Chapter 2

Linear Optimization (Introduction)

Solutions

2. 1.
Denote by

$x_i, \quad i = 1,\ 2,\ 3:$ Amount of the product produced by the activity i,

The costs to produce one unit of $x_i,\ i = 1,2,3$ are:

$$\begin{aligned} 1:&\quad 14\cdot 25 + 34\cdot 35 + 12\cdot 30 = 1900 \\ 2:&\quad 14\cdot 32 + 34\cdot 25 + 12\cdot 36 = 1730 \\ 3:&\quad 14\cdot 43 + 34\cdot 20 + 12\cdot 40 = 1762 \end{aligned}$$

The profits to the firm of producing one unit of $x_i,\ i = 1,2,3$ are given by:

Profit = price – cost.

Therefore, we have the following profit figures:

$$\begin{aligned} 1:&\quad 2610 - 1900 = 710 \\ 2:&\quad 2400 - 1730 = 670 \\ 3:&\quad 2250 - 1762 = 488 \end{aligned}$$

The Model:

$$\pi(x_1, x_2, x_3) = 710x_1 + 670x_2 + 488x_3 -> \text{max!}$$

$$\begin{aligned} 25x_1 + 32x_2 + 43x_3 &\le 850 \\ 35x_1 + 25x_2 + 20x_3 &\le 800 \\ 30x_1 + 36x_2 + 40x_3 &\le 980 \\ x_1,\ x_2, x_3 &\ge 0 \end{aligned}$$

2. 2.
The decision variables here are the number of 1-bedroom, 2-bedroom, and 3-bedroom apartments we should design. Let us call them x_1, x_2 and x_3 respectively. We can write the objective (in thousands of dollars) in terms of these variables as maximising:

$$p = 20x_1 + 24x_2 + 27x_3$$

Subject to the following constraints:

$$\begin{aligned} 90x_1 + 180x_2 + 220x_3 &\le 1800 \\ 120x_1 + \ \ 60x_2 + \ \ 20x_3 &\le \ 960 \\ x_2 \qquad\qquad &\le 6 \\ x_3 &\le 3 \\ x_1,\ x_2, x_3 &\ge 0: \quad \text{integer} \end{aligned}$$

2. 3.
Denote by

x_1: the number of cups of dietary drink X,

x_2: the number of cups of dietary drink Y,

C: the cost.

<u>*The model*</u>:

$$C(x_1, x_2) = 0.12x_1 + 0.15x_2 \to \text{min!}$$
$$\begin{aligned} 60x_1 + 60x_2 &\ge 300 \\ 12x_1 + \ \ 6x_2 &\ge 36 \\ 10x_1 + 30x_2 &\ge 90 \\ x_1, \qquad x_2 &\ge 0 \end{aligned}$$

2. 4.
Denote by

x_1: the number of acres of fruit A,

x_2: the number of acres of fruit B,

P: the profit.

<u>*The model*</u>:

$$P(x_1, x_2) = 140x_1 + 235x_2 \to \text{max!}$$
$$\begin{aligned} x_1 + \quad x_2 &\le 150 \\ x_1 + \ \ 2x_2 &\le 240 \\ 0.3x_1 + 0.1x_2 &\le 30 \\ x_1, \qquad x_2 &\ge 0 \end{aligned}$$

2. 5.
Let us put the data in the tabular form:

Type	Inputs/unit			Selling price/unit
	A	B	C	
I	30	20	6	14
II	25	5	15	15
Availability	6000	3000	3000	

Denote by

x_i : the number of the $i-$th type of soap $(i=1,2)$.

The model:

$$\begin{aligned}
z = 14x_1 + 15x_2 &\to \max! \\
30x_1 + 25x_2 &\le 6000 \\
20x_1 + \ \ 5x_2 &\le 3000 \\
6x_1 + 15x_2 &\le 3000 \\
x_1, x_2 &\ge 0, \text{ integer.}
\end{aligned}$$

2. 6.
Denote by

x_i : the number of buses starting at the beginning of the $i-$th period, $i=1,2,...,5.$

Note that each bus operates during two consecutive shifts. Buses which join the crew at 5 AM and 9 AM will be in the operation between 9 AM and 1 PM. As the minimum number of buses required in this interval is 13, we have $x_1 + x_2 \ge 13$, and similarly others.

The model:

$$\begin{aligned}
z = x_1 + x_2 + x_3 + x_4 + x_5 &\to \min! \\
x_1 + x_2 &\ge 13 \\
x_2 + x_3 &\ge 11 \\
x_3 + x_4 &\ge 14 \\
x_4 + x_5 &\ge 4 \\
x_5 + x_1 &\ge 5 \\
x_i \ge 0, \ i=1,2,...,5; &\text{ integer}.
\end{aligned}$$

2. 7.
Denote by

x_j : the level of the stock at the the beginning of the $j-$th week;

y_j : the amount bought during the $j-$th week;

z_j : the amount sold during the $j-$th week.

The model:

$$\begin{aligned}
z = \sum_{j=1}^{6} p_j \cdot \left(z_j - y_j\right) - 15x_j &\to \max! \\
x_{j+1} = x_j + y_j - z_j, &\quad j=1,2,...,5 \\
x_j \le 2000, &\quad j=1,2,...,6
\end{aligned}$$

$$x_1 = 0, \quad x_6 + y_6 - z_6 = 0$$
$$x_j \geq 0, \; y_j \geq 0, \quad z_j \geq 0, \quad j = 1,2,...,6.$$

2. 8.

Denote by

x_i : number of napkins purchased on the $i-$th day, $i = 1,2,...,5.$

y_j : number of napkins given for washing on $j-$th day under express service, $j = 1,2,3,4.$

z_k : number of napkins given for washing on the $k-$th day under ordinary service, $k = 1,2,3.$

v_l : number of napkins left in the stock on $k-$th day after the napkins have been given for washing, $l = 1,2,...,5.$

The data is tabulated as

Type	Number of napkins required on days				
	1	2	3	4	5
New napkins	x_1	x_2	x_3	x_4	x_5
Express service	-	y_1	y_2	y_3	y_4
Ordinary service	-	-	z_1	z_2	z_3
Napkins required	80	50	100	80	150

We have to minimise

$$2(x_1 + x_2 + x_3 + x_4 + x_5) + y_1 + y_2 + y_3 + y_4 + 0.5(z_1 + z_2 + z_3).$$

From the table:

$$x_1 = 80, \; x_2 + y_1 = 50, \; x_3 + y_2 + z_1 = 100, \; x_4 + y_3 + z_2 = 80, \; x_5 + y_4 + z_3 = 150.$$

Also, there is another set of constraints, which shows the total number of napkins which may be given for washing and some napkins which were not given for washing just on the day these have been used. These constraints are:

$$y_1 + z_1 + v_1 = 80,$$
$$y_2 + z_2 + v_2 = 50 + v_1,$$
$$y_3 + z_3 + v_3 = 100 + v_2,$$
$$y_4 + v_4 = 80 + v_3,$$
$$v_5 = 150 + v_4.$$

<u>*The model*</u>:

$$z = 160 + 2(x_1 + x_2 + x_3 + x_4 + x_5) + y_1 + y_2 + y_3 + y_4 + 0.5(z_1 + z_2 + z_3) -> \text{min!}$$
$$x_2 + y_1 = 50$$

$$x_3 + y_2 + z_1 = 100$$
$$x_4 + y_3 + z_2 = 80$$
$$x_5 + y_4 + z_3 = 150$$
$$y_1 + z_1 + v_1 = 80$$
$$y_2 + z_2 + v_2 - v_1 = 50$$
$$y_3 + z_3 + v_3 - v_2 = 100$$
$$y_4 + v_4 - v_3 = 80$$
$$v_5 - v_4 = 150$$
$$x_i, y_j, z_k, v_l \geq 0,\ \forall i, j, k, l.$$

2. 9.
A standard roll may be cut according to the following patterns:

Widths ordered in cm	Number of subrolls cut on different patterns							
	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
30	6	4	3	2	2	1	1	0
50	0	1	0	1	2	0	3	2
70	0	0	1	1	0	2	0	1
Trim loss	0	10	20	0	20	10	0	10

Denote by

x_i : number of the standard print rolls piece to cut on the patterns p_i, $i = 1,2,...,8$.

<u>*The model*</u>:

$$z = 10x_2 + 20x_3 + 20x_5 + 10x_6 + 10x_7 \rightarrow \text{min!}$$
$$6x_1 + 4x_2 + 3x_3 + 2x_4 + 2x_5 + x_6 + x_7 = 40$$
$$x_2 + x_4 + 2x_5 + 3x_7 + 2x_8 = 60$$
$$3x_3 + x_4 + 2x_6 + x_8 = 30$$
$$x_i \geq 0,\ i = 1,2,...,8 : \text{ integer.}$$

Here, in the constraints the equality is desired due to the fact that any left thing is of no use.

2. 10.
Denote by

x_{ij} : tons of ore i allocated to alloy j; $i = 1,2,3$; $j = A, B$,

w_j : tons of alloy j produced, $j = A, B$.

<u>*The model*</u>:

$$z = 200w_A + 300w_B - 30(x_{1A} + x_{1B}) - 40(x_{2A} + x_{2B}) - 50(x_{3A} + x_{3B}) \rightarrow \text{max!}$$

Specification constraints:

$$0.2x_{1A} + 0.1x_{2A} + 0.5x_{3A} \le 0.8w_A$$
$$0.1x_{1A} + 0.2x_{2A} + 0.05x_{3A} \ge 0.3w_A$$
$$0.3x_{1A} + 0.3x_{2A} + 0.2x_{3A} \ge 0.5w_A$$
$$0.1x_{1B} + 0.2x_{2B} + 0.05x_{3B} \ge 0.4w_B$$
$$0.1x_{1B} + 0.2x_{2B} + 0.05x_{3B} \le 0.6w_B$$
$$0.3x_{1B} + 0.3x_{2B} + 0.7x_{3B} \ge 0.3w_B$$
$$0.3x_{1B} + 0.3x_{2B} + 0.2x_{3B} \le 0.7w_B$$

Ore constraints:

$$x_{1A} + x_{1B} \le 1000$$
$$x_{2A} + x_{2B} \le 2000$$
$$x_{3A} + x_{3B} \le 3000$$

Alloy constraints:

$$x_{1A} + x_{2A} + x_{3A} \ge w_A$$
$$x_{1B} + x_{2B} + x_{3B} \ge w_B$$
$$x_{ij},\ w_j \ge 0,\ \ i = 1,2,3;\ j = A, B.$$

2. 11.
Denote by

x_1 : the number of cattle,
x_2 : the amount of wheat,
x_3 : the amount of corn,
x_4 : the amount of tomatoes.

<u>*The model*</u>:

$$z = 1600(1/4)x_1 + 5(50)x_2 + 6(80)x_3 + 1/2\,(1000)x_4 \to Max$$

whose units are:

[$] = [$/head][heads/acre][acres] + [$/bushel][bushel/acre][acres]+

+ [$/bushel][bushel/acre][acres] + [$/lbs][lbs/acre][acres]

s.t.

$$x_1 + x_2 + x_3 + x_4 \le 1000$$
$$40(1/4)x_1 + 10x_2 + 12x_3 + 25x_4 \le 12000$$
$$x_1 \ge 0.2(x_1 + x_2 + x_3 + x_4)$$
$$x_4 \le 0.3(1000)$$

$$x_2 / (1000 - x_1 - x_2 - x_3 - x_4) \le 2/1$$
$$x_1, x_2, x_3, x_4 \ge 0, \qquad x_1: \text{ integer.}$$

A few routine transformations yield:

$$z = 400x_1 + 250x_2 + 480x_3 + 500x_4 \to Max$$

s.t.

$$x_1 + x_2 + x_3 + x_4 \le 1000$$
$$10x_1 + 10x_2 + 12x_3 + 25x_4 \le 12000$$
$$0.8x_1 - 0.2x_2 - 0.2x_3 - 0.2x_4 \ge 0$$
$$x_4 \le 300$$
$$2x_1 + 3x_2 + 2x_3 + 2x_4 \le 2000$$
$$x_1, x_2, x_3, x_4 \ge 0, \qquad x_1: \text{ integer.}$$

(The optimal solution is: $x_1^* = 200,\ x_2^* = 0,\ x_3^* = 769.2308,\ x_4^* = 30.7692,\ z^* = 464615.40.$*)*

2. 12.
Denote by

x_1 : investment in bonds,
x_2 : investment in stocks,
x_3 : investment in term deposits,
x_4 : investment in saving accounts,
x_5 : investment in real estates,
x_6 : investment in gold,
x_7 : loan.

<u>*The model*</u>:

$$z = 1.05x_1 + 1.09x_2 + 1.04x_3 + 1.03x_4 + 1.07x_5 + 1.11x_6 - 1.06x_7 \to Max!$$

s. t.

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 1000000 + x_7$$
$$x_1 \ge 0.3(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)$$
$$x_2 \le 0.1(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)$$
$$x_3 + x_4 \ge 0.1(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)$$
$$x_7 \le 0.5x_5$$
$$x_7 \le 150000$$
$$\frac{3x_1 + 10x_2 + 2x_3 + x_4 + 5x_5 + 20x_6}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} \le 4.5$$
$$\frac{5x_1 + 2x_2 + 4x_3 + 3x_4}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} \ge 2.5$$

$x_6 \le 100000$
$x_6 \le 0.08(1000000 + x_7)$
$x_i \ge 0,\ i = 1,2,...,7.$

(The optimal solution is: $x_1^* = 437000,\ x_2^* = 115000,\ x_3^* = 115000,\ x_4^* = 0,$
$x_5^* = 478400,\ x_6^* = 4600,\ x_7^* = 150000,\ z^* = 1061794$ *)*

2. 13.
Denote by

x_i : production in month $i = 1,2,3,4$
I_j : inventory in at the beginning of the months $j = 1,2,3,4,5$.

The Model:

$$z = x_1 + 1.1x_2 + 1.2x_3 + 1.2x_4 + 0.3I_1 + 0.2I_3 + 0.2I_4 \to Min!$$

s.t.

$x_1 \le 100$
$x_2 \le 100$
$x_3 \le 160$
$x_4 \le 150$
$I_1 = 0$
$I_2 = 0 + x_1 - 50$
$I_3 = I_2 + x_2 - 120$
$I_4 = I_3 + x_3 - 150$
$I_5 = 0 = I_4 + x_4 - 160$
$x_i \ge 0,\ I_j \ge 0,\ \forall i, j.$

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