Chapter One

Matrix Algebra

<u>D. 1. 1.</u> (*Matrix*)
A *matrix* is a rectangular two-dimensional array

$$A := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j}. & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

of $m \cdot n$ *elements* $a_{ij} \in R, i = 1, 2,..., m;\ j = 1, 2,..., n$. The ordered pair (m, n) is called the *dimension* of the matrix A.

(When we need to be explicit about a matrix's dimension, we often append it as a subscript: $A_{(m,n)}$.)

<u>R. 1. 1.</u>
The matrix A will often be denoted by its *general element* a_{ij}:

$$A := (a_{ij}), i = 1, 2,..., m;\ j = 1, 2,..., n.$$

<u>Ex. 1. 1.</u>
The following table represents the distances between the two sources S_1 and S_2 and the three destinations D_1, D_2 and D_3 in a "transportation problem":

	D_1	D_2	D_3
S_1	40	50	34
S_2	70	23	80

The table could just as well be represented by the following matrix:

$$A := \begin{pmatrix} 40 & 50 & 34 \\ 70 & 23 & 80 \end{pmatrix}$$

Ex. 1. 2.
An enterprise uses three raw materials R_1 and R_2 to produce the final products P_1, P_2 and . P_3. The following table shows the raw material consumption per product unit:

	P_1	P_2
R_1	4	2
R_2	0	5
R_3	3	4

The table can also be represented by a matrix:

$$B := \begin{pmatrix} 4 & 2 \\ 0 & 5 \\ 3 & 4 \end{pmatrix}.$$

D. 1. 2. (***Matrix Types***)
The matrix A is called

1.
a *zero matrix* if $a_{ij} = 0, \ \forall i, j$. A zero matrix will be denoted as 0.

2.
a *square matrix* if $m = n$.
(In this case we speak of a matrix of the *order* m or . n)

2. 1.
a *diagonal matrix* if $m = n$ and $a_{ij} = 0, \ \forall i \neq j$. The elements $a_{ii}, \ i = 1, 2, ..., n$ are called *diagonal elements*.

2. 1. 1.
a *scalar matrix* if $m = n$ and

$$a_{ij} := \begin{cases} a = const & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}.$$

2.1.1.1.
an **identity** *matrix* if $m = n$ and

$$a_{ij} := \begin{cases} 1 & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}.$$

The identity matrix is usually written as

$$I := (\delta_{ij}), i, j = 1, 2, ..., n \ ,$$

$$\delta_{ij} := \begin{cases} 1 & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}$$

The symbol $\delta_{ij}, i, j = 1, 2, ..., n$ is called the *Kronecker delta*.

2.2.
a *lower triangular matrix*, if $m = n$ and $a_{ij} = 0$ for $i < j$. It is called an *upper triangular matrix* if $m = n$ and $a_{ij} = 0$ for $i > j$.

2.3.
a *symmetric matrix*, if $m = n$ and $a_{ij} = a_{ji}, i, j = 1, 2, ..., n$. It is called a *skew-symmetric matrix* if $m = n$ and $a_{ij} = -a_{ji}, i, j = 1, 2, ..., n$.

3.
a *row vector* if $m = 1$ and a *column vector*, if $n = 1$.
(Vectors will be usually denoted by small letters.)

4.
a *scalar* if $m = n = 1$.

<u>D. 1. 3.</u>
Let A and B be matrices of the same dimension.

$$\begin{aligned} A := B \quad &\Leftrightarrow \quad a_{ij} = b_{ij}, \quad \forall i, j, \\ A :< B \quad &\Leftrightarrow \quad a_{ij} < b_{ij}, \quad \forall i, j, \\ A :\leq B \quad &\Leftrightarrow \quad a_{ij} \leq b_{ij}, \quad \forall i, j. \end{aligned}$$

<u>D. 1. 4.</u> (*Transpose of a Matrix*)
Let $A := (a_{ij}) : i = 1, 2, ..., m; \; j = 1, 2, ..., n$. The matrix $A^T := (a_{ji}), \quad j = 1, 2, ..., n; i = 1, 2, ..., m$ is called the *transpose* of the matrix A.

<u>Ex. 1. 3.</u>
Transpose the matrix in the example Ex.1.1.

Solution:

$$A := \begin{pmatrix} 40 & 50 & 34 \\ 70 & 23 & 80 \end{pmatrix}, \qquad A^T = \begin{pmatrix} 40 & 70 \\ 50 & 23 \\ 34 & 80 \end{pmatrix}.$$

<u>D. 1. 5.</u> (*Sum of Matrices*)
Let $A := (a_{ij}) : i = 1, 2, ..., m; \; j = 1, 2, ..., n$ and $B := (b_{ij}) : i = 1, 2, ..., m; \; j = 1, 2, ..., n$ be matrices.

$$C := A + B \quad \text{with} \quad C := (c_{ij}) := (a_{ij} + b_{ij}), \quad i = 1, 2, ..., m; \quad j = 1, 2, ..., n.$$

Ex. 1. 4.
Find the sum of the matrices

$$A := \begin{pmatrix} 13 & -50 & 17 \\ 10 & 11 & -80 \end{pmatrix}, \qquad B := \begin{pmatrix} -10 & 90 & 0 \\ 15 & -8 & 92 \end{pmatrix}.$$

Solution:

$$C := A + B = \begin{pmatrix} 3 & 40 & 17 \\ 25 & 3 & 12 \end{pmatrix}.$$

D. 1. 6. (***Multiplication of a Matrix by a Real Number***)
Let $A := (a_{ij}) : i = 1, 2, ..., m;\ j = 1, 2, ..., n$ be a matrix and $\alpha \in R$.

$$B := \alpha \cdot A := (\alpha \cdot a_{ij}), i = 1, 2, ..., m;\ j = 1, 2, ..., n\ .$$

Ex. 1. 5.
Multiply the number $\alpha := -2$ by the matrix

$$A := \begin{pmatrix} 0 & -1 & 2 \\ 3 & 2 & 0 \\ -3 & 1 & 1 \end{pmatrix}.$$

Solution:

$$B := \alpha \cdot A = \begin{pmatrix} 0 & 2 & -4 \\ -6 & -4 & 0 \\ 6 & -2 & -2 \end{pmatrix}.$$

D. 1. 7. (***Comformable Matrices***)
The matrices $A := (a_{ij}) : i = 1, 2, ..., m;\ j = 1, 2, ..., n$ and $B := (b_{kl}) : k = 1, 2, ..., p;\ l = 1, 2, ..., q$ are said to be comformable, if

$$\boxed{n = p}$$

D. 1. 8. (***Matrix Multiplication***)
Let $A := (a_{ij}) : i = 1, 2, ..., m;\ j = 1, 2, ..., n$ and $B := (b_{kl}) : k = 1, 2, ..., p;\ l = 1, 2, ..., q$ be given with $n = p$ (comformability).

$$A \cdot B =: C := (c_{il}),\ \ i = 1, 2, ..., m;\ \ l = 1, 2, ..., q\ ,$$

$$c_{il} := \sum_{j=1}^{n} a_{ij}.b_{jl}\ , \qquad i = 1, 2, ..., m;\ \ l = 1, 2, ..., q\ .$$

Ex. 1. 6.
Determine:

1.

$$C := A \cdot B$$

with

$$A := \begin{pmatrix} -2 & 4 & 0 \\ 1 & -5 & 3 \end{pmatrix}, \qquad B := \begin{pmatrix} -1 & 3 & 7 \\ 2 & -4 & 7 \\ 0 & 1 & -2 \end{pmatrix}$$

Solution:

$$C := A \cdot B = \begin{pmatrix} -2 \cdot (-1) + 4 \cdot 2 + 0 \cdot 0 & -2 \cdot 3 + 4 \cdot (-4) + 0 \cdot 1 & -2 \cdot 7 + 4 \cdot 7 + 0 \cdot (-2) \\ 1 \cdot (-1) - 5 \cdot 2 + 3 \cdot 0 & 1 \cdot 3 - 5 \cdot (-4) + 3 \cdot 1 & 1 \cdot 7 - 5 \cdot 7 + 3 \cdot (-2) \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -22 & 14 \\ -11 & 26 & -34 \end{pmatrix}$$

2.

$$c := A \cdot b$$

with

$$A := \begin{pmatrix} -2 & 4 & 0 \\ 1 & -1 & 3 \\ 6 & 0 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Solution:

$$c := A \cdot b = \begin{pmatrix} -2 \cdot 2 + 4 \cdot (-1) + 0 \cdot 3 \\ 1 \cdot 2 + (-1) \cdot (-1) + 3 \cdot 3 \\ 6 \cdot 2 + 0 \cdot (-1) + 1 \cdot 3 \end{pmatrix} = \begin{pmatrix} -8 \\ 12 \\ 15 \end{pmatrix}$$

Ex. 1. 7.
Given

$$a := \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

determine the following products:

1. $a \cdot b$
2. $a^T \cdot b$
3. $a \cdot b^T$

Solution:

1.
The product is not defined, since the matrices are not comformable.

2.

$$a^T \cdot b = \begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 2 \cdot 3 + 1 \cdot 4 = 10 .$$

(This kind of product is called *scalar product*.)

3.

$$a \cdot b^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 & 2 \cdot 4 \\ 1 \cdot 3 & 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 3 & 4 \end{pmatrix}$$

(This kind of product is called *dyadic* or *dot product*.)

<u>R. 1. 2.</u> (*Some Laws of Matrix Algebra*)
Let A, B, C be matrices.

• Associative:

$$(A + B) + C = A + (B + C)$$
$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

• Commutative:

$$A + B = B + A$$

• Distributive:

$$A \cdot (B + C) = A \cdot B + A \cdot C$$
$$(A + B) \cdot C = A \cdot C + B \cdot C$$

<u>T. 1. 1.</u> (*Some Properties of Matrix Operations*)
Let A, B, C be matrices. It can be shown:

1. $(A. + B)^T = A^T + B^T$
2. $(A^T)^T = A$
3. $A. \cdot B \neq B \cdot A$
4. $(A. \cdot B)^T = B^T \cdot A^T$

<u>Ex. 1. 8.</u> (*An Economic Application of Matrix Multiplication*)
Consider a two-stage production process. In the first stage the raw materials R_1, R_2 and R_3 will be needed to produce the semi-products S_1, S_2. In a second stage the semi-products will be used to produce the final products P_1, P_2.

The following tables contain the necessary informations concerning the input-outpt-coeffeicients:

Raw Material Consumption per Semi-product Unit

	S_1	S_2
R_1	2	4
R_2	0	1
R_3	3	2

Semiproduct Consumption per Final Product Unit

	P_1	P_2
S_1	4	1
S_2	0	2

1. Determine the raw material consumption per final product unit.
2. How much raw material will be needed in order to produce 100 units of P_1 and 80 units of P_2 ?
3. The raw materials R_1, R_2 and R_3 cost 2 €, 4 € and 5 € per unit respectively. Calculate the raw materials costs for the above production programme.

Solution:
Let the above tables of technical coefficients be represented by the following matrices:

$$M_{RS} := \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}, \qquad M_{SP} := \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}.$$

1.
Let M be the matrix, whose elements give the raw material consumption needed to produce one unit of the final products. Then we have

$$M = M_{RS} \cdot M_{SP} = \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 0 & 2 \\ 12 & 7 \end{pmatrix}.$$

2
Let r denote the raw material vector. We obtain

$$r = M \cdot p = \cdot \begin{pmatrix} 8 & 10 \\ 0 & 2 \\ 12 & 7 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 80 \end{pmatrix} = \begin{pmatrix} 1600 \\ 160 \\ 1760 \end{pmatrix}.$$

3.

The costs of the raw material, denoted by C_R, will be calculated as follows:

$$C_R = \begin{pmatrix} 2 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1600 \\ 160 \\ 1760 \end{pmatrix} = 12640 \text{ €}.$$

<u>D. 1. 9</u> (*Determinant of a Matrix*)

The determinant of a (square) matrix $A := (a_{ij}), i, j = 1, 2, ..., n$, denoted by $\det(A)$, will be defined as follows:

$$\det(A) := \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det\left(A_{ij}\right)$$

(cofactor expansion of A by the $i-$th row)

or

$$\det(A) := \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det\left(A_{ij}\right)$$

(cofactor expansion of A by the $j-$th column)

Here is A_{ij} a *submatrix* of A obtained by eliminating the $i-$th row and the $j-$th column. $(-1)^{i+j}.A_{ij}$ are called *adjoints* or *adjuncts*.

For $n = 1$ we obtain:

$$\det(a) := a$$

For $n = 2$ we obtain:

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := a_{11} \cdot a_{22} - a_{12}.a_{21}$$

<u>Ex. 1. 9.</u>

Find the determinant of the matrix

$$A := \begin{pmatrix} 1 & 2 & -3 \\ -2 & -1 & 8 \\ 5 & 17 & -11 \end{pmatrix}$$

Solution:

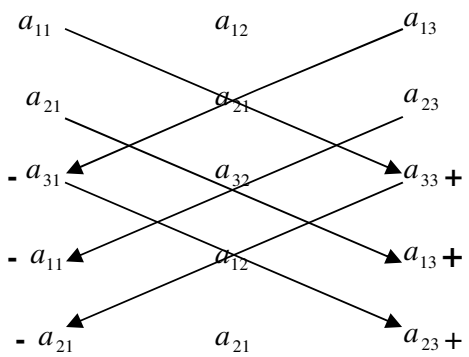
Let us expand the matrix according to the second row. We obtain:

$$\det(A) = (-1)^{2+1} \cdot (-2) \cdot \det\begin{pmatrix} 2 & -3 \\ 17 & -11 \end{pmatrix} + (-1)^{2+2} \cdot (-1) \cdot \det\begin{pmatrix} 1 & -3 \\ 5 & -11 \end{pmatrix} + (-1)^{2+3} \cdot 8 \cdot \det\begin{pmatrix} 1 & 2 \\ 5 & 17 \end{pmatrix}$$

$$= 2 \cdot (2 \cdot (-11) - (-3) \cdot 17) - (1 \cdot (-11) - (-3) \cdot 5) - 8 \cdot (1 \cdot 17 - 2 \cdot 5)$$

$= -2$

<u>R. 1. 3.</u> (***Sarrus Scheme***)
There is a convenient scheme, the so-called Sarrus scheme to find the determinant of a three-dimensional matrix .
This is illustrated below:



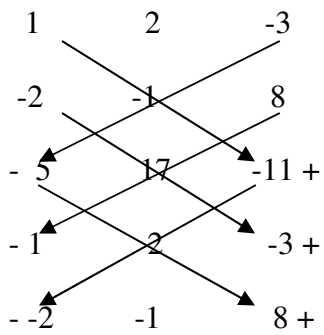
$$\det A = +(a_{11} \cdot a_{21} \cdot a_{33} + a_{21} \cdot a_{33} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23}) - \left(a_{13} \cdot a_{21} \cdot a_{31} + a_{23} \cdot a_{33} \cdot a_{11} + a_{33} \cdot a_{12} \cdot a_{21}\right)$$

<u>Ex. 1. 11.</u>
Find the determinant of the matrix

$$A := \begin{pmatrix} 1 & 2 & -3 \\ -2 & -1 & 8 \\ 5 & 17 & -11 \end{pmatrix}$$

by applying the Sarrus scheme.

Solution:



$$\det(A) = (11 + 102 + 80) - (15 + 136 + 44) = -2$$

R. 1. 4. (***Properties of Determinants***)

1. If a row (or column) of a matrix is multiplied by a constant c, its determinant is also multiplied by c.
2. If a multiple of one row (or column) is added to another the determinant is unchanged.
3. If two rows are interchanged, the determinant is multiplied by -1.
4. $\det I = 1$.
5. $\det A.B = \det A + \det B$.

D. 1. 10 (***Singular and Regular Matrices***)

The matrix $A := (a_{ij}), i, j = 1, 2, ..., n$ is called *singular*, if $\det(A) = 0$; otherwise it is called *regular*.

Ex. 1. 12.

It can be shown that the matrix

$$A := \begin{pmatrix} 1 & 2 & -3 \\ -2 & -1 & 8 \\ 0 & 3 & 2 \end{pmatrix}$$

is singular (show it!)

D. 1. 11 (***Inverse of a Matrix***)

Let $A := (a_{ij}), i, j = 1, 2, ..., n$ be a given matrix. If there is a matrix $B := (b_{ij}), i, j = 1, 2, ..., n$ such that

$$A \cdot B = B \cdot A = I$$

then B is called the *inverse* of A, written A^{-1}. A is then said to be *invertible*.

T. 1. 1.

Let $A := (a_{ij}), i, j = 1, 2, ..., n$ be a regular matrix. Then

$$A^{-1} = \frac{1}{\det(A)} A_{adj}^{T},$$

$$A_{adj} := ((-1)^{i+j} \cdot \det(A_{ij})) .$$

Ex. 1. 13.

Invert the following matrix:

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 2 \\ 1 & 5 & 3 \end{pmatrix}.$$

Solution:

$$\det A = -28\,,$$

$$A_{11} = -7, \qquad A_{12} = -7, \qquad A_{13} = 14,$$

$$A_{21} = -7, \qquad A_{22} = 5, \qquad A_{23} = -6,$$

$$A_{31} = 7, \qquad A_{32} = -1, \qquad A_{33} = -10$$

$$A^{-1} = -\frac{1}{28}\begin{pmatrix} -7 & -7 & 7 \\ -7 & 5 & -1 \\ 14 & -6 & -10 \end{pmatrix}.$$

<u>R. 1. 5.</u>

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

<u>D. 1. 12.</u>(*Trace of a Matrix)*
The *trace* of a (square) matrix is the sum of the main diagonal elements:

$$trA = \sum_{i=1}^{n} a_{ii}\ .$$

<u>Ex. 1. 14.</u>

$$trA := \begin{pmatrix} 1 & 2 & -3 \\ -2 & -1 & 8 \\ 5 & 17 & -11 \end{pmatrix} = 1 - 1 - 11 = -11.$$

<u>R. 1. 5.</u>

1. $trA \cdot B = trB \cdot A$
2. $trA \cdot B \cdot C = trC \cdot A \cdot B = trB \cdot C.A$

(Last updated: 15.04.22)