Introduction to the Sets Theory

<u>D. 1</u>. (*Set, Elements of a Set*)
A *set* is any well defined collection of "objects". The *elements* of a set are the object in a set.

<u>R. 1.</u> (*Notation*)
Usually we denote sets with upper-case letters, elements with lower-case letters.
$x \in S$ means the element x belongs to the set S ; $x \notin S$ means the element x does not belong to the set S .

<u>D. 2.</u> (*Empty or Null Set*)
An *empty* or a *null set* has no members at all. It is denoted by $\varnothing$.

<u>D. 3.</u> (*Universal Set*)
The set of all elements currently under consideration is called a *universal set*. It is symbolized by Ω .

<u>D. 4.</u> (*Singleton*)
A set with only one element is called a *singleton*.

<u>R. 2.</u> (*Specification of Sets*)
There are three main ways to specify a set:

1. by listing all its elements (*list notation*)
2. by listing properties of its elements (*predicate notation*)
3. by defining a set of rules which defines its elements (*recursive rules notation*)

<u>R. 3.</u> (*Operations on Sets*)

1. Subset
2. Union
3. Intersection
4. Difference
5. Complement
6. Cartesian product (will be dealt with a little later).

<u>D. 5.</u> (*Subset, Proper Subset*)
Given two sets A and B, B is a *subset* of A , denoted by $B \subseteq A$, if every element in B belongs also to A :

$$B \subseteq A := \{x \mid x \in B \Rightarrow x \in A\}.$$

Set B is a *proper set* of A , denoted by $B \subset A$, if there exists an element in A that does not belong to B.

D. 3. (***Equality***)

$$A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A).$$

D. 6. (***Union***)
Given two sets A and B, the *union* of A and B, denoted by $A \cup B$, is the set that contains elements belonging to *at least* of the two sets:

$$A \cup B := \{x \mid x \in A \vee x \in B\}.$$

D. 7. (***Intersection***)
Given two sets A and B, the *intersection* of A and B, denoted by $A \cap B$, is the set that contains elements belonging to *both* sets:

$$A \cap B := \{x \mid x \in A \wedge x \in B\}.$$

D. 8. (***Disjoint Sets***)
Two sets A and B are called *disjoint*, if

$$A \cap B = \emptyset.$$

D. 9. (***Difference***)
Given two sets A and B, the *difference* of A and B, denoted by $A \setminus B$, is the set of elements of A which are not also elements of B:

$$A \setminus B := \{x \mid x \in A \wedge x \notin B\}.$$

D. 10. (***Complement***)

Given a set A, the *complement* of A, denoted by $\bar{A}$ is the set of all elements in the universal set Ω, but not in A:

$$\bar{A} := \{x \in \Omega \mid x \notin A\}.$$

Th. 1. (***Some Laws***)
For any arbitrary sets A, B, C, we have:

1. The distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

2. The de Morgan's laws:

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Ex. 1.
Given the sets

$$A = \{x \in R^1 \mid -1 < x < 3\}$$
$$B = \{x \in R^1 \mid -8 < x\}$$
$$C = \{x \in R^1 \mid -8 < x < 1\},$$

construct the following sets:

$$A \cap B, \qquad A \cup C, \qquad \bar{B}, \qquad \bar{A} \cup \bar{C}, \qquad \bar{A} \cap \bar{C}, \qquad B \setminus A, \qquad C \setminus B.$$

Solution:

$$A \cap B = A,$$

$$A \cup C = \{x \in R^1 \mid -8 < x < 3\},$$

$$\bar{B} = \{x \in R^1 \mid x \leq -8\},$$

$$\bar{A} \cup \bar{C} = \{x \in R^1 \mid x \leq -1 \vee x \geq 1\}, \quad \bar{A} \cap \bar{C} = \{x \in R^1 \mid x \leq -8 \vee x \geq 3\},$$

$$B \setminus A = \{x \in R^1 \mid -8 < x \leq -1 \vee x \geq 3\},$$

$$C \setminus B = \varnothing$$

D. 11. (*Cardinality*)
If a set S has $n \in N$ distinct elements, n is the *cardinality* of S and S is a *finite* set. The cardinality of S is denoted by $|S|$.

Ex. 2.
The set $S = \{1,3,5,7,9\}$ has the cardinality $|S| = 5$.

Th. 2.
For an arbitrary set S the following assertions hold:

1. $S \subseteq \Omega$
2. $\varnothing \in S$
3. $S \subseteq S$.

D. 12. (*Power Set*)
The set of all subsets of a set S is called the *power set* of S and is denoted by 2^S or $P(S)$.

Ex. 3.
Give a list notation of the following set and its power set:

$$S = \{x \in \mathbb{R}^1 \mid (x-1)\cdot(x+2)\cdot(x-3) = 0\}.$$

Solution:

$$S=\{1,\text{-}2,\ 3\}$$

$$2^S=\{\{\ \},\{1\},\{-2\},\{3\},\{1,-2\},\{1,3\},\{-2,3\},\{1,-2,3\}\}$$

Th. 3.

For an arbitrary set S, the number of subsets of S is $2^{|s|}$.

D. 13. (***Ordered Pair***)

An *ordered pair* is the set of two elements of two elements a and b in which the order of elements matters: (a,b).

D. 14. (***Cartesian Product***)

Given two sets A and B, the *(Cartesian) product* of A and B, denoted by AxB, is the set of all ordered pairs such that $a\in A$ and $b\in B$:

$$\{(a,b)\,|\,a\in A\ \wedge\ b\in B\}=:AxB\ .$$

Ex. 4.

Given the sets $A=\{a,\ \ b\},\ \ B=\{c,\ \ d\}$, we have:

$$AxB=\{(a,\ c),\ (a,\ d),\ (b,\ c),\ (b,\ d)\},.$$

$$BxA=\{(c,\ \ a),\ \ (d,\ \ a),\ \ (c,\ \ b),\ \ (d,\ \ b)\}.$$

Th. 4.

$$AxB\neq BxA\ .$$

R. 4.

Given the elements $a_1,a_2,...,a_n$, the Cartesian product can be generalized to:

$$\{(a_1,\ a_2,\ ...,a_n)\,|\,a_1\in A,\ a_2\in A,\ ...,a_n\in A\}=:A^n\ .$$

For real numbers $x_i,\ i=1,2,...,n$, we have

$$R^n=\{(x_1,\ x_2,\ ...,x_n)\ |\ x_i\in R^1\}.$$

D. 15. (***Mapping***)

Given the sets X and Y, the set $A\subseteq XxY$ is called a *mapping of* X in Y. In an arbitrary ordered pair $(x,y)\in A,\ y$ is the *image* of x and x is the *preimage* or *inverse image* of y.

D. 16. (***Domain, Range***)

Let $A\subseteq XxY$ be a mapping of X in Y.

The set

$$D(A) := \{x \in X \mid \exists y \in Y \text{ with } (x, y) \in A\}$$

is called the *domain of* A.

The set

$$R(A) := \{y \in Y \mid \exists x \in X \text{ with } (x, y) \in A\}$$

is called the *range of* A.

D. 17. (***Function***)

A *function* is a special mapping where every element of X is associated with a unique element of Y.

We write

$$f : X \to Y \quad (\text{or } f(x) = y)$$

D. 18. (***Injective, Surjective and Bijective Functions***)**:**

1.

A function is *injective* (*one-to-one*) if

$$\langle \forall x_1, x_2 \in X, f(x_1) = f(x_2) \rangle \;\Rightarrow\; x_1 = x_2$$

2.

A function is *surjective* (*onto)* if

$$\forall y \in Y, \; \exists x \in X : f(x) = y$$

3.

A function is *bijective* if it is both injective and surjective. A bijective function is also called a *bijection* (a *one-to-one correspondence*.):

$$\forall y \in Y, \; \exists \text{ unique } x \in X : f(x) = y$$

D. 19. (***Inverse Mapping***)**:**

Let $A \subseteq XxY$ be a bijective mapping of X in Y. The set

$$A^{-1} := \{(y, x) \mid (y, x) \in YxX \wedge (x, y) \in A\}$$

is called the *inverse mapping of* A.

Th. 5.
Let $A \subseteq XxY$ be a bijective mapping of X in Y.

1.

$$D(A^{-1}) = R(A) \text{ and } D(A) = R(A^{-1}).$$

2.

$$(A^{-1})^{-1} = A.$$

Ex. 5.

1.

$$X = \{a, b, c\}, \qquad Y = \{1, 2, 3, 4\}$$

$A = \{(a,1), (a,2), (b,3), (c,4)\} \subseteq AxB$ no function

2.

$$X = \{a, b, c, d\}, \qquad Y = \{1, 2, 3, 4, 5\}$$

$A = \{(a,2), (b,1), (c,3), (d,4)\} \subseteq AxB$ injective (not surjective)

$$D(A) = \{a, b, c\} = X; \quad R(A) = \{1, 2, 3\} \subset Y$$

3.

$$X = \{a, b, c, d, e\}, \qquad Y = \{1, 2, 3, 4\}$$

$A = \{(a,2), (b,1), (c,3), (d,3), (e,4)\} \subseteq AxB$ surjective (not injective)

$$D(A) = \{a, b, c, d, e\} = X; \quad R(A) = \{1, 2, 3, 4\} = Y$$

4.

$$X = \{a, b, c, d, e\}, \qquad Y = \{1, 2, 3, 4, 5\}$$

$A = \{(a,2), (b,1), (c,5), (d,3), (e,4)\} \subseteq AxB$ bijective (injective and surjective)

$$D(A) = \{a, b, c, d, e\} = X; \quad R(A) = \{1, 2, 3, 4\} \subset Y$$

R. 5. ***(Real functions).***
For $X = R^n$, $n = 1,2...$ we speak of a real function.

D. 20. **(*Graph of a function of a single variable*):**
Consider $f : X \to R^1$. The following set is called the graph of the function f:

$$graph\, f := \{(x, y) \in XxY \,|\, f(x) = y\}$$

Ex. 6.

$$f(x) = \begin{cases} -1 & x \le -1 \\ x & -1 < x \le 1 \\ 1 & x > 1 \end{cases}$$

Ex. 7
:

1. Consider the function

   $f : R^1 \to R^1$ with $Y = f(x) = 1 + x.$

   The function is bijective with $x = f^{-1}(y) = y - 1$

2. Consider the function

   $f : R^1 \to R^1$ with $Y = f(x) = x^2.$

   The function is not bijective.

   However, if we decompose the original domain in the following subdomains:

   $D_1 = \{x \in R^1 \,|\, x \ge 0\}$ and $D_2 = \{x \in R^1 \,|\, x < 0\}$,

   then both functions

   $y = f(x) = x^2,\ x \in D_1$ and $y = f(x) = x^2,\ x \in D_2$

   will be bijective with the inverse functions:

   $x = f_1^{-1}(y) = \sqrt{y},\ \forall x \in D_1$ and $x = f_2^{-1}(y) = -\sqrt{y},\ \forall x \in D_2$

D. 21. (***Relation***)
A binary relation on a set A is a mapping $R \subseteq AxA$. Instead of writing $a, b \in A$ it will be denoted by $a\ R\ b$

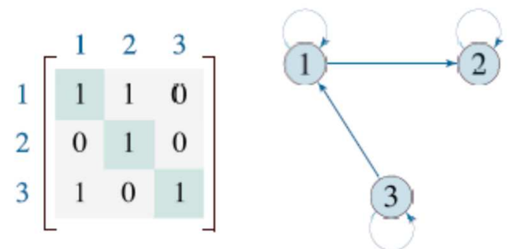
D. 22. (***Properties of Relations***)
Consider the set A and a relation $R \subseteq AxA$
The relation is called

- *reflexiv* if $x\ R\ x$ for all $x \in A$,
- *symmetric* if $x\ R\ y \Rightarrow y\ R\ x$ for all $x, y \in A$,
- *antisymmetric* if $[x\ R\ y \wedge y\ R\ x] \Rightarrow x = y, \forall x, y \in A$,
- *transitiv* if $[x\ R\ y \wedge y\ R\ z] \Rightarrow x\ R\ z, \forall x, y, z \in A$.

Ex. 8.
Consider the set $A = \{1, 2, 3\}$
1. The relation

$$R = \{(1,1), (1,2), (2,2), (3,3), (3,1)\}$$

is on set A is reflexive.



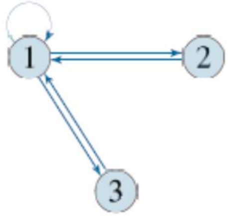
Reflexive relations are always represented by a matrix that has 1 on the main diagonal. The digraph of a reflexive relation has a loop from each node to itself.

2. The relation

$$R = \{(1,1), (1,2), (2,1), (1,3), (3,1)\}$$

is on set A symmetric.

	1	2	3
1	1	1	1
2	1	0	0
3	1	0	0



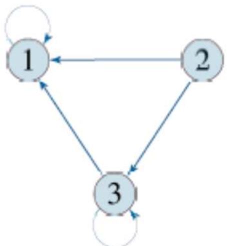
For a symmetric relation, the logical matrix is symmetric about the main diagonal. The transpose of the matrix is always equal to the original matrix. In a digraph of a symmetric relation, for every edge between distinct nodes, there is an edge in the opposite direction.

3. The relation

$$R = \{(1,1),(2,1),(2,3),(3,1),(3,3)\}$$

is on set A is <u>antisymmetric</u>.

	1	2	3
1	1	0	0
2	1	0	1
3	1	0	1



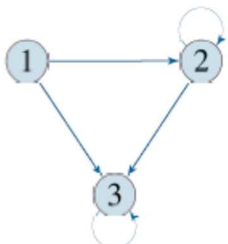
In a matrix representing an antisymmetric relation, all elements symmetric about the main diagonal are not equal to each other. The digraph of an antisymmetric relation may have loops, however connections between two distinct vertices can only go one way.

4. The relation

$$R = \{(1,2),(1,3),(2,2),(2,3),(3,3)\}$$

is on set A is <u>transitive</u>.

	1	2	3
1	0	1	1
2	0	1	1
3	0	0	1



In a matrix of a transitive relation, for each pair of (i,j)- and (j,k) entries with value 1 there exists the (i,k)- entry with value 1. The presence of $1'$s on the main diagonal does not violate transitivity.

Ex. 9.
Let $A = \boldsymbol{N}$ (set of natural numbers).

a) Let $x\,R\,y \;\Leftrightarrow\; x \le y$. This relation is reflexive, antisymmetric und transitive, because

$x \le x,\ \forall x \in \boldsymbol{N}$ (reflexiv),
$(x \le y \;\wedge\; y \le x) \;\Rightarrow\; x = y,\ \forall x, y \in \boldsymbol{N}$ (antisymmetric),
$(x \le y \;\wedge\; y \le z) \;\Rightarrow\; x \le z,\ \forall x, y, z \in \boldsymbol{N}$ (transitiv).

b) Let $x\,R\,y \;\Leftrightarrow\; x < y$. This Relation is not reflexive, not symmetric but antisymmetric and transitive.

$(x < y \;\wedge\; y < z) \;\Rightarrow\; x < z,\ \forall x, y, z \in \boldsymbol{N}$ (transitiv).

c) Let $x\,R\,y$ if only if x is a divisor of y. This Relation is reflexive, antisymmetric und transitive.

d) $x\,R\,y$ if and only if $x = y$. This relation is reflexive, antisymmetric and transitive

D. 23. (***Equivalence Relation)***
The relation $R \subseteq AxA$ is called an *equivalence relation* if it is reflexive, transitive and symmetric.

Ex. 10.
The relation $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,2),(3,2),(1,3),(3,1)\}$ on set $A = \{1,2,3\}$ is an equivalence relation since it is reflexive, symmetric, and transitive

D. 24. (***Equivalence Classes)***
Let R be an equivalence relation on a set A, and let $a \in A$. The *equivalence class* of a is called the set of all elements of A which are equivalent to a. The equivalence class of an element a is denoted by $[a]$. Thus, by definition,

$$[a] = \{b \in A \,|\, aRb\} = \{b \in A \,|\, a \sim b\}$$

If $b \in [a]$ then the element b is called a *representative of the equivalence* class. Any element of an equivalence class may be chosen as a representative of the class $[a]$.

The set of all equivalence classes of A is called the *quotient set* of A by the relation R . The quotient set is denoted as A/R:

$$A/R = \{[a] | a \in A\}$$

<u>Ex. 11.</u>

A well-known sample equivalence relation is congruence modulo n .Two integers a and b are equivalent if they have the same remainder after dividing by n .

(Last updated: 15.12.20)