Chapter 3

Linear Optimization Some Fundamental Definitions and Theorems

D. 3. 1. (***Feasible Solution***)
Consider a linear optimisation problem in the standard form:

$$\begin{aligned} z = c^T x &\to \max! \\ Ax &= b \\ x &\geq 0\,. \end{aligned}$$

A vector x is called a *feasible solution* of the linear optimisation problem if it satisfies the constraints and the non-negativity condition.

Ex. 3. 1.
Consider the problem

$$\begin{aligned} &\pi(x_1, x_2) = 50x_1 + 20x_2 \to \max \\ &5x_1 + 3x_2 + x_3 = 45 \\ &2x_1 + 3x_2 + x_4 = 36 \\ &x_1 + x_5 = 6 \\ &x_i \geq 0,\ i = 1, 2, \ldots, 5. \end{aligned}$$

Show that the vector $x = (5,\ 4,\ 8,\ 14,\ 1)^T$ is a feasible solution of the above problem.

Solution:
We have

$$\begin{aligned} &5 \cdot 5 + 3 \cdot 4 + 8 = 45\,, \\ &2 \cdot 5 + 3 \cdot 4 + 14 = 36\,, \\ &5 + 1 = 6\,, \\ &x = (5,\ 4,\ 8,\ 14,\ 1)^T \geq 0 \end{aligned}$$

D. 3. 2. (***Set of Feasible Solution***)
Define

$$M := \left\{ x \in R^n \mid Ax = b,\ x \geq 0 \right\}.$$

The set M is called the *set of feasible solutions*.

<u>D. 3. 3.</u> (*Set of Optimal Solutions*)
The set

$$M_{opt} := \left\{ x \in R^n \mid c^T x^* \geq c^T x, \ \forall x \in M \right\}$$

is called the *set of optimal solutions*.

<u>D. 3. 4.</u> (*Basic Solution, Basic and Non-Basic Variables*)
Consider the system of equations

$$Ax = b$$

with $n > m$ .Let B be any non singular $m \cdot m$ submatrix which is formed by appropriate columns of the matrix A . Further let x be a solution to $Ax = b$. Suppose all its $n - m$ components which are not associated with columns of B are zero.
Then, x is called a *basic solution with respect to the basis* B .
The components of x which are associated with columns of B are called *basic variables*, and the remaining components are called *non-basic variables*.

<u>D. 3. 5.</u> (*Degenerate and Non-Degenerate Basic Solutions*)
If one or more of the basic variables in a basic solution are zero, then the basic solution is called a *degenerate basic solution*. Otherwise, this solution is called a *non-degenerate basic solution*.

<u>D. 3. 6.</u> (*Basic Feasible Solution*)
Let x be a basic solution of $Ax = b$. If $x \geq 0$, then it is called a *basic feasible solution* of the linear optimisation problem.

<u>Ex. 3. 2.</u>

$$\begin{aligned} 5x_1 + 3x_2 + x_3 \qquad &= 45 \\ 2x_1 + 3x_2 \qquad + x_4 \quad &= 36 \\ x_1 \qquad\qquad\qquad + x_5 &= 6 . \end{aligned}$$

Let

$$A := \begin{pmatrix} 5 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$x = (6,\ 0,\ 15,\ 24,\ 0)^T$ is a basic solution with the corresponding basis matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

<u>Ex. 3. 3.</u>

$$\begin{aligned} 3x_1 + 3x_2 \quad + x_3 + x_4 \qquad &= 2 \\ x_1 + 3x_2 + 2x_3 + x_4 + x_5 &= 2 . \end{aligned}$$

Let

$$A := \begin{pmatrix} 3 & 3 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 \end{pmatrix}.$$

We see that $x = (0,\ 0,\ 0,\ 2,\ 0)^T$ is a degenerate basic solution with the corresponding basis matrix

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

R. 3. 1.

Maximum possible number of basic solutions for an $m \cdot n$ matrix A with rank m is determined by the number of ways of choosing m columns out of n:

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}.$$

Th. 3. 1.

Consider a linear optimisation problem. We have:

1. If there is a feasible solution, then there exists a basic feasible solution.
2. If there is an optimal solution, then there exits an optimal basic solution.

Proof:

1.

Let the columns of the matrix A be denoted by $a^1, a^2, ..., a^n$. Furthermore, let $x = (x_1, x_2, ..., x_n)^T$ be a feasible solution. Then

$$x_1 a^1 + x_2 a^2 + ... + x_n a^n = b, \quad x_i \geq 0, \quad i = 1,2,...,n.$$

Let r bet he number of components of x which are positive. For convenience, we can assume that

$$x_i > 0, \quad i = 1,2,...,r,$$

and

$$x_i = 0, \quad i = r+1...,n.$$

This implies that

$$x_1 a^1 + x_2 a^2 + ... + x_r a^r = b.$$

There are two cases to be considered:

Case 1: The vectors $a^1, a^2, ... a^r$ are linearly independent, and $r \leq m$. (Note that if $r > m$, then $a^1, a^2, ... a^r$ must be linearly independent.)

a) If $r = m$, then the solution is basic.

b) If $r < m$, then we have a degenerate case.

Case 2: The vectors $a^1, a^2, ... a^r$ are linearly dependent. In this case, there exist $y_1, y_2, ..., y_r$, and at least one of them is positive, such that

$$y_1 a^1 + y_2 a^2 + ... + y_r a^r = 0.$$

Multiplying the above equation by α, we obtain:

$$\alpha y_1 a^1 + \alpha y_2 a^2 + ... + \alpha y_r a^r = 0.$$

Subtracting the last equation from the one before,

$$(x_1 - \alpha y_1) a^1 + (x_2 - \alpha y_2) a^2 + ... + (x_r - \alpha y_r) a^r = b.$$

Define $y := (y_1, y_2, ..., y_r, 0, ..., 0)^T$. Then,

$$x - \alpha y = (x_1 - \alpha y_1, ..., x_r - \alpha y_r, 0, ..., 0)^T$$

is a solution of $Ax = b$, where $x = (x_1, ..., x_r, 0, ..., 0)^T$. If $\alpha = 0$, then $x - \alpha y$ reduces to x, and hence is a feasible solution. If $\alpha > 0$, we have:

$$\begin{cases} x_i - \alpha y_i & \text{decreases} & \text{if } y_i > 0 \\ x_i - \alpha y_i & \text{remains constant} & \text{if } y_i = 0 \\ x_i - \alpha y_i & \text{increases} & \text{if } y_i < 0 \end{cases}.$$

Since there exists at least one $y_i > 0$, it is easy to see that $\alpha > 0$ cannot increase indefinitely. In fact the maximum value that it can take is

$$\alpha_m = \min\{x_i / y_i \mid \text{those } i \text{ such that } y_i > 0\}.$$

By taking $\alpha = \alpha_m$, the corresponding $x - \alpha y$ is a feasible solution with the number of positive components $\leq (r-1)$.
We can repeat this process until the remaining columns are linearly independent, and so we have a basic feasible solution.

2.
Let $x^* = (x_1^*, ..., x_n^*)$ be an optimal (feasible) solution. Thus

$$x_1^* a^1 + x_2^* a^2 + ... + x_n^* a^n = b, \quad x^* \geq 0,$$

and

$$c^T x^* \geq c^T x$$

for all x satisfying $Ax = b$ and $x \geq 0$. We assume that the first r components of x^* are positive, and the rest of the components are zero, i.e.,

$$x_i^* > 0, \text{ for } i = 1, 2, ..., r; \;\; x_i^* = 0, \text{ for } i = r+1, ..., n.$$

Hence, we have

(3.1.) $$x_1^* a^1 + x_2^* a^2 + ... + x_r^* a^r = b.$$

There are two cases to be considered:

Case 1:

The vectors $a^1, a^2, ..., a^r$ are linearly independent. In this case, $r \leq m$.

(a) If $r = m$, then the solution is basic.
(b) If $r < m$, then we have a degenerate case.

Case 2:

The vectors $a^1, a^2, ..., a^r$ are linearly dependent. Then, there exist $y_1, ..., y_r$ and at least one of them is positive, such that

$$y_1 a^1 + y_2 a^2 + ... + y_r a^r = 0.$$

Multiplying by α, we obtain

$$\alpha y_1 a^1 + \alpha y_2 a^2 + ... + \alpha y_r a^r = 0.$$

Subtracting the last equation from (2. 1.),

$$\left(x_1^* - \alpha y_1\right) a^1 + \left(x_2^* - \alpha y_2\right) a^2 + ... + \left(x_r^* - \alpha y_r\right) a^r = b.$$

Define $y := \left(y_1, y_2, ..., y_r, 0, ..., 0\right)^T$. Then,

$$x^* - \alpha y = \left(x_1^* - \alpha y_1, ..., x_r^* - \alpha y_r, 0, ..., 0\right)^T$$

is a solution of $Ax = b$, where $x = (x_1^*, ..., x_r^*, 0, ..., 0)^T$. If $\alpha = 0$, then $x^* - \alpha y$ reduces to x^*, and hence is a feasible solution. If $\alpha > 0$, we have:

$$\begin{cases} x_i^* - \alpha y_i & \text{decreases} \quad \text{if } y_i > 0 \\ x_i^* - \alpha y_i & \text{remains constant} \quad \text{if } y_i = 0 \\ x_i^* - \alpha y_i & \text{increases} \quad \text{if } y_i < 0 \end{cases}.$$

Since there exists at least one $y_i > 0$, it is easy to see that $\alpha > 0$ cannot increase indefinitely. In fact the maximum value that it can take is

$$\alpha_m = \min\left\{x_i^* / y_i \mid \text{those } i \text{ such that } y_i > 0\right\}.$$

Thus, $x^* - \alpha y$ is a feasible solution for all $\alpha \in [0,\ \alpha_m]$. Let

$$\alpha_l = \min\left\{x_i^* / y_i \mid \text{those } i \text{ such that } y_i < 0\right\}.$$

Clearly, if there exist i such that $y_i < 0,$ then $\alpha_l < 0.$ If there does not exist i such that $y_i < 0,$ we let $\alpha_l = 0.$ We can easily prove that $x^* - \alpha y$ is also a feasible solution for all $\alpha \in [0,\ \alpha_l]$.

Our remaining task is to show that $x^* - \alpha y,\ \alpha_l \le \alpha \le \alpha_m$, is optimal. For this, we note that

$$c^T\left(x^* - \alpha y\right) = c^T x^* - \alpha c^T.$$

Now we prove $c^T y$ must be 0. Let us look at the following two situations.

(i) $c^T y > 0.$ In this case, we may choose $\alpha > 0$ such that

$$c^T\left(x^* - \alpha y\right) > c^T x^*.$$

This implies that $c^T x^*$ is not the maximum, and hence x^* is not an optimal solution. This is a contradiction. Thus, we cannot have $c^T y > 0$.

(ii) $c^T y < 0.$ In this case, we may choose $\alpha < 0$ such that

$$c^T\left(x^* - \alpha y\right) > c^T x^*.$$

This also implies that $c^T x^*$ is not the maximum, and hence x^* is not an optimal solution. This is again a contradiction. This, we also cannot have $c^T y < 0$.

Combining these two situations together, we see that if x^* is an optimal situation, then we have

$$c^T y = 0.$$

Therefore,

$$c^T\left(x^* - \alpha y\right) = c^T x^* - \alpha c^T y = c^T x^*,$$

and hence, $x^* - \alpha_{\max} y$ is an optimal solution with positive components x^* whose number is less than $r-1$.
We can repeat this process until the remaining columns of A are linearly independent, and so we have an optimal basic solution.
This completes the proof.

Th. 3. 2.
The set of feasible solutions

$$M := \left\{ x \in R^n \mid Ax = b,\ x \geq 0 \right\}$$

is convex.

Proof:
Let $x^1, x^2 \in M$, and $\alpha \in [0,\ 1]$ be arbitrary. Then,

$$Ax^1 = b \ \Rightarrow\ A\alpha x^1 = \alpha b$$

$$x^1 \geq 0 \ \Rightarrow\ \alpha x^1 \geq 0$$

and

$$Ax^2 = b \ \Rightarrow\ A\left(1-\alpha\right)x^2 = (1-\alpha)b$$

$$x^2 \geq 0 \ \Rightarrow\ (1\text{-}\alpha)x^2 \geq 0.$$

Thus,

$$A(\alpha x^1 + (1-\alpha)x^2) = A\alpha x^1 + A(1-\alpha)x^2 = \alpha b + (1-\alpha)b = b$$

$$\alpha x^1 + (1-\alpha)x^2 \geq 0.$$

Therefore,

$$\alpha x^1 + (1-\alpha)x^2 \in M,$$

and hence M is convex.

Th. 3. 3.

$$\langle x \text{ is an extreme point of } M \rangle \quad \Leftrightarrow \quad \langle x \text{ is a basic feasible solution} \rangle$$

Proof:
$(\Leftarrow)$:
Let $x = \left(x_1, ..., x_m, 0, ..., 0\right)^T$ be a basic feasible solution, where $x_i,\ i = 1, ..., m,$ are basic variables, and $x_i = 0,\ i = m+1, ..., n,$ are nonbasic variables. Then,

(3. 2.) $$x_1a^1 + ... + x_ma^m = b$$

and $a^1,...,a^m$ are linearly independent.

Assume that x is not an extreme point. Then, there exit $y, z \in M$ with

(3. 3.) $$y \neq z,$$

and $\alpha \in \left]0,\ 1\right[$ such that

(3. 4.) $$x = \alpha y + (1-\alpha)z .$$

Note that all components of the vectors $x, y,$ and z are nonnegative, and $\alpha \in \left]0,\ 1\right[$. Since $x_j = 0,\ i = m+1,...,n,$ we have that

(3. 5.) $$y_j = 0,\ i = m+1,...,n$$

and

(3. 6.) $$z_j = 0,\ i = m+1,...,n.$$

Let

$$y = \left(y_1,...,y_m,0,...,0\right)^T$$

and

$$z = \left(z_1,...,z_m,0,...,0\right)^T .$$

Recall that y is chosen such that

$$y \in M.$$

Thus,

(3. 7.) $$y_1a^1 + ... + y_ma^m = b .$$

Subtracting (3. 7.) from (3. 2.), we get

$$(x_1 - y_1)a^1 + ... + (x_m - y_m)a^m = 0.$$

Since $a^1,...,a^m$ are linearly independent, it follows that

$$x_i = y_i,\quad \forall i = 1,...,m.$$

This, in turn, implies that

$$x = y.$$

Repeating this we also have

$$x = z.$$

Thus

$$x = y = z.$$

This is a contradiction, and hence x is an extreme point.

$(\Rightarrow)$:

Let x be an extreme point of M . We assume that the first k components of x are non-zero. Then, we have

(3. 8.) $$x_1 a^1 + x_1 a^2 + ... + y_k a^k = b$$

with

(3. 9.) $$x_i > 0, \;\; i = 1,...,k; \; x_i = 0, \;\; i = k+1,...,n.$$

If we can show that $a^1,...,a^k$ are linearly independent, then x is a basic feasible solution. We suppose $a^1,...,a^k$ are not linearly independent. Then, there exist $y_1,...y_k$, not all zero, such that

$$y_1 a^1 + y_1 a^2 + ... + x_k a^k = 0.$$

Thus, for any $\alpha > 0$, we have

(3. 10.) $$\alpha y_1 a^1 + \alpha y_1 a^2 + ... + \alpha x_k a^k = 0.$$

Define

$$y = \left(y_1,...,y_k,0,...,0 \right)^T .$$

Since $x_i > 0, \;\; i = 1,...,k,$ we can find an $\alpha > 0$ such that

$$x + \alpha y \geq 0, \quad x - \alpha y \geq 0.$$

Thus, we have

$$A(x + \alpha y) = (x_1 + \alpha y_1)a^1 + (x_2 + \alpha y_2)a^2 + ... + (x_k + \alpha y_k)a^k = b$$

and

$$A(x - \alpha y) = (x_1 - \alpha y_1)a^1 + (x_2 - \alpha y_2)a^2 + ... + (x_k - \alpha y_k)a^k = b.$$

Thus,

$$x + \alpha y,\ x - \alpha y \in M$$

and hence

$$x = \frac{1}{2}(x + \alpha y) + \frac{1}{2}(x - \alpha y).$$

Since

$$\frac{1}{2}(x + \alpha y) \neq \frac{1}{2}(x - \alpha y),$$

x is not an extreme point. This is a contradiction. Thus, $a^1,...,a^k$ are linearly independent, and hence x is a basic feasible solution if $k = m$, and it is a degenerate basic feasible solution if $k < m$.
This completes the proof.

C. 3. 1.
If the constraint set M is non-empty, then, it contains at least one extreme point.

Proof:
Since M is non-empty, then there exists a basic feasible solution. Thus, by theorem T. 3. 1., there exists a basic feasible solution. However, from theorem T. 3. 3., basic feasible solutions are extreme points. Therefore, the set M contains at least one extreme point.

C. 3. 2.
Consider the linear optimisation problem in standard form. If it has a finite optimal solution, then there exists a finite optimal solution which occurs at an extreme point of the constraint set M .

Proof:
The conclusion follows from theorems T. 3. 1. and T. 3. 3.

C. 3. 3.
There are only a finite number of extreme points in the constraint set M .

Proof:
The maximum possible number of basic solutions is

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} < \infty.$$

Since the number of basic feasible solutions cannot be larger than that of basic solutions, the conclusion follows readily.

Th. 3. 4.
If the constraint set M is bounded, then it is a convex polytope.

Th. 3. 5.
If the constraint set M is bounded, then at least one optimal feasible solution is an extreme point of M .

Proof:
By theorem T. 3. 4., the constraint set M is a convex polytope. Let $x^1,...,x^k$ be extreme points of M . Then, any point $x = \left(x_1^0,...,x_n^0\right)^T \in M$ can be expressed as a convex combination of these points as follows:

$$x^0 = \alpha_1 x^1 + ... + \alpha_k x^k$$

where

$$\alpha_i \geq 0,\ i = 1,...,k;\quad \sum_{i=1}^{k} \alpha_i = 1.$$

This, in turns, implies that

(3. 11.)
$$c^T x^0 = \sum_{i=1}^{k} \alpha_i c^T x^i .$$

Let

$$z_{\max} := \max\left\{c^T x^i \mid i = 1,...,k \right\}.$$

Then it follows from (2. 11.) that

$$c^T x^0 = \sum_{i=1}^{k} \alpha_i x^i \leq \sum_{i=1}^{k} \alpha_i \max\left\{c^T x^i \mid i = 1,...,k\right\}$$

$$= \sum_{i=1}^{k} \alpha_i z_{\max} = z_{\max} \cdot \sum_{i=1}^{k} \alpha_i = z_{\max}.$$

This is true for any $x^0 \in M$. Thus, $z_{\max}$ is the maximum value over M , and this maximum value is occurred at one of the extreme points $x^i,\ i = 1,...,n,$ of M .
This completes the proof.

Th. 3. 6. (***Fundamental Theorem of Linear Optimisation***)
A linear programming problem satisfies exactly one of the following:

1. The linear optimisation has no feasible solution.
2. The linear optimisation problem has an optimal solution.
3. The objective function is unbounded over the set of feasible solutions
4. If there is a feasible solution then there is a basic feasible solution and if there is an optimal solution then there is a basic feasible solution which is optimal (*Chvátal*).

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