Chapter 4

Linear Optimization

Graphical Method

A linear optimization problem can be solved graphically if it contains at most 3 variables. We describe the most important steps of such a solution method in the case of two variables:

- Plot the (linear) constraints.
- Determine the region of feasible solutions.
- Find the coordinates of the vertices of the set of feasible solutions.
- Calculate the values of the objective function at each vertex.
- Select the vertex (vertices) yielding the highest (maximum) or the lowest (minimum) of the objective function. Those coordinates determine the optimal solution (s).

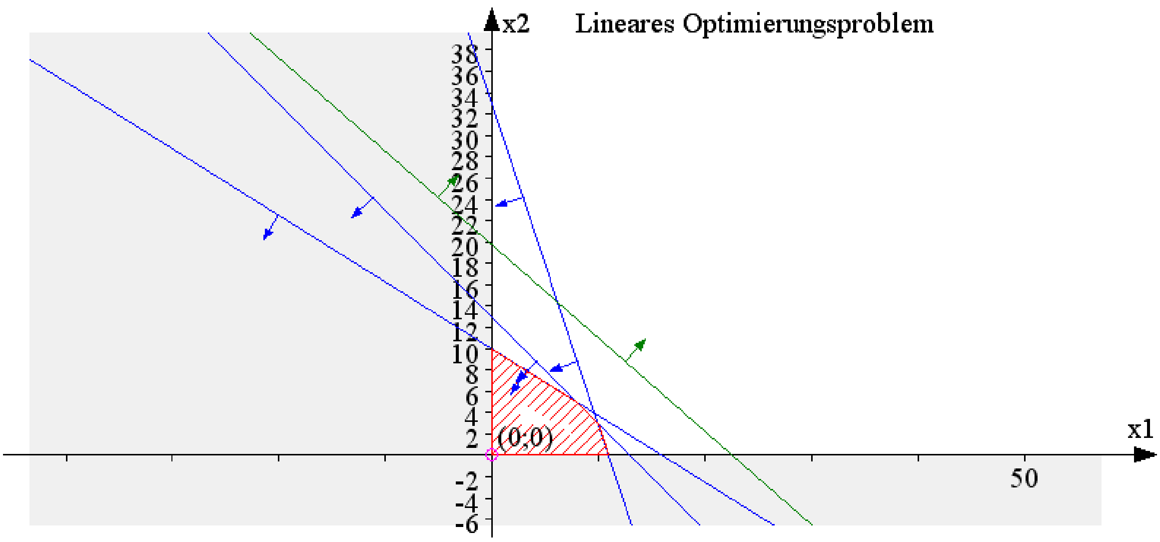
The following examples represent the most important cases occurring in solving a linear optimization problem:

Case 1: (A unique optimum)

$$z = 21x_1 + 24x_2 -> \max$$

$$\begin{aligned} 3x_1 + x_2 &\le 33 \\ x_1 + x_2 &\le 13 \\ 5x_1 + 8x_2 &\le 80 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Solution:



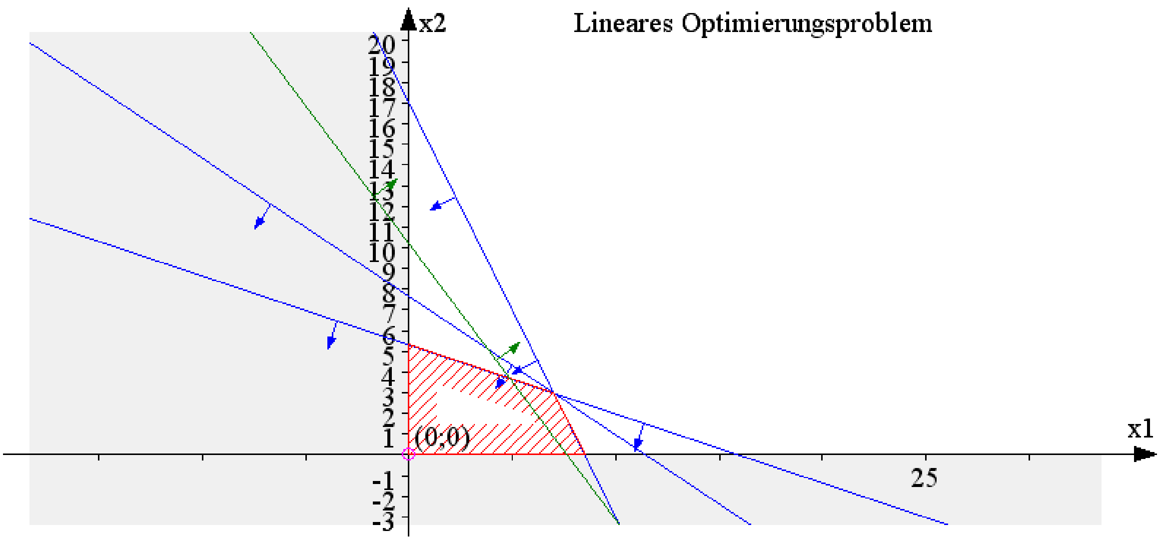
$$x^* = (8,\ 5)^T, \qquad z^* = 288$$

Case 2: (A degenerate optimum)

$$z = 40x_1 + 30x_2 -> \max$$

$$\begin{aligned} x_1 \quad + 3x_2 &\le 16 \\ 2x_1 \quad + x_2 &\le 17 \\ 2x_1 + 3\,x_2 &\le 23 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Solution:



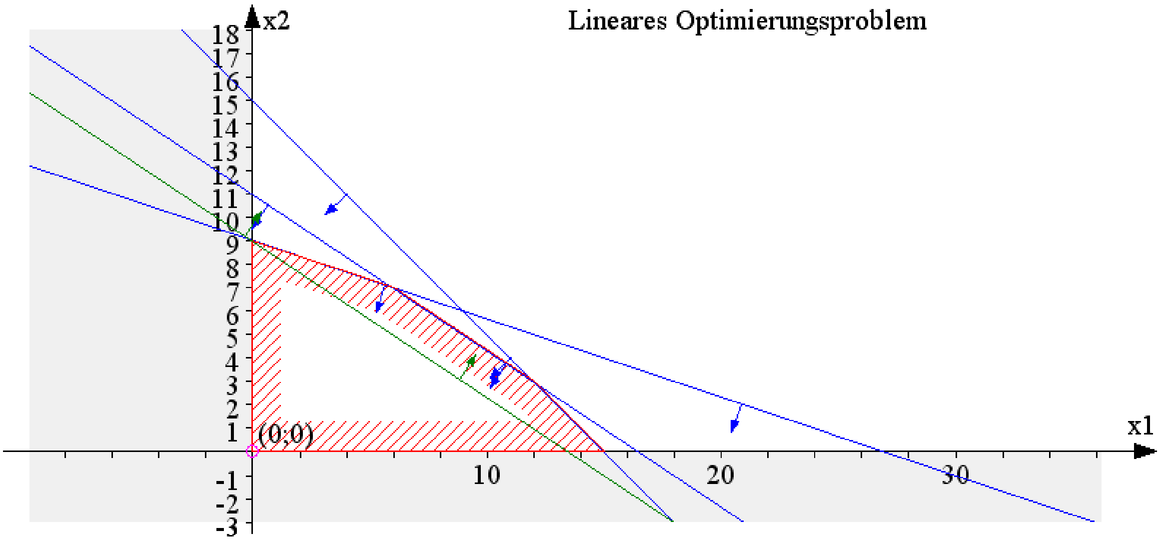
$$x^* = (7,\ 3)^T, \qquad z^* = 370$$

Case 3: (Multiple optima)

$$z = 12x_1 + 18x_2 -> \max$$

$$\begin{aligned} 2x_1 + 3x_2 &\le 33 \\ x_1 \quad + x_2 &\le 15 \\ x_1 + 3x_2 &\le 27 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Solution:



Multiple optimal solutions:

$$x^* = \alpha x^{*1} + (1-\alpha)x^{*2}, \quad 0 \le \alpha \le 1, \qquad z^* = 198,$$

with

$$x^{*1} = \begin{pmatrix} 6 & 7 \end{pmatrix}^T, \quad x^{*2} = \begin{pmatrix} 12 & 3 \end{pmatrix}^T$$

Case 4: (Objective function unbounded over the set of feasible solutions)

$$z = 18x_1 + 6x_2 -> \max$$

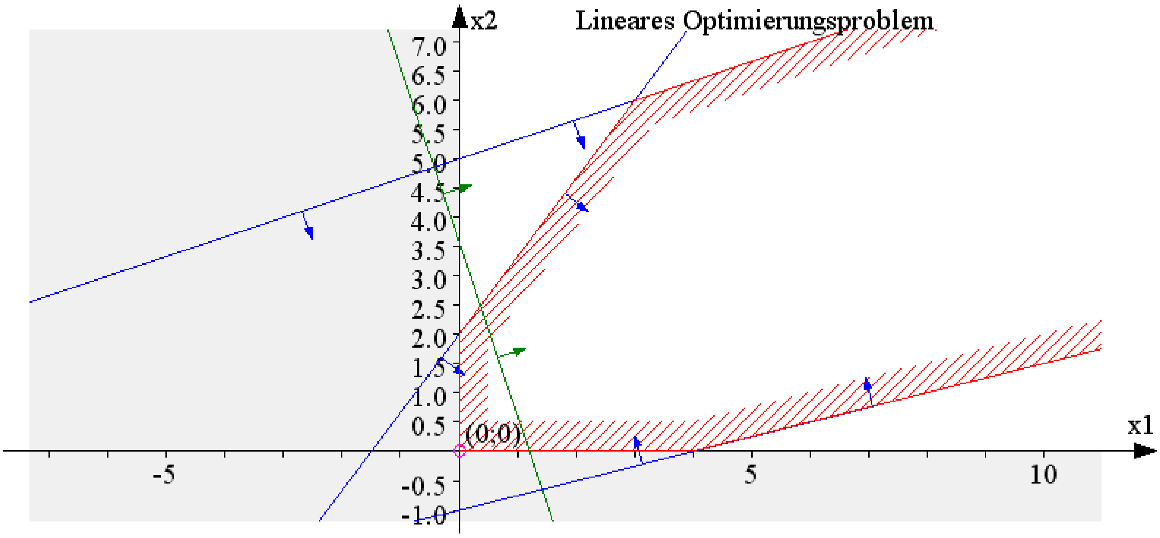
$$-4x_1 \ + 3x_2 \le 6$$

$$-x_1 \ + 3x_2 \le 15$$

$$x_1 - 4\, x_2 \le 4$$

$$x_1,\, x_2 \ge 0$$

Solution:

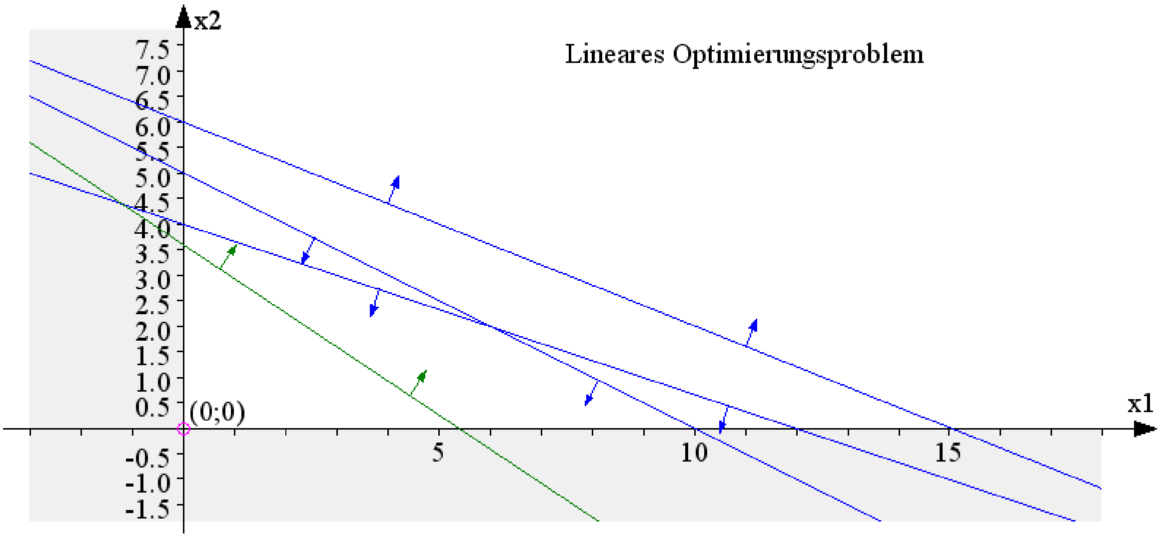


The objective function is not bounded on the set of feasible solutions. Therefore, there is no optimal solution.

Case 5: (No feasible solution)

$$z = 12x_1 + 18x_2 -> \max$$
$$x_1 + 3x_2 \leq 12$$
$$x_1 + 2x_2 \leq 10$$
$$2x_1 + 5x_2 \geq 30$$
$$x_1,\ x_2 \geq 0$$

Solution:



There set of feasible solutions is empty. Therefore, there is no optimal solution.

Case 6: (A minimization problem)

$$z = 20x_1 + 40x_2 -> \min$$

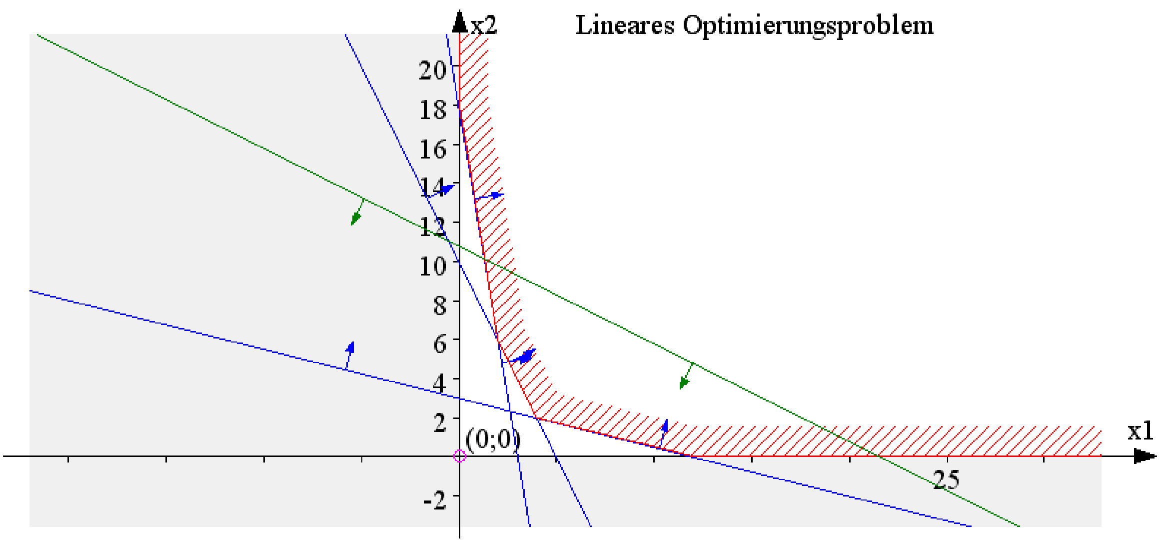
$$6x_1 \quad + x_2 \geq 18$$

$$x_1 + 4x_2 \geq 12$$

$$2\,x_1 + \quad x_2 \geq 10$$

$$x_1,\ x_2 \geq 0$$

Solution:



$$x^* = (4,\ 2)^T, \qquad z^* = 160$$

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