Chapter 5

Linear Optimization

The Simplex Method

R. 5. 1.
The simplex method provides a way of moving one basic feasible solution (extreme point) to another basic feasible solution so that the value of the objective function (in standard form) will systematically increase.
The *simplex algorithm* consists of the following steps:

Step 1:
Find an initial basic feasible solution.

Step 2:
Check if it is an optimal solution. If YES, stop.

Step 3:
If NO, exchange a basic variable with a non-basic variable so that a new basic feasible is obtained and the value of the objective function increased.

Step 4:
Repeat Steps 2 and 3 until no further improvement is possible.

R. 5. 2.
Let us assume that a linear optimisation problem has the following form:

$$f(x) = c_1x_1 + c_2x_2 + ... + c_nx_n \rightarrow \max!$$

s.t.

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1,n-m}x_{n-m} + x_{n-m+1} &&= b_1\\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2,n-m}x_{n-m} \qquad + x_{n-m+2} &&= b_2\\
&.\\
&.\\
&.\\
&a_{m1}x_1 + a_{m2}x_2 + ... + a_m x_{n-m} \qquad\qquad + x_n &&= b_m\\
&x_1, x_2, ..., x_n \geq 0; \quad b_1, b_2, ..., b_m \geq 0.
\end{aligned}$$

In matrix notation

$$f = c^T x \rightarrow \max!$$

$$\text{s.t.} \quad Ax_N + x_B = b$$

$$x \geq 0; \; b \geq 0,$$

where

$$x = \left(x_1, ..., x_n\right)^T,$$
$$x_N = \left(x_1, ..., x_{n-m}\right)^T,$$
$$x_B = \left(x_{n-m+1}, ..., x_n\right)^T$$
$$c = \left(x_1, ..., c_n\right)^T,$$
$$b = \left(b_1, ..., b_m\right)^T,$$

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1,n-m} \\ a_{21} & a_{22} & . & . & . & a_{2,n-m} \\ & & & . & & \\ & & & . & & \\ & & & . & & \\ a_{m1} & a_{m2} & & & & a_{m,n-m} \end{pmatrix}.$$

In this case, $x_B = b \geq 0,\ x_N = 0$ is a basic feasible solution of the LOP.

Now we can write the problem as follows:

(5. 1.) $$f = c_N^T x_N + c_B^T x_B \to \max!$$

(5. 2.) s.t. $$x_B + Ax_N = b$$

(5. 3.) $$x_B, x_N \geq 0.$$

From (5. 2.) we have

$$x_B = b - Ax_N.$$

Substituting this into (5. 1.) we have

$$\begin{aligned} f &= c_B^T\left(b - Ax_N\right) + c_N^T x_N \\ &= c_B^T b - c_B^T A x_N + c_N^T x_N \\ &= c_B^T b - \left(c_B^T A + c_N^T\right) x_N. \end{aligned}$$

Let

$$z := c_B^T A - c_N^T = \left(c_B^T a^1 - c_1,\ c_B^T a^2 - c_2, ...,\ c_B^T a^{n-m} - c_{n-m}\right),$$

where a^j is the $j-$th column of $A,\ j = 1, 2, ..., n-m.$

<u>Th. 5. 1.</u> (*Optimality*)

The basic feasible solution $x_B = b \geq 0,\ x_N = 0$ is optimal if

$$z_i = c_B^T a^j - c_j \geq 0 \text{ for } j = 1, 2, ..., n-m.$$

Proof:

The proof will be given later.

<u>R. 5. 2.</u> (*Simplex Tableau*)

The corresponding initial simplex tableau is as follows:

x_B	x_1	. . .	x_{n-m}	x_{n-m+1}	x_{n-m+2}	. . .	x_n	RHS
x_{n-m+1}	a_{11}		$a_{1,n-m}$	1	0	. . .	0	b_1
x_{n-m+2}	a_{21}		$a_{2,n-m}$	0	1	. . .	0	b_2
.	.			.	.	.	.	.
.	.			.	.	.	.	.
.	.			.	.	.	.	.
x_n	a_{m1}		$a_{m,n-m}$	0	0	. . .	1	b_m
z	z_1	. . .	z_{n-m}	0	0	. . .	0	$f = c_B^T b$

We can also write the above initial simplex tableau in matrix form as follows:

x_B	x_N	x_B	RHS
x_B	A	I	b
z	$c_B^T A - c_N^T$	0	$c_B^T b$

<u>Al. 5. 1.</u> (***Simplex Algorithm***)

Step 1: Choose a non-basic variable to become a basic variable.

The non-basic variable x_r is chosen according to the following ("can") rule:

$$z_r = \min_{j \in J} \left\{ z_j = c_B^T a^j - c_j \right\}$$

and $z_r < 0,$ where J is the set of indices of non-basic variables. This x_r will become a basic variable.

If all $z_j \geq 0,\ j \in J,$ then this basic feasible solution is an optimal solution.

Step 2: Find a basic solution to become a non-basic variable.

Let x_B^i be the i^{th} component of x_B. Then, x_B^p is chosen to become a non-basic variable if

$$\varepsilon = \frac{x_B^p}{a_{pr}} = \min \left\{ \frac{x_B^i}{a_{ir}} \mid a_{ir} > 0,\ i = 1, ..., m \right\}.$$

If $a_{ir} < 0$ **for all** $i = 1, 2, ..., m$**, then the objective function is unbounded on the set of feasible solutions.**

Step 3: Pivot on the element a_{pr}. Let the coefficient of the old tableau be denoted by a_{ij}, then the coefficient $a_{ij}^{'}$, of the old tableau is given by

$$a_{pj}^{'} = \frac{a_{pj}}{a_{pr}}, \quad b_{p}^{'} = \frac{b_p}{a_{pr}}$$

$$a_{ij}^{'} = a_{ij} - a_{ir}\frac{a_{pj}}{a_{pr}}, \quad b_{i}^{'} = b_i - a_{ir}\frac{b_p}{a_{pr}}, \quad i \neq p$$

$$z_{j}^{'} = z_j - z_r\frac{a_{pj}}{a_{pr}}, \quad f' = f - z_r\frac{b_p}{a_{pr}}.$$

We can find a solution by repeating Step 1-Step 3.

Ex. 5. 1.

$$f(x_1, x_2) = 21x_1 + 24x_2 -> \max$$

$$\begin{aligned} 3x_1 + x_2 &\leq 33 \\ x_1 + x_2 &\leq 13 \\ 5x_1 + 8x_2 &\leq 80 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

Introducing slack variables $x_3, x_4,$ and x_5, we have

$$\begin{aligned} f(x_1, x_2) = 21x_1 &+ 24x_2 -> \max \\ 3x_1 + x_2 + x_3 &= 33 \\ x_1 + x_2 + x_4 &= 13 \\ 5x_1 + 8x_2 + x_5 &= 80 \\ x_i \geq 0,\ i &= 1, 2, ..., 5 \end{aligned}$$

Obviously,

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 5 & 8 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 33 \\ 13 \\ 80 \end{pmatrix}.$$

We choose $x_B = (x_3, x_4, x_5)^T$, $x_N = (x_1, x_2)^T$ so that

$$x_B = \begin{pmatrix} 33 \\ 13 \\ 80 \end{pmatrix}, \quad x_N = 0$$

is a basic feasible solution and $c_N^T = (21,\ 24)$, $z = c_B^T b - c_N^T = (-21,\ -24)$. The simplex tableaux will now follow:

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_0
x_3	3	1	1	0	0	33
x_4	1	1	0	1	0	13
x_5	5	8	0	0	1	80
z	-21	-24	0	0	0	0
x_3	$\frac{19}{8}$	0	1	0	$-\frac{1}{8}$	23
x_4	$\frac{3}{8}$	0	0	1	$-\frac{1}{8}$	3
x_2	$\frac{5}{8}$	1	0	0	$\frac{1}{8}$	10
z	-6	0	0	0	3	240
x_3	0	0	1	0	$\frac{2}{3}$	4
x_1	1	0	0	$\frac{8}{3}$	$-\frac{1}{3}$	8
x_2	0	1	0	1	$\frac{1}{3}$	5
z	0	0	0	16	1	288

$x^* = \begin{pmatrix} 8 & 5 & 4 & 0 & 0 \end{pmatrix}^T$, $\qquad z^* = 288$

Ex. 5. 2.

$$z = 40x_1 + 30x_2 -> \max$$

$$\begin{aligned} x_1 \ + 3x_2 &\le 16 \\ 2x_1 \ + x_2 &\le 17 \\ 2x_1 + 3x_2 &\le 23 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Solution:

Standard form:

$$z = 40x_1 + 30x_2 -> \max$$

$$\begin{aligned} x_1 + 3x_2 + x_3 \quad\quad &= 16 \\ 2x_1 + x_2 \quad + x_4 \quad &= 17 \\ 2x_1 + 3x_2 \quad\quad + x_5 &= 23 \\ x_i \ge 0,\ i = 1, 2, \ldots, 5 & \end{aligned}$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_0
x_3	1	3	1	0	0	16
x_4	2	1	0	1	0	17
x_5	2	3	0	0	1	23
z	-40	-30	0	0	0	0
x_3	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	0	$\frac{15}{2}$
x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{17}{2}$
x_5	0	2	0	-1	1	6
z	0	-10	0	20	0	340
x_2	0	1	$\frac{2}{5}$	$-\frac{1}{5}$	0	3
x_1	1	0	$-\frac{1}{5}$	$\frac{3}{5}$	0	7
x_5	0	0	$-\frac{4}{5}$	$-\frac{3}{5}$	1	0
z	0	0	4	18	0	370

$$x^* = \begin{pmatrix}7 & 3 & 0 & 0 & 0\end{pmatrix}^T, \qquad z^* = 370$$

The optimal solution is degenerate.

<u>Ex. 5. 3.</u>

$$z = 12x_1 + 18x_2 -> \max$$

$$2x_1 + 3x_2 \le 33$$
$$x_1 + x_2 \le 15$$
$$x_1 + 3x_2 \le 27$$
$$x_1,\ x_2 \ge 0$$

Solution:

Standard form:

$$z = 12x_1 + 18x_2 -> \max$$

$$\begin{aligned} 2x_1 + 3x_2 + x_3 \quad\quad &= 33 \\ x_1 \;\; + x_2 \quad + x_4 \quad &= 15 \\ x_1 + 3x_2 \quad\quad + x_5 &= 27 \end{aligned}$$

$$x_i \geq 0, \; i = 1, 2, ..., 5.$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_0
x_3	2	3	1	0	0	33
x_4	1	1	0	1	0	15
x_5	1	3	0	0	1	27
z	-12	-18	0	0	0	0
x_3	1	0	1	0	-1	6
x_4	$\frac{2}{3}$	0	0	1	$-\frac{1}{3}$	6
x_2	$\frac{1}{3}$	1	0	0	$\frac{1}{3}$	9
z	-6	0	0	0	6	162
x_1	1	0	1	0	-1	6
x_4	0	0	$-\frac{2}{3}$	1	$\frac{1}{3}$	2
x_2	0	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	7
z	0	0	6	0	**0**	198
x_1	1	0	-1	3	0	12
x_5	0	0	-2	3	1	6
x_2	0	1	1	-2	0	3
z	0	0	6	**0**	0	198

The problem has the multiple solution:

$$x^* = \alpha\begin{pmatrix}6 & 7 & 0 & 2 & 0\end{pmatrix}^T + (1-\alpha)\begin{pmatrix}12 & 3 & 0 & 0 & 6\end{pmatrix}^T, \quad 0 \le \alpha \le 1$$

$$z^* = 198.$$

Ex. 5. 4.

$$z = 18x_1 + 6x_2 -> \max$$

$$\begin{aligned} -4x_1 + 3x_2 &\le 6 \\ -x_1 + 3x_2 &\le 15 \\ x_1 - 4x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution:

Standard form:

$$z = 18x_1 + 6x_2 -> \max$$

$$\begin{aligned} -4x_1 + 3x_2 + x_3 &= 6 \\ -x_1 + 3x_2 + x_4 &= 15 \\ x_1 - 4x_2 + x_5 &= 4 \\ x_i \ge 0,\ i &= 1,2,\dots,5. \end{aligned}$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_0
x_3	-4	3	1	0	0	6
x_4	-1	3	0	1	0	15
x_5	1	-4	0	0	1	4
z	-18	-6	0	0	0	0
x_3	0	**-13**	1	0	4	22
x_4	0	**-1**	0	1	1	19
x_1	1	**-4**	0	0	1	4
z	0	-78	0	0	18	72

The objective function is not bounded on the set of feasible solutions. Therefore, there is no optimal solution.

<u>R. 5. 3.</u> (*Two-Phase Method*)
Consider the standard linear optimisation problem:

$$f = c^T x \to \max!$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

and we assume that $b \geq 0$. We construct an „*artificial*“ linear optimisation problem

$$\tilde{f} = -\sum_{i=1}^{m} y_i \to \max!$$
$$\text{s.t.} \quad Ax + y = b$$
$$x, y \geq 0$$

where $y = (y_1, ..., y_m)^T$ denotes the vector of „*artificial*“ variables. Obviously, if x is a feasible solution to the original problem, $y = 0$ is the optimal solution to the „*artificial*“ linear optimisation problem and $\tilde{f} = 0$, otherwise $\tilde{f} < 0$, indicating that there is no feasible solution to the original problem.
Based on this observation we construct the following two-phase algorithm:

Phase 1:
Solve the “*artificial*” linear optimisation problem by the simplex method to find x and y.
If $\tilde{f} = 0$, then x is a feasible solution to the original problem. And then go to Phase 2.
Otherwise, there is no feasible solution to the optimisation problem. Stop.

Phase 2:
Starting from the feasible basic solution found in Phase 1 solve the original linear optimisation problem by the simplex method. This can start from the final tableau in Phase 1 by dropping the artificial variable y.

<u>Ex. 5. 5.</u>

$$z = 12x_1 + 18x_2 -> \max$$
$$x_1 + 3x_2 \leq 12$$
$$x_1 + 2x_2 \leq 10$$
$$2x_1 + 5x_2 \geq 30$$
$$x_1,\ x_2 \geq 0$$

Solution:

<u>Standard form</u>:

$$
\begin{aligned}
& z = 12x_1 + 18x_2 -> \max \\
& x_1 + 3x_2 + x_3 = 12 \\
& x_1 + 2x_2 + x_4 = 10 \\
& 2x_1 + 5x_2 - x_5 = 30 \\
& x_i \ge 0,\ i = 1, 2, ..., 5.
\end{aligned}
$$

$$
\begin{aligned}
& \tilde{z} = -x_6 > \max \\
& x_1 + 3x_2 + x_3 = 12 \\
& x_1 + 2x_2 + x_4 = 10 \\
& 2x_1 + 5x_2 - x_5 + x_6 = 30 \\
& x_i \ge 0,\ i = 1, 2, ..., 6.
\end{aligned}
$$

$$x_6 = 30 - 2x_1 - 5x_2 + x_5$$

$$\tilde{z} = -30 + 2x_1 + 5x_2 - x_5 \to \max!$$

Simplex Tableau (Phase 1)

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_0
x_3	1	3	1	0	0	0	12
x_4	1	2	0	1	0	0	10
x_6	2	5	0	0	-1	1	30
z	-12	-18	0	0	0	0	(0)
$\tilde{z}$	**-2**	-5	0	0	1	0	-30
x_2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	0	4
x_4	$\frac{1}{3}$	0	$-\frac{2}{3}$	1	0	0	2
x_6	$\frac{1}{3}$	0	$-\frac{5}{3}$	0	-1	1	10
z	-6	0	6	0	0	0	(72)
$\tilde{z}$	$-\frac{1}{3}$	0	$\frac{5}{3}$	0	1	0	-10
x_2	0	1	1	-1	0	0	2
x_1	1	0	-2	3	0	0	6
x_6	0	0	-1	-1	-1	1	8
z	0	0	-6	18	0	0	(108)
$\tilde{z}$	0	0	1	1	1	0	-8

Because of $\max \tilde{z} \neq 0$ there is no (basic) feasible solution. Therefore, there is also no optimal solution

Ex. 5. 6.

$$
\begin{aligned}
z = 20x_1 + 40x_2 -> \min \\
6x_1 + x_2 \geq 18 \\
x_1 + 4x_2 \geq 12 \\
2x_1 + x_2 \geq 10 \\
x_1,\ x_2 \geq 0
\end{aligned}
$$

Solution:

Standard form:

$$
-20x_1 - 40x_2 -> \max
$$

$$
\begin{aligned}
6x_1 + x_2 - x_3 &= 18 \\
x_1 + 4x_2 - x_4 &= 12 \\
2x_1 + x_2 - x_5 &= 10 \\
x_i \geq 0,\ i = 1, 2, \ldots, 5.
\end{aligned}
$$

$$
\tilde{z} = -(x_6 + x_7 + x_8) -> \max
$$

$$
\begin{aligned}
6x_1 + x_2 - x_3 + x_6 &= 18 \\
x_1 + 4x_2 - x_4 + x_7 &= 12 \\
2x_1 + x_2 - x_5 + x_8 &= 10 \\
x_i \geq 0,\ i = 1, 2, \ldots, 8.
\end{aligned}
$$

$$
\begin{aligned}
x_6 &= 18 - 6x_1 - x_2 + x_3 \\
x_7 &= 12 - x_1 - 4x_2 + x_4 \\
x_8 &= 10 - 2x_1 - x_2 + x_5 \\
x_6 + x_7 + x_8 &= 40 - 9x_1 - 6x_2 + x_3 + x_4 + x_5
\end{aligned}
$$

$$
\tilde{z} = -40 + 9x_1 + 6x_2 - x_3 - x_4 - x_5 -> \max
$$

Simplex Tableau (Phase 1)

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_0
x_6	**6**	1	-1	0	0	1	0	0	18
x_7	1	4	0	-1	0	0	1	0	12
x_8	2	1	0	0	-1	0	0	1	10
z	20	40	0	0	0	0	0	0	(0)
$\tilde{z}$	**-9**	-6	1	1	1	0	0	0	-40
x_1	1	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$	0	0	3
x_7	0	$\frac{23}{6}$	$\frac{1}{6}$	-1	0	$-\frac{1}{6}$	1	0	9
x_8	0	$\frac{2}{3}$	$\frac{1}{3}$	0	-1	$-\frac{1}{3}$	0	1	4
z	0	$\frac{110}{3}$	$\frac{10}{3}$	0	0	$-\frac{10}{3}$	0	0	(-60)
$\tilde{z}$	0	$-\frac{9}{2}$	$-\frac{1}{2}$	1	1	$\frac{3}{2}$	0	0	-13
x_1	1	0	$-\frac{4}{23}$	$\frac{1}{23}$	0	$\frac{4}{23}$	$-\frac{1}{23}$	0	$\frac{60}{23}$
x_2	0	1	$\frac{1}{23}$	$-\frac{6}{23}$	0	$-\frac{1}{23}$	$\frac{6}{23}$	0	$\frac{54}{23}$
x_8	0	0	$\frac{7}{23}$	$\frac{4}{23}$	-1	$-\frac{7}{23}$	$-\frac{4}{23}$	1	$\frac{56}{23}$
z	0	0	$\frac{40}{23}$	$\frac{220}{23}$	0	$-\frac{40}{23}$	$-\frac{220}{23}$	0	$\left(-\frac{3360}{23}\right)$
$\tilde{z}$	0	0	$-\frac{7}{23}$	$-\frac{4}{23}$	1	$\frac{30}{23}$	$\frac{27}{23}$	0	$-\frac{56}{23}$
x_1	1	0	0	$\frac{1}{7}$	$-\frac{4}{7}$	0	$-\frac{1}{7}$	$\frac{4}{7}$	4
x_2	0	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	0	$\frac{2}{7}$	$-\frac{1}{7}$	2
x_3	0	0	1	$\frac{4}{7}$	$-\frac{23}{7}$	-1	$-\frac{4}{7}$	$\frac{23}{7}$	8
z	0	0	0	$\frac{60}{7}$	$\frac{40}{7}$	*	*	*	-160
$\tilde{z}$	0	0	0	0	0	1	1	1	0

$$x^* = \begin{pmatrix} 4 & 2 & 8 & 0 & 0 \end{pmatrix}^T, \qquad z^* = 160$$

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