\left\{Z = b^T \lambda \mid A^T \lambda \ge c,\ \lambda \ge 0\right\}$

D. 6. 2. (**A*symmetric Duality***)
The following problems are said to be *asymmetrically dual* to one another:

(ASD) $\min\left\{z = c^T x \mid Ax = b,\ x \ge 0\right\}$

(AP) $\max\left\{Z = b^T \lambda \mid A^T \lambda \ge c\right\}$

R. 6. 1. (***Primal and Dual***)
Usually the given problem is called the *primal* and the one to be constructed is called the *dual*.

Th. 6. 1.
Dual of dual is primal.

Th. 6. 2. (***Weak Duality***)[1]

If $\bar{x}$ is feasible for (SP) and $\bar{\lambda}$ is feasible for (DP), then

$$c^T \bar{x} \le \bar{\lambda}^T b\,.$$

Proof:

$$c^T \bar{x} \le (\bar{\lambda}^T A)\,\bar{x} = \bar{\lambda}^T (A\bar{x}) \le \bar{\lambda}^T b$$

C. 6. 1.

If $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively, and if $c^T \bar{x} = \bar{\lambda}^T b$, then $\bar{x}$ and $\bar{\lambda}$ are optimal for (SP) and (SD), respectively.

[1] The following statements will be proved only for the symmetric duality. They also hold for the asymmetric duality.

Proof:

Suppose $x^{\bullet}$ is any feasible solution for (SP). Then $c^T x^{\bullet} \le \bar{\lambda}^T b = c^T \bar{x}$. Similarly, if $\lambda^{\bullet}$ is any solution for (SD), then $\lambda^{\bullet T} b \ge \bar{\lambda}^T b$.

C. 6. 2.
If (SP) has unbounded objective function value, then (SD) is infeasible. If (SD) has unbounded objective value, then (SP) is infeasible

Proof:

Suppose (SD) is feasible. Let $\bar{\lambda}$ be a particular feasible solution. Then for all $\bar{x}$ feasible for (SP) we have $c^T \bar{x} \le \bar{\lambda}^T b$. So (SP) has bounded objection value if it is feasible, and therefore cannot be unbounded. The second statement is proved similarly.

Th. 6. 3.
Assume (SP) is feasible. Then (SP) is unbounded if and only if the following system is feasible:

(USP) $$\begin{cases} Aw \le 0 \\ c^T w > 0 \\ w \ge 0 \end{cases}$$

Proof:

Suppose $\bar{x}$ is feasible for (SP).

First assume that $\bar{w}$ is feasible for (USP) and $t \ge 0$ is a real number. Then

$$A(\bar{x} + t\bar{w}) = A\bar{x} + tA\bar{w} \le b + 0 = b$$

$$\bar{x} + t\bar{w} \ge 0 + t0 = 0$$

$$c^T(\bar{x} + t\bar{w}) = c^T \bar{x} + tc^T \bar{w}$$

Hence $\bar{x} + t\bar{w}$ is feasible for (SP), and by choosing t approximately large, we can make $c^T(\bar{x} + t\bar{w})$ as large as desired since $c^T \bar{w}$ is a positive number.
Conversely, suppose that (SP) has unbounded objective function value, then by C. 6. 2., (SD) is infeasible. That is, the following system has no solution:

$$\begin{cases} \lambda^T A \ge c^T \\ \lambda \ge 0 \end{cases}$$

or

$$\begin{cases} A^T \lambda \ge c^T \\ \lambda \ge 0 \end{cases}$$

By the "Theorem of Alternatives"[2], the following system is feasible:

$$\begin{cases} w^T A^T \leq 0^T \\ w^T c > 0 \\ w \geq 0 \end{cases}$$

or

$$\begin{cases} Aw \leq 0 \\ c^T w > 0 \\ w \geq 0 \end{cases}$$

Hence (USP) is feasible.

Th. 6. 4.

Assume (SD) is feasible. Then (SD) is unbounded if and only if the following system is feasible:

(USD) $$\begin{cases} v^T A \geq 0^T \\ v^T b < 0 \\ v \geq 0 \end{cases}$$

C. 6. 3.

(SP) is feasible if and only if (USD) is infeasible. (SD) is feasible if and only if (USP) is infeasible

C. 6. 4.

If (SP) is infeasible, then either (SD) is infeasible or (SD) is unbounded. If (SD) is infeasible, then either (SP) is infeasible or (SP) is unbounded.

Th. 6. 5.

Suppose (SP) and (SD) are both feasible. Then (SP) and (SD) each have finite objective function values, and moreover these two values are equal

Proof:

We know by Weak Duality Theorem that if $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively, then $c^T \bar{x} \leq \bar{\lambda}^T b$. In particular, neither (SP) nor (SD) is unbounded. So it suffices to show that the following system is feasible:

[2] Theorem of Alternatives: Either the system $Ax \geq b$, $x \geq 0$ has a solution, or the system

$$\begin{cases} \lambda^T A \leq 0^T \\ \lambda^T b > 0 \\ \lambda \geq 0 \end{cases}$$

has a solution, but not both.

$$(I) \qquad \begin{cases} Ax \le b \\ x \ge 0 \\ \lambda^T A \ge c^T \\ \lambda \ge 0 \\ c^T x \ge \lambda^T b \end{cases}$$

For if $\bar{x}$ and $\bar{\lambda}$ are feasible for this system, then by Weak Duality Theorem in fact it would have to be the case that $c^T \bar{x} \ge \bar{\lambda}^T b$

We now write the above system in matrix form:

$$\begin{pmatrix} A & 0 \\ 0 & -A^T \\ -c^T & b^T \end{pmatrix} \cdot \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ -c \\ 0 \end{pmatrix}, \qquad x, \lambda \ge 0$$

We assume that this system is infeasible and derive a contradiction. If it is not feasible, then by the Theorem of Alternatives the following system has a solution $\bar{v}, \bar{w}, \bar{t}$,

$$(II) \qquad \begin{cases} \begin{pmatrix} v^T & w^T & t \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -A^T \\ -c^T & b^T \end{pmatrix} \ge \begin{pmatrix} 0^T & 0^T \end{pmatrix} \\ \begin{pmatrix} v^T & w^T & t \end{pmatrix} \begin{pmatrix} b \\ -c \\ 0 \end{pmatrix} < 0 \\ v, w, t \ge 0 \end{cases}$$

So we have

$$\bar{v}^T A - \bar{t} c^T \ge 0^T$$
$$-\bar{w}^T A^T + \bar{t} b^T \ge 0^T$$
$$\bar{v}^T b - \bar{w}^T c < 0^T$$
$$v, w, t \ge 0$$

Case 1: Suppose $\bar{t} = 0$. Then

$$\bar{v}^T A \ge 0^T$$
$$A\bar{w} \le 0$$
$$\bar{v}^T b < c^T \bar{w}$$
$$\bar{v}, \bar{w} \ge 0$$

Now we cannot have both $c^T \bar{w} \leq 0$ and $\bar{v}^T b \geq 0$; otherweise $0 \leq \bar{v}^T b < c^T \bar{w} \leq 0$, which is a contradiction.

Case 1a: Suppose $c^T \bar{w} > 0$. Then $\bar{w}$ is a solution to (USP), so (SD) is infeasible by C. 6.3., a contardiction.

Case 1b: Suppose $\bar{v}^T b < 0$. Then $\bar{v}$ is a solution to (USD), so (SP) is infeasible by C. 6.3., a contardiction.

Case 2: Suppose $\bar{t} > 0$. Set $\bar{x} = \bar{w} / \bar{t}$ and $\bar{\lambda} = \bar{v} / \bar{t}$. Then

$$A\bar{x} < b$$

$$\bar{x} \geq 0$$

$$\bar{\lambda}^T A \geq c^T$$

$$\bar{\lambda} \geq 0$$

$$c^T \bar{x} \geq \bar{\lambda}^T b$$

Hence we have a pair of feasible solutions to (SP) and (SD), repectively, that violates Weak Duality, a contradiction.
We have now shown that (*II*) has no solution, Therefore, (*I*) has a solution.

<u>C. 6. 5.</u>
Suppose (SP) has a finite optimal objective value. Then so does (SD), and these two values are equal.
Suppose (SD) has a finite optimal objective function value. Then so does (SP), and these two values are equal.

Proof:
We will prove the first statement only.. If (SP) has a finite optimal objective function value, then it is feasible, but not unbounded. So (USP) has no solution by Theorem Th. 6. 3. Therefore (SD) is feasible by C. 6. 3. Now apply Theorem Th. 6. 5.

R. 6. 2.
The results gained above can now be summarised in the following central theorem.

Th. 6. 6. **(*Strong Duality*)**
Exactly one of the following holds for the pair (SP) and (SD):

1. They are both infeasible.
2. One is infeasible and the other unbounded.
3. They are both fesible and have equal finite optimal objective function values.

C. 6. 8.

If $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively, then $\bar{x}$ and $\bar{\lambda}$ are optimal for (SP) and (SD), respectively, if and only if $c^T \bar{x} = \bar{\lambda}^T b$.

C. 6. 9.

Suppose $\bar{x}$ is feasible for (SP). Then $\bar{x}$ is optimal for (SP) if and only if there exists $\bar{\lambda}$ feasible for (SD) such that $c^T \bar{x} = \bar{\lambda}^T b$.

Similarly suppose $\bar{\lambda}$ is feasible for (DS). Then $\bar{\lambda}$ is optimal for (SD) if and only if there exists $\bar{x}$ feasible for (SP) such that $c^T \bar{x} = \bar{\lambda}^T b$.

D. 6. 3. (*Complementary Slackness*)

Suppose $\bar{x} \in R^n$ and $\bar{\lambda} \in R^m$. Then $\bar{x}$ and $\bar{\lambda}$ satisfy *complementary slackness* if

1. For all j, either $\bar{x}_j = 0$ or $\sum_{i=1}^{m} a_{ij} \bar{\lambda}_i = c_j$ or both; and
2. For all i, either $\bar{\lambda}_i = 0$ or $\sum_{i=1}^{n} a_{ij} \bar{x}_j = b_i$ or both.

Th. 6. 7.

Suppose $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively. Then $c^T \bar{x} = \bar{\lambda}^T b$ if and only if $\bar{x}$ and $\bar{\lambda}$ satisfy complementary slackness.

C. 6. 10.

Suppose $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively, then $\bar{x}$ and $\bar{\lambda}$ are optimal for (SP) and (SD), respectively if and only if they $\bar{\lambda}$ satisfy complementary slackness.

C. 6. 11.

Suppose $\bar{x}$ is feasible for (SP). Then $\bar{x}$ is optimal for (SP) if and only if there exists $\bar{\lambda}$ feasible for (SD) such that $\bar{x}$ and $\bar{\lambda}$ satisfy complementary slackness.

Similarly suppose $\bar{\lambda}$ is feasible for (SD). Then $\bar{\lambda}$ is optimal for (SD) if and only if there exists $\bar{x}$ feasible for (SP) such that $\bar{x}$ and $\bar{\lambda}$ satisfy complementary slackness.

Ex. 6. 1. (*See Ex. 5. 6.*)

1. Solve the folloowing problem by dualisation:

$$
\begin{aligned}
z = 20x_1 + 40x_2 -> \min \\
6x_1 + x_2 \geq 18 \\
x_1 + 4x_2 \geq 12 \\
2x_1 + x_2 \geq 10 \\
x_1, x_2 \geq 0.
\end{aligned}
$$

2. Check the complementary slackness for this problem.

Solution:
1.

$$
\begin{aligned}
Z = 18\lambda_1 + 12\lambda_2 + 10\lambda_3 &\to \max \\
6\lambda_1 \quad + \lambda_2 + 2\lambda_3 &\le 20 \\
\lambda_1 + 4\lambda_2 \quad + \lambda_3 &\le 40 \\
\lambda_1, \lambda_2, \lambda_3 &\ge 0
\end{aligned}
$$

<u>Standard form</u>:

$$
\begin{aligned}
Z = 18\lambda_1 + 12\lambda_2 + 10\lambda_3 &\to \max \\
6\lambda_1 \quad + \lambda_2 + 2\lambda_3 + \lambda_4 \qquad &= 20 \\
\lambda_1 + 4\lambda_2 \quad + \lambda_3 \qquad + \lambda_5 &= 40 \\
\lambda_i \ge 0,\ i = 1, 2, \ldots, 5.
\end{aligned}
$$

Simplex Tableau

BV	λ_1	λ_2	λ_3	λ_4	λ_5	λ_0
λ_4	6	1	2	1	0	20
λ_5	1	4	1	0	1	40
Z	-18	-12	-10	0	0	0
λ_1	1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{10}{3}$
λ_5	0	$\frac{23}{6}$	$\frac{2}{3}$	$-\frac{1}{6}$	1	$\frac{110}{3}$
Z	0	-9	-4	3	0	60
λ_1	1	0	$\frac{7}{23}$	$\frac{4}{23}$	$-\frac{1}{23}$	$\frac{40}{23}$
λ_2	0	1	$\frac{4}{23}$	$-\frac{1}{23}$	$\frac{6}{23}$	$\frac{220}{23}$
Z	0	0	$-\frac{56}{23}$	$\frac{60}{23}$	$\frac{54}{23}$	$\frac{3360}{23}$
λ_3	$\frac{23}{7}$	0	1	$\frac{4}{7}$	$-\frac{1}{7}$	$\frac{40}{7}$
λ_2	$-\frac{4}{7}$	1	0	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{60}{7}$
Z	8	0	0	4	2	160

$$x^* = \begin{pmatrix}4 & 2 & 8 & 0 & 0\end{pmatrix}^T, \quad \lambda^* = \begin{pmatrix}0 & 60/7 & 40/7 & 0 & 0\end{pmatrix}^T, \ z^* = Z^* = 160$$

2.

x_j^*	λ_j^*	Constraint j
4	0	> 0
2	1380/161	$= 0$
8	40/7	$= 0$
0	0	> 0
0	0	> 0

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