Chapter 2

Linear Optimization Introduction

Ex. 2. 1. (***Profit Maximisation***)
A firm produces two products P_1 and P_2 using the raw materials R_1, R_2, R_3. The availability of the raw materials, and the amount of each raw material used to produce one unit of each product (the so-called "technical coefficients") and the profit to the firm of producing one unit of these products are given in the following table:

Raw Material	P_1	P_2	Availability
R_1	5	3	45
R_2	2	3	36
R_3	1	-	6
Profit (€)	50	20	

The firm would like to maximize its profit.

Ex. 2. 2. (***The Diet Problem***)
A dietician has to construct meals from two foods: type F_1 and type F_2. Each 100 gm of type F_1 costs 3 € and has 3 energy units, 1 unit of vitamins and 1 unit of minerals. Each type of F_2 costs 5 € and has 4 energy units, 1 unit of vitamins and 5 units of minerals. For a balanced diet, a patient needs at least 340 energy units, 100 units of vitamins and 100 units of minerals. What is the least expensive balanced meal?

Ex. 2. 3. (***Transportation Problem***)
A certain commodity is to be transported from the sources S_1, S_2 and S_3 to the destinations D_1, D_2, D_3 and D_4. The following table shows the availabilities of the commodity, the requirements of the destinations and the costs per unit for the transportation of the commodity from the three sources to the four destinations:

Destination $\rightarrow$ Sources $\downarrow$	D_1	D_2	D_3	D_4	Availabilities
S_1	5	2	4	3	30
S_2	6	4	9	5	40
S_3	2	3	8	1	55
Requirements $\rightarrow$	15	20	40	50	

Determine a transportation plan that minimises the total transportation cost.

R. 2. 1

The problems formulated in Ex. 1. – Ex. 1. 3. represent the first civil application of linear optimisation.

Ex. 2. 4.

Formulate the problems in Ex. 2. 1. – Ex. 2. 3. as linear optimisation models.

1.

Let

$x_i,\ i = 1,2$: output of P_i

$\pi(x_1, x_2)$: profit function.

The Model:

$$\begin{aligned}
\pi(x_1, x_2) = 50x_1 + 20x_2 &\to \max \\
5x_1 + 3x_2 &\le 45 \\
2x_1 + 3x_2 &\le 36 \\
x_1 \qquad &\le 6 \\
x_1, x_2 &\ge 0 .
\end{aligned}$$

2.

Let

$x_i,\ i = 1,2$: amount of F_i

$C(x_1, x_2)$: cost function.

The Model:

$$\begin{aligned}
C(x_1, x_2) = 3x_1 + 5x_2 &\to \min \\
3x_1 + 4x_2 &\ge 340 \\
x_1 + x_2 &\ge 100 \\
x_1 + 5x_2 &\ge 100 \\
x_1, x_2 &\ge 0 .
\end{aligned}$$

3.

Let

$x_{ij},\ i = 1,2,3;\ j = 1,2,3,4$: amount of commodity transported from S_i to D_j function.

$C(...)$: cost function.

$$C = 5x_{11} + 2x_{12} + ... + 8x_{33} + x_{34} \to \min$$

$$\sum_{j=1}^{4} x_{1j} = 30, \quad \sum_{j=1}^{4} x_{2j} = 40, \quad \sum_{j=1}^{4} x_{3j} = 55,$$

$$\sum_{i=1}^{3} x_{i1} = 15, \quad \sum_{i=1}^{3} x_{i2} = 20, \quad \sum_{i=1}^{3} x_{i3} = 40, \quad \sum_{i=1}^{3} x_{i4} = 50$$

$x_{ij} \geq 0,\ \forall i, j.$

<u>D. 2. 1.</u> (*Optimisation Problem*)
The *Optimisation Problem* (*OP*) is defined as:

(2. 1.) $$f(x) \rightarrow opt$$

subject to (s.t.)

(2. 2.) $$g_i(x) \leq,\ =,\ \geq 0,\ i = 1, 2, ..., m,$$

(2. 3.) $$x \geq 0.$$

Here is: $x := \left(x_1, x_2, ..., x_n\right)^T$.

Now, we define:

(1) The function $f(x)$ to be optimised is termed as *objective function*;
(2) The relations in (1. 1.) are *constraints*;
(3) The conditions in (1. 2.) are *nonnegative restrictions*;
(4) Variables $x_1, x_2, ..., x_n$ are *decision variables*;
(5) The terminology *optimise* stands for *minimise* or *maximise*.

The symbol $\geq,\ =,\ \leq$ means that one and only one of these is involved in each constraint.

The problem (OP) is further classified into two classes:

1. *Linear Optimisation Problem* (*LOP*)
   If the objective function $f(x)$ and all the constraints $g_i(x)$ are linear in an optimisation problem, we call the problem a *linear optimisation problem* (*LOP*).

2. *Nonlinear Optimisation Problem*
   If the objective function $f(x)$ or at least one of the constraints $g_i(x)$ or both are nonlinear functions in an optimisation problem, then the problem is termed as *nonlinear optimisation problem* (*NLOP*).

<u>D. 2. 2.</u> (*Linear Optimisation Problem*)
A *linear optimisation problem* (*LOP*) has the general form:

$$z = \sum_{j=1}^{n} c_j x_j \rightarrow opt$$

s.t. $$\sum_{j=1}^{n} a_{ij} x_j \leq,\ =,\ \geq b_i, \quad i = 1, 2, ..., m$$

$$x_1, x_2, ..., x_n \geq 0,$$

where $c_j,\ a_{ij},\ b_i \in R^1,\ i = 1, 2, ..., m;\ j = 1, 2, ..., n.$

D. 2. 3. (***Standard Form of a Linear Optimisation Problem***)
The *standard form* of a LOP is defined as

$$z = \sum_{j=1}^{n} c_j x_j \to \text{max!}$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1,2,...,m$$

$$x_1, x_2, ..., x_n \geq 0,$$

or, in the matrix form

$$\begin{aligned} z = c^T x &\to \text{max!} \\ Ax &= b \\ x &\geq 0, \end{aligned}$$

where

$$\begin{aligned} &c = (c_1, c_2, ..., c_n)^T, \\ &c = (x_1, x_2, ..., x_n)^T, \\ &A = (a_{ij}),\ i = 1,2,...,m;\ j = 1,2,...,n, \\ &b = (b_1, b_2, ..., b_m)^T. \end{aligned}$$

Ex. 2. 4.
Write the linear optimisation problem in Ex. 2. 1. into the standard form.

Solution:

$$\begin{aligned} \pi(x_1, x_2) = 50x_1 + 20x_2 &\to \text{max} \\ 5x_1 + 3x_2 + x_3 \qquad &= 45 \\ 2x_1 + 3x_2 \quad + x_4 \quad &= 36 \\ x_1 \qquad\qquad + x_5 &= 6 \\ x_i \geq 0,\ i = 1,2,...,5. \end{aligned}$$

The variables x_3, x_4, x_5 are termed as the *slack variables*.

Ex. 2 5.
Write the linear optimisation problem in Ex. 2. 2. into the standard form.

Solution:

$$\begin{aligned} -C(x_1, x_2) = -x_1 - 5x_2 &\to \text{max} \\ 3x_1 + 4x_2 - x_3 \qquad &= 340 \\ x_1 + x_2 \quad - x_4 \quad &= 100 \end{aligned}$$

$$
\begin{aligned}
x_1 + 5x_2 \qquad\quad - x_5 &= 100 \\
x_i \geq 0,\ i &= 1,2,...,5.
\end{aligned}
$$

The variables x_3, x_4, x_5 are termed as the *surplus variables*.

Ex. 2. 6.

Write the following linear optimisation problem into the standard form

$$
\begin{aligned}
z = x_1 + 2x_2 &\to \max \\
-3 \leq x_1 &\leq +3 \\
-2 \leq x_2 &\leq +2\,.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
z = x_1 + 2x_2 &\to \max \\
x_1 &\geq -3 \\
x_1 &\leq 3 \\
x_2 &\geq -2 \\
x_2 &\leq 2\,.
\end{aligned}
$$

$$
\begin{aligned}
z = x_1 + 2x_2 &\to \max \\
x_1 \quad - x_3 \qquad\qquad &= -3 \\
x_1 \qquad + x_4 \qquad &= \ \ 3 \\
x_2 \qquad - x_5 \quad &= -2 \\
x_2 \qquad\qquad + x_6 &= \ \ 2\,.
\end{aligned}
$$

Let

$$x_1 := x_1^+ - x_1^-\,, \qquad\qquad x_2 := x_2^+ - x_2^-\,.$$

$$
\begin{aligned}
z = x_1^+ - x_1^- + 2x_2^+ - 2x_2^- &\to \max \\
x_1^+ - x_1^- \qquad - x_3 \qquad\qquad &= -3 \\
x_1^+ - x_1^- \qquad\qquad + x_4 \qquad &= \ \ 3 \\
x_2^+ - x_2^- \qquad - x_5 \quad &= -2 \\
x_2^+ - x_2^- \qquad\qquad + x_6 &= \ \ 2
\end{aligned}
$$

$$x_1^+, x_1^-, x_2^+, x_2^-, x_3, x_4, x_5, x_6 \geq 0\,.$$

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