I

Random Events and Events Algebra

D. 1. 1. (***Random Trial***)
A trial whose outcome cannot be predicted in advance is called a *random trial*.

D. 1. 2. (***Random Event***)
The outcome of a random trial is called a *random event*.

Ex. 1. 1.
Random trial: Rolling an unbiased die

Random events: $j = 1, 2, ..., 6$: Number appearing above

Ex. 1. 2.
Random trial: Quality control

Random events: $k = 0, 1, ...$: Number of defective products

Ex. 1. 3.
Random trial: Examining the lifetime of a certain kind of tyre

Random events: $t \in [0, \quad +\infty[$: lifetime of a certain kind of tyre.

R. 1. 1.
The algebra of events is the application of the set theory to random events.

D. 1. 3. (***Impossible and Certain Events***)
An event that in all repetitions of a certain random trial will never happen is called an *impossible event*. It will be denoted by $\varnothing$.
An event that in all repetitions of a certain event will always occur is called a *certain event*. It will be denoted by Ω.

Ex. 1. 4.
Random trial: Rolling an unbiased die.
The random event "Throw the number 8" or $\{8\}$ is an impossible event.
The random event "Throw at most the number 6"or $\{1, 2, 3, 4, 5, 6\}$ is a certain event.

D. 1. 4. (***Subevent***)
The event E_1 is called a *subevent of the event* E_2 if it always accompanies the event E_2.
We write

$$E_1 \subseteq E_2.$$

<u>Ex. 1. 5.</u>
Random trial: Rolling an unbiased die.

Let

E_1**:** "throw an odd number",
E_2**:** "throw at most 5".

We have:

$$E_1 = \{1,\ 3,\ 5\} \subseteq \{1, 2, 3, 4, 5\} = E_2.$$

<u>D. 1. 5.</u> (*Equivalent Events*)

$$E_1 \coloneqq E_2 \quad \Leftrightarrow \quad (E_1 \subseteq E_2 \ \wedge \ E_2 \subseteq E_1)$$

<u>D. 1. 6.</u> (*Sum of Events*)
E is said to be the *sum* of the events $E_i, i = 1, 2, \ldots, n$, if <u>at least</u> one of the events E_i occurs:

$$E \coloneqq \bigcup_{i=1}^{n} E_i$$

<u>Ex. 1. 6.</u>
Random trial: Rolling an unbiased die.

Let

E_1**:** "throw either 2 or 4",
E_2**:** "throw either 2 or 6".

We have:

$$E_1 = \{2,\ 4\}, \qquad E_2 = \{2,\ 6\},$$

$$E = E_1 \cup E_2 = \{2,\ 4, 6\}$$

<u>D. 1. 7.</u> (*Product of Events*)
E is said to be the *product* of the events $E_i, i = 1, 2, \ldots, n$, if the events E_i occurs <u>at the same time:</u>

$$E \coloneqq \bigcap_{i=1}^{n} E_i$$

<u>Ex. 1. 7.</u>
Random trial: Rolling an unbiased die.

Let

E_1: "throw 1 or 4",
E_2: "throw a prime number".

We have:

$$E_1 = \{1,\ 4\}, \qquad E_2 = \{1, 2, 3, 5\},$$

$$E = E_1 \cap E_2 = \{1\}$$

<u>D. 1. 8.</u> (*Mutually exclusive events*)
The events E_1 and E_2 are said to be *mutually exclusive* if

$$E_1 \cap E_2 = \varnothing$$

<u>Ex. 1. 8.</u>
Random trial: Rolling an unbiased die.

Let

E_1: "throw an even number",
E_2: "throw an odd number".

We have

$$E_1 \cap E_2 = \varnothing$$

<u>D. 1. 9.</u> (*Complementary Events*)

The events E and $\bar{E}$ are said to be *complementary* if

$$E \cup \bar{E} = \Omega \quad \wedge \quad E \cap \bar{E} = \varnothing$$

<u>Ex. 1. 9.</u>
Random trial: Rolling an unbiased die.

Let

E_1: "throw an even number",
E_2: "throw an odd number".

We have:

$$E_1 = \{2, 4,\ 6\}, \qquad E_2 = \{1, 2, 3\},$$

$$E_1 \cup E_2 = \{1, 2, 3, 4, 5, 6\} = \Omega \ , \qquad E_1 \cap E_2 = \varnothing$$

<u>D. 1. 10.</u> (*Difference*)
The *difference* of the events E_1 and E_2, denoted by $E_1 \setminus E_2$ is defined as the case in which E_1 occurs while E_2 does not occur.

<u>Ex. 1. 10.</u>
Random trial: Rolling an unbiased die.

Let

E_1**:** "throw either 1 or 4",
E_2**:** "throw the number 4".

We have:

$$E_1 = \{1, 4\}, \qquad E_2 = \{1\},$$

$$E_1 \setminus E_2 = \{4\}$$

<u>T. 1. 1.</u>
Let $E_i, i = 1, 2, \ldots, n$, be random events. Then we have:

The Commutative Laws:

$$E_1 \cup E_2 = E_2 \cup E_1$$
$$E_1 \cap E_2 = E_2 \cap E_1.$$

The Associative Laws:

$$E_1 \cup (E_2 \cup E_3) = (E_2 \cup E_1) \cup E_3 = E_1 \cup E_2 \cup E_3$$
$$E_1 \cap (E_2 \cap E_3) = (E_2 \cap E_1) \cap E_3 = E_1 \cap E_2 \cap E_3$$

The Distributive Laws:

$$E_1 \cup (E_2 \cap E_3) = (E_1 \cup E_2) \cap (E_1 \cup E_3)$$
$$E_1 \cap (E_2 \cup E_3) = (E_1 \cap E_2) \cup (E_1 \cap E_3).$$

Further we have:

$$E \cup E = E, \qquad E \cap E = E$$

$$E \cup \bar{E} = \Omega \qquad E \cap \bar{E} = \varnothing$$

$$E \cup \varnothing = E \qquad\qquad E \cap \varnothing = \varnothing$$

$$E \cup \Omega = \Omega \qquad\qquad E \cap \Omega = E$$

De Morgan Laws.

$$\overline{\bigcap_{i=1}^{n} E} = \bigcup_{i=1}^{n} \bar{E}_i \,,$$

$$\overline{\bigcup_{i=1}^{n} E_i} = \bigcap_{i=1}^{n} \bar{E}_i \,.$$

<u>Ex. 1. 11</u>.
A factory consists of three departments D_1, D_2, D_3. Let

$E_i,\ i = 1, 2, 3$: "There is <u>no</u> disturbance in D_i.

Describe the following events:

A: "There is no disturbance in the three departments:"

B: "There is disturbance in all departments."
C: "There is no disturbance in at least one department."

D: "There is disturbance in at least one department."

E: "There is disturbance in at most one department."

F: "There is disturbance only in department D_3 "
G: "There is disturbance in department D_3."

Solution:

$$A = E_1 \cap E_2 \cap E_3$$

$$B = \bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3 = \overline{E_1 \cup E_2 \cup E_3}$$

$$C = E_1 \cup E_2 \cup E_3$$

$$= (E_1 \cap E_2 \cap E_3)$$

$$\cup (E_1 \cap E_2 \cap \bar{E}_3) \cup (E_1 \cap \bar{E}_2 \cap E_3) \cup (\bar{E}_1 \cap E_2 \cap E_3)$$

$$\cup(E_1\cap\bar{E}_2\cap\bar{E}_3)\cup(\bar{E}_1\cap\bar{E}_2\cap E_3)\cup(\bar{E}_1\cap E_2\cap\bar{E}_3)$$

$$=\Omega\setminus(\bar{E}_1\cap\bar{E}_2\cap\bar{E}_3)$$

$$D=\bar{E}_1\cup\bar{E}_2\cup\bar{E}_3$$

$$E=(E_1\cap E_2\cap E_3)\cup(\bar{E}_1\cap E_2\cap E_3)\cup(E_1\cap\bar{E}_2\cap E_3)\cup(E_1\cap E_2\cap\bar{E}_3)$$

$$F=E_1\cap E_2\cap\bar{E}_3$$

$$G=(E_1\cap E_2\cap\bar{E}_3)\cup(\bar{E}_1\cap E_2\cap\bar{E}_3)\cup(E_1\cap\bar{E}_2\cap\bar{E}_3)\cup(\bar{E}_1\cap\bar{E}_2\cap\bar{E}_3)$$

D. 1. 11. (*System of Mutually Exclusive and Exhaustive Events*)
The events E_i, $i=1,2,\dots,n$, form a *mutually exclusive and exhaustive* system if the following conditions are fulfilled:

1. $E_i\neq\varnothing,\qquad i=1,2,\dots,n$
2. $\bigcup_{i=1}^{n}E_i=\Omega$
3. $E_i\cap E_j=\varnothing,\quad i\neq j,\quad i,j=1,2,\dots,n$

Ex. 1. 12.
Determine if the following events form a *mutually exclusive and exhaustive* system:

1. Random trial: Rolling an unbiased die

E_1 : : „throw 1 or 4“
E_2 : „throw an odd number > 2“
E_3 : “throw an even number $\neq 4$“.

$E_1=\{1,\ 4\},\quad E_2=\{3,\ 5\},\quad E_3=\{2,\ 6\}.$

The events form a *mutually exclusive and exhaustive* system, since

$$E_i\neq\varnothing,\quad i=1,2,3$$

$$E_1\cup E_2\cup E_3=\Omega$$

$$E_1\cap E_2=\varnothing,\quad E_1\cap E_3=\varnothing,\qquad E_2\cap E_3=\varnothing$$

2. Random trial: Rolling three unbiased dice.

E_1 : : „throw at least 17“

E_2 : „throw at most 5“

$E_1 = \{17,\ 18\},\ E_2 = \{3,\ 4,\ 5\}.$

The events do not form a *mutually exclusive and exhaustive* system, since among other things

$$E_1 \cup E_2 \neq \Omega$$

<u>D. 1. 12.</u> (*Field of Events*)
The *field of events* F is a set with the following properties:

1. $\varnothing\ ,\Omega \in F$
2. $E_1, E_2 \in F \quad \Rightarrow \quad (E_1 \cup E_2 \in F)\ \wedge\ (E_1 \cap E_2 \in F)$
3. $E \in F\ \Rightarrow\ \bar{E} \in F$
4. $E_i \in F,\ i = 1, 2, ...\ \Rightarrow\ \bigcup\limits_i E_i \in F\ \wedge\ \bigcap\limits_{\mathrm{i}} E_i \in F$

<u>Ex. 1. 13</u>.
Random trial: Rolling an unbiased die. Let

E_1 : : „throw an odd number“

E_2 : „throw an even number“

$F = \ \{ E_1, E_2, \Omega, \varnothing \}$

Ω : „throw either 1 or 2 or…6”

$\varnothing$: „throw neither of the numbers 1, 2, ..., 6”

<u>D. 1. 13.</u> (*Elementary or Atomic Event*)
An *elementary* or *atomic event* is an event which is not further genuinely decomposable.

<u>Ex. 1. 14</u>.
Random trial: Rolling an unbiased die. Let

E_1 : : „throw an odd number“

E_2 : „throw a 6“.

E_1 is decomposable: $E_1 = \{1, 3, 5\} = \{1\} \cup \{3\} \cup \{5\}$, E_2 is not.

(Last updated: 10.05.2009)

II

Introduction to Probability

D. 2. 1. (***Absolute and Relative Frequency***)
The *absolute frequency of an event* E in n trials, denoted by $F(E)$**,** is the number of cases in which it occurs.
The *relative frequency of an event* E in n trials is defined as:

$$f(E) := \frac{F(E)}{n}.$$

Ex. 2. 1.
A symmetrical coin will be tossed. We denote the two possible outcomes of the experiment H (for head) and T (for tail).
If we repeat the experiment 35 times and the coin lands 21 times heads-up, then we have

$$F(H) = 21, \quad f(H) = \frac{21}{35} = 0.6.$$

Ex. 2. 2. (***A historical example)***
The following table shows the results of an experiment with tossing a coin and observing the event H (for head):

Experimented by	n	$F(H)$	$f(H)$
Buffon	4040	2048	0.5069
K. Pearson	12000	6019	0.5016
K. Pearson	24000	12012	0.5005

It can be seen that with the increasing number of trial the relative frequencies will be stabilised about the number 0.5.

D. 2. 2. (***Statistical or Empirical Definition of Probability***)
Let the probability of the event E be denoted by $P(E)$. Then

$$P(E) := \lim_{n\to\infty} \frac{F(E)}{n}.$$

E. 2. 3.
On February 1, 2003, the Space shuttle Columbia exploded. This was the second disaster in 113 space missions for NASA.
On the basis of this information, what is the probability that a future mission is successfully completed?

Solution:

$$\text{Probability of successful flights} \approx \frac{111}{113} \approx 0.98.$$

<u>D. 2. 3.</u> (*Classical Definition of Probability*)

$$P(E) := \frac{k}{n}.$$

Here are:

k : number of favourable outcomes,
n : total number of possible outcomes.

<u>Assumption:</u> All outcomes are equally likely.

<u>Ex. 2. 4.</u>
A die will be thrown. Calculate the probabilities for the following events:

A : „An even umber will be thrown".
B : „At least a 3 will be thrown".
C : „At most a 5 will be thrown".

Solution:

$$P(A) = \frac{3}{6} = 0.5, \qquad P(B) = \frac{4}{6} \approx 0.67, \qquad P(C) = \frac{5}{6} \approx 0.83.$$

<u>D. 2. 4.</u> (*Subjective Probability*)
A probability that is derived from an individual's personal judgement about how likely is an event to occur is called *subjective probability*.

<u>Ex. 2. 5.</u>
The probability that a certain country will win the next football championship.

<u>D. 2. 5.</u> (*Axiomatic Definition of Probability- Kolmogorov*)
<u>Axiom 1</u>:
Let $P(E)$ be a unique mapping of $E \in F$ into $[0, 1]$ called *probability*.

<u>Axiom 2</u>:

$$P(\Omega) = 1.$$

<u>Axiom 3</u>:

$$\left\langle E_i \cap E_k = \varnothing \quad , \ i \neq k,\ i,k = 1,\ 2,\dots,n \right\rangle \quad \Rightarrow \quad \left\langle P\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} P\left(E_j\right) \right\rangle .$$

T. 2. 1.

$$\langle E_1 = E_2 \rangle \quad \Rightarrow \quad \langle P(E_1) = P(E_2) \rangle$$

T. 2. 2.

$$P(\bar{E}) = 1 - P(\bar{E}) .$$

Proof:

$$P(E \cup \bar{E}) = P(E) + P(\bar{E}) \qquad (\because\ E \cap \bar{E} = \varnothing,\ \text{axiom 3})$$

$$1 = P(E) + P(\bar{E}) \qquad (\because\ E \cup \bar{E} = \Omega,\ \text{axiom 2})$$

T. 2. 3.

$$P(\varnothing) = 0 .$$

Proof:

$$\begin{aligned} P(\varnothing) &= 1 - P(\Omega) && (\because\ T.2.2 \text{ with } E = \Omega) \\ &= 0 && (\because\ \text{axiom 2}) \end{aligned}$$

T. 2. 4.

$$\langle E_1 \subseteq E_2 \rangle \quad \Rightarrow \quad \langle P(E_1) \le P(E_2) \rangle .$$

T. 2. 5.

Let $E_j,\ j = 1,\ 2, \ldots, n$ be mutually exclusive and exhaustive events. Then

$$\sum_{j=1}^{n} P(E_j) = 1 .$$

Proof:

$$P\left(\bigcup_{j=1}^{n} E_j \right) = \sum_{j=1}^{n} P\left(E_j\right) \qquad (\because\ \text{axiom 3})$$

$$P(\Omega) = \sum_{j=1}^{n} P\left(E_j\right)$$

$$1 = \sum_{j=1}^{n} P\left(E_j\right) \qquad (\because\ \text{axiom 2})$$

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Chapter III

Probability Algebra

<u>T. 3. 1.</u> (*Addition Rule*)
Let A and B be two *arbitrary* events. Then, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

<u>Ex. 3. 1.</u>
At a certain university, men engage in various sports in the following proportions:

Soccer (A): 60%
Basketball (B): 50%
Both soccer and basketball: 30%

If a man is selected at random for an interview, what is the chance that he will

1. play soccer or basketball?
2. play neither sport?

Solution:

1.
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.60 + 0.50 - 0.30 \\ &= 0.80 \end{aligned}$$

2.
$$\begin{aligned} 1 - P(A \cup B) &= 1 - 0.80 \\ &= 0.20 \end{aligned}$$

<u>R. 3. 1.</u>
Axiom III of the axiomatic definition of probability is a special case of the addition theorem for two mutually exclusive events.

<u>D. 3. 1.</u> (*Conditional Probability*)
Let $A, B \in S$. The *conditional probability* of A given B is:

$$P(A/B) := \begin{cases} \dfrac{P(A \cap B)}{P(B)} & \text{if} \quad P(B) > 0 \\ 0 & \text{if} \quad P(B) = 0 \end{cases}$$

<u>Ex. 3. 2.</u>
A math lecturer gave his group two tests. 25% of the group passed both tests and 42% passed the first test.
What percent of those who passed the first test passed also the second test?

Solution:

Let

A : „The group passed the second test. "

B : „The group passed the first test. "

We have:

$$P(B) = 0.42\,, \qquad P(A \cap B) = 0.25\,.$$

Therefore

$$P(A/B) = \frac{0.25}{0.42} \approx 0.60$$

R. 3. 2.
Similarly, we define the conditional probability of B given A **:**

$$P(B/A) := \begin{cases} \dfrac{P(B \cap A)}{P(A)} & \text{if} \quad P(A) > 0 \\ 0 & \text{if} \quad P(A) = 0 \end{cases}$$

R. 3. 3.
Since $P(A \cap B) = P(B \cap A)$, we have

$$P(A) \cdot P(A/B) = P(B) \cdot P(B/A)\,.$$

T. 3. 2. (*Multiplication Rule for Two Events*)
Let $A, B \in S$ be two *arbitrary* events. Then

$$P(A \cap B) = P(B) \cdot P(A/B)$$

$$= P(A) \cdot P(B/A)$$

Ex. 3. 3.
Three defective light bulbs inadvertently got mixed with 6 good ones. If two bulbs are chosen at random for a ceiling lamp, what is the probability that they both are good?

Solution:

A : „Second bulb is good. "

B : „First bulb is good. "

$$P(A \cap B) = \frac{6}{9} \cdot \frac{5}{8} \approx 0.42$$

R. 3. 3.
The multiplication rule can also be extended to *n arbitrary events* $E_i, i = 1, 2, ..., n$:

$$P(\bigcap_{i=1}^{n} E_i) = P(E_1) \cdot P(E_2 / E_1) \cdot P(E_3 / E_1 \cap E_2) ... P(E_n / \bigcap_{i=1}^{n-1} E_i) .$$

D. 3. 2. **(*Independent Events*)**
The event A is said to be *independent* of B if

$$P(A) = P(A / B) .$$

Otherwise, A is said to be *dependent* of B .

T. 3. 2.

$$\langle A \text{ is independen t of } B \rangle \Rightarrow \langle B \text{ is independen t of } A \rangle .$$

Proof:

We have

$$P(B / A) = \frac{P(A / B) \cdot P(B)}{P(A)}$$
$$= \frac{P(A) \cdot P(B)}{P(A)}$$
$$= P(B)$$

R. 3. 3.
Based on the last result, we usually speak of two *independent events*.

Ex. 3. 4.
Consider choosing a card from a well-shuffled Skat deck of 32 playing cards. Let:

A : „The chosen card is an ace. "

B : „The chosen card is a diamond card. "

C : „The chosen card is ace of diamonds. "

Determine if the following pairs of events are independent:

1. (A, B)
2. (C, A)

Solution:
1.

$$P(A) = \frac{1}{8} = P(A / B)$$

Therefore, the events A, B are independent.

2.

$$P(C) = \frac{1}{32} \neq \frac{1}{4} P(C / A)$$

Therefore, the events A, C are dependent.

<u>T. 3. 3.</u> (*Multiplication Rule for Two Independent Events*)
Let $A, B \in S$ be two *independent* events. Then

$$P(A \cap B) = P(A) \cdot P(B)$$

Proof:

$$\begin{aligned} P(A \cap B) &= P(B) \cdot P(A / B) \\ &= P(A) \cdot P(B) \end{aligned}$$

<u>R. 3. 4.</u>
The notion independence can also be extended to $n > 2$ events.

<u>T. 3. 4.</u> (*Multiplication Rule for Independent Events*)
Let $E_i, i = 1, 2, ..., n$, be *independent* events. Then

$$P\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} P(E_i).$$

<u>Ex. 3. 5.</u>
Consider three machines $M_i, i = 1, 2, 3,$ with the reliabilities

M_1: 0.9, M_2: 0.8, M_3: 0.85.

Assuming that the three machines work independently of one another, calculate the probability that

1. no machine will fall out.
2. all machines will fall out.

Solution:
Let

E_i : „Machine M_i will <u>not</u> fall out”, $i = 1, 2, 3$.

1.

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2) \cdot P(E_3)$$

$$= 0.9 \cdot 0.8 \cdot 0.85 = 0.612$$

2.

$$P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3) = P(\bar{E}_1) \cdot P(\bar{E}_2) \cdot P(\bar{E}_3)$$

$$= (1-0.9) \cdot (1-0.8) \cdot (1-0.85) = 0.003$$

T. 3. 5.
Let $E_i, i = 1, 2, ..., n$, be *independent* events. Then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = 1 - \prod_{i=1}^{n} \left(1 - P(E_i)\right)$$

Ex. 3. 5. (*continued*)

3. at least one of the machines does not fall out.
4. at least one of the machines falls out.

Solution:

3. $P(E_1 \cup E_2 \cup E_3) = 1 - (1 - P(E_1)) \cdot (1 - P(E_2)) \cdot (1 - P(E_3))$

$$= 1 - (1-0.9) \cdot (1-0.8) \cdot (1-0.85)$$

$$= 1 - 0.003 = 0.997$$

4. $P(\bar{E}_1 \cup \bar{E}_2 \cup \bar{E}_3) = 1 - (1 - P(\bar{E}_1)) \cdot (1 - P(\bar{E}_2)) \cdot (1 - P(\bar{E}_3))$

$$= 1 - (1-0.1) \cdot (1-0.2) \cdot (1-0.15)$$

$$= 1 - 0.612 = 0.388$$

Ex. 3. 6.
The probability to hit a plane by a single shot is equal to 0.004. What is the chance of bringing down a plane by 250 simultaneous shots. It will be assumed that the plane will be brought down if it is hit at least once.

Solution:
Let

E_i, $i = 1, 2, ..., 250$: "hit the plane by the i – th shot".

$$P\left(\bigcup_{i=1}^{250} E_i\right) = 1 - \prod_{i=1}^{250} (1 - P(E_i))$$

$$= 1 - (1 - 0.004)^{250}$$

$$= 1 - 0.367142 = 0.632858$$

<u>T. 3. 6.</u> (*Total probability*)

Let $B_i, i = 1, 2, ..., n$, form a group of mutually exclusive and exhaustive events. For an arbitrary event *A* we have

$$P(A) = \sum_{i=1}^{n} P(A / B_i) \cdot P(B_i) .$$

Proof:

$$A = \Omega \cap A$$

$$= \left(\bigcup_{i=}^{n} B_i \right) \cap A$$

$$= \bigcup_{i=1}^{n} (B_i \cap A)$$

$$P(A) = P\left(\bigcup_{i=1}^{n} (B_i \cap A) \right)$$

$$= \sum_{i=1}^{n} P(B_i \cap A)$$

$$= \sum_{i=1}^{n} P(A / B_i) \cdot P(B_i) .$$

<u>Ex. 3. 7.</u>

Consider three groups of jars.
The first group consists of three jars each containing 2 white and 3 red marbles.
The second group consists of 2 jars each containing 4 white and 1 red marbles
The third group consists of only 1 jar containing 0 white and 8 red marbles.
A jar will be chosen at random and a marble taken out.

1. What is the chance of having chosen a white marble?
2. What is the chance of having chosen a red marble?

Solution:

Let

A: „a white marble will be chosen",

$B_i, \; i = 1, 2, 3$: „the jar chosen belongs to the group i".

1.

$$P(A) = \sum_{i=1}^{3} P(A / B_i) \cdot P(B_i)$$
$$= \frac{3}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{4}{5} + \frac{1}{6} \cdot 0 = \frac{7}{15}$$

2.

$$P(\bar{A}) = \sum_{i=1}^{3} P(\bar{A}/B_i)\cdot P(B_i)$$
$$= \frac{3}{6}\cdot\frac{3}{5}+\frac{2}{6}\cdot\frac{1}{5}+\frac{1}{6}\cdot 1 = \frac{8}{15}$$

<u>T. 3. 7.</u> (*Bayes*)
Let $B_i, i = 1, 2, ..., n$, form a group of mutually exclusive and exhaustive events. For an arbitrary event $A \neq \varnothing$ we have:

$$P(B_i / A) = \frac{P(A/B_i)\cdot P(B_i)}{\sum_{i=1}^{n} P(A/B_i)\cdot P(B_i)}, \qquad i = 1, 2, ..., n .$$

Proof:

$$P(B_i \cap A) = P(A)\cdot P(B_i / A) = P(B_i)\cdot P(A/B_i)$$

$$P(B_i / A) = \frac{P(B_i)\cdot P(A/B_i)}{P(A)}$$
$$= \frac{P(A/B_i)\cdot P(B_i)}{\sum_{i=1}^{n} P(A/B_i)\cdot P(B_i)}, \qquad i = 1, 2, ..., n .$$

<u>Ex. 3. 8.</u>
A factory produces a product on three machines. The first machine produces 25%, the second 35% and the third 40% of the total production. Experience shows that 5% of the products produced on the first, 4% on the second and 2% on the third machine are defective. Determine the probability that a randomly selected defective product has been produced on

1. the first
2. the second
3. the third

machine.

Solution:
Let

A: „a product is defective.”,

B_i: „a product has been produced on the machine i" .

We have:

$P(B_1) = 0.25$, $P(B_2) = 0.35$, $P(B_3) = 0.40$,

$P(A/B_1) = 0.05$, $P(A/B_2) = 0.04$, $P(A/B_3) = 0.02$

1.

$$P(B_1 / A) = \frac{0.05 \cdot 0.25}{0.05 \cdot 0.25 + 0.04 \cdot 0.35 \cdot 0.02 \cdot 0.40} = \frac{25}{69},$$

2.

$$P(B_2 / A) = \frac{0.04 \cdot 0.35}{0.05 \cdot 0.25 + 0.04 \cdot 0.35 \cdot 0.02 \cdot 0.40} = \frac{28}{69},$$

3.

$$P(B_3 / A) = \frac{0.02 \cdot 0.35}{0.05 \cdot 0.25 + 0.04 \cdot 0.35 \cdot 0.02 \cdot 0.40} = \frac{16}{69}.$$

R. 3. 5.
In Bayes Theorem, $P(A/B_i)$ are called prior *probabilities* and posterior $P(B_i / A)$ *probabilities*.

R. 3. 6. (*Contingency Tables*)
A *contingency table* is a table used to classify sample observations according to two or more identifiable characteristics.

Ex. 3. 9.
A sample of executives was surveyed about their loyalty to their company. One of the questions was "If you were given an offer by another company equal or slightly better than your present position, would you remain with the company or take the other position?"
The responses of the 200 executives in the survey were cross-classified with their length of service with the company:

	Length of Service				
Loyalty	Less than 1 year B_1	1 – 5 years B_2	6 – 10 years B_3	More than 10 years B_4	Total
Would remain A_1	10	30	5	75	120
Would not remain A_2	25	15	10	30	80
	35	45	15	105	200

What is the probability of randomly selecting an executive who is loyal to the company and who has more than 10 years of service?

Solution:

$$P(B_1 \cap A_4) = P(A_1) \cdot P(B_4 / A_1) = \frac{120}{200} \cdot \frac{75}{120} = \frac{75}{200} = 0.375.$$

T. 3. 8.
Consider a finite set of N elements, $M \le N$ elements of which have a certain property. Let us choose a sample of $n \le N$ elements.
The probability that the sample contains $m \le n$ elements with the above-mentioned property is in case of

1. *nonreplacement:*

$$P_{noplacement} = \frac{\binom{M}{m} \cdot \binom{N-M}{n-m}}{\binom{N}{n}}$$

2. *replacement*

$$P_{nreplacet} = \binom{n}{m} \cdot p^m \cdot q^{n-m},$$

$$\frac{M}{N} =: p, \qquad q := 1 - p$$

Ex. 3. 10.
A box contains 25 items, 10 of which are defective. A sample of two items will be taken.

1. without replacement
2. with replacement.

Determine the probability that

i) both items are good
ii) exactly one item is defective
iii) both items are defective.

Solution:

$$N = 25, \qquad M = 10, \qquad n = 2, \qquad p = \frac{10}{25} = \frac{2}{5}$$

		1.	2.
i)	$m = 0$	$P = \frac{7}{20}$	$P = \frac{9}{25}$
ii)	$m = 1$	$P = \frac{1}{2}$	$P = \frac{12}{25}$
iii)	$m = 2$	$P = \frac{3}{20}$	$P = \frac{4}{25}$

(Last revised: 08.04.09)

Chapter IV

Random Variables

<u>D. 4. 1.</u> (*Random Variable*)
A *random variable* is a function defined on the sample space.

$$X = X(E), \quad E \in S$$

<u>Ex. 4. 1.</u> (see Ex. 3.9.)
A box contains 25 items, 10 of which are defective. A sample of two items will be taken, without replacement:
Let

X : „number of defective items".

E_1 : „both items are good" $\Rightarrow$ $X = 0$

E_2 : „exactly one item is defective" $\Rightarrow$ $X = 1$

E_3 : „both items are defective" $\Rightarrow$ $X = 2$

<u>Ex. 4. 2.</u>
Let

X : „temperature on 01.01.2005"
E : $[0, \quad 24]$: time

<u>D. 4. 2.</u> (*Discrete and Continuous Random Variables*)

1. A random variable is called a *discrete random variable* if it takes only a finite or countably infinite number of values. (See ex. 4. 1.).
2. A random variable is called a *continuous random variable* if it takes every real value within an interval. (See Ex. 4. 2.).

<u>Ex. 4. 3.</u>
1. Examples of *discrete* random variables:

- The number of cars entering a carwash an hour.
- The number of home mortgages approved by a bank.

2. Examples of *continuous* random variables:

- The time it takes an executive to drive to work.
- The length of time of a particular phone call.

D. 4. 3. (***Distribution Function***)
A *distribution function* is defined as:

$$F(X) = P(X < x), \quad x \in R^1$$

Ex. 4. 4. (See Ex. 3. 9 and Ex. 4. 1.)
We had :

$$P(X = 0) = \frac{7}{20},$$

$$P(X = 1) = \frac{10}{20},$$

$$P(X = 2) = \frac{3}{20}.$$

Therefore, we have

$$F(0) = P(X < 0) = 0,$$
$$F(1) = P(X < 1) = P(X = 0) = \frac{7}{20},$$
$$F(2) = P(X < 2) = P(X = 0) + P(X = 1) = \frac{7}{20} + \frac{10}{20} = \frac{17}{20},$$
$$F(3) = P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{7}{20} + \frac{10}{20}\frac{3}{23} = 1.$$

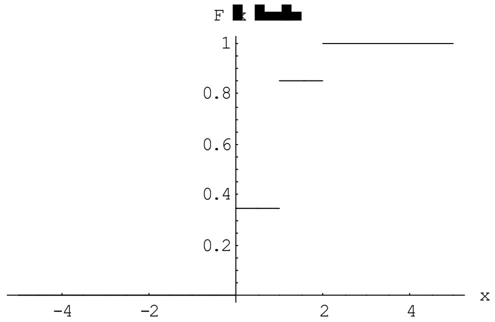
Analytic form of the distribution function:

$$F(x) = \begin{cases} 0 & \text{when} \quad -\infty < x \le 0 \\ \frac{7}{20} & \text{when} \quad 0 < x \le 1 \\ \frac{17}{20} & \text{when} \quad 1 < x \le 2 \\ 1 & \text{when} \quad 2 < x < +\infty \end{cases}$$

Tabular form of the distribution function:

x	$F(x)$
$]-\infty, \quad 0]$	0
$]0, \quad 1]$	$\frac{7}{20}$
$]1, \quad 2]$	$\frac{17}{20}$
$]2, \quad +\infty[$	1

Graphical form of the distribution function:



T. 4. 1. (***Important Properties of the Distribution Function***)

1.

$$0 \le F(x) \le 1, \quad \forall x \in R^1 .$$

2.

$$\forall x_1, x_2 : x_1 < x_2 \quad \Rightarrow \quad F(x_1) \le F(x_2) .$$

3.

$$\forall x_1, x_2 : x_1 < x_2 \quad \Rightarrow \quad P(x_1 \le X < x_2) = F(x_2) - F(x_1) .$$

4.

$$x \to -\infty \quad \Rightarrow \quad F(x) \to 0$$

$$x \to +\infty \quad \Rightarrow \quad F(x) \to 1 .$$

5.

$F(x)$ is at least left-sided continuous and has at most a finite number of jump discontinuities.

Ex. 4. 5.

Let X be the time in hours elapsed between the arrival of two ships at a certain port with the following distribution function:

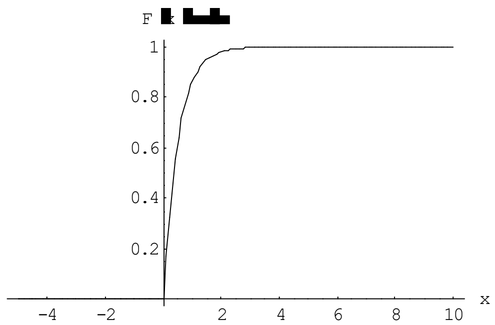
$$F(x) := \begin{cases} 0 & \text{when} \quad x \le 0 \\ 1 - e^{-2x} & \text{when} \quad x > 0 \end{cases} .$$

Find the probability that

1. the time elapsed between the arrival of two ships is less than 90 minutes.
2. 15 minutes elapse without any ship arriving.

3. the time elapsed between the arrival of two ships is at least 6 minutes but less than 30 minutes.

Solution:



1.

$$P\left(X < \frac{3}{2}\right) = F\left(\frac{3}{2}\right) = 1 - e^{-3} = 0.9502$$

2.

$$P\left(X \geq \frac{1}{4}\right) = 1 - P\left(X < \frac{1}{4}\right) = 1 - F\left(\frac{1}{4}\right) = 1 - 1 + e^{-0.5} = 0.6065$$

3.

$$P\left(\frac{1}{10} \leq X < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = e^{-0.5} - e^{-1} = 0.8187 - 0.3679 = 0.4508$$

(Last revised: 15.04.09)

Chapter V

Discrete and Continuous Random Variables

D. 5. 1. (***Probability Function***)
If X is a discrete random variable, then the function

$$p(x) = P(X = x)$$

defined on the outcomes of X is called the *probability function* of the discrete random variable X .
If X has the outcomes x_i, $i = 1, 2, ..., n$, then we can write:

x_i	x_1	x_2	$\dots$	x_n
$p_i = P(X = x_i) = f(x_i)$	p_1	p_2	$\dots$	p_n

Ex. 5. 1. (see Ex. 4. 1.)

x_i	0	1	2
$p_i = P(X = x_i)$	$\frac{7}{20}$	$\frac{10}{20}$	$\frac{3}{20}$

T. 5. 1.

$$\sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} p_i = 1 .$$

T. 5. 2.

$$F(x) = P(X < x) = \begin{cases} 0 & \text{for} \quad x \le x_1 \\ \sum_{i=1}^{k} p_i & \text{for} \quad x_k < x \le x_{k+1}, \quad k = 1,\ 2,\ ...,n-1 \\ 1 & \text{for} \quad x > x_n \end{cases}$$

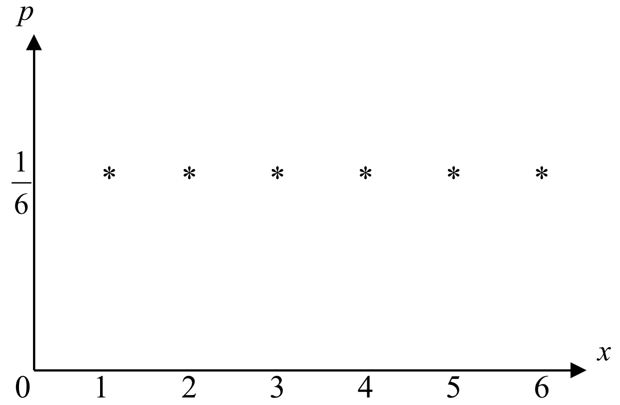
Ex. 5. 2.
A die will be thrown. Let

X : „the number appearing above“.

Probability function:
1) in tabular form:

x_i	1	2	3	4	5	6
p_i	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

2) in graphical form:



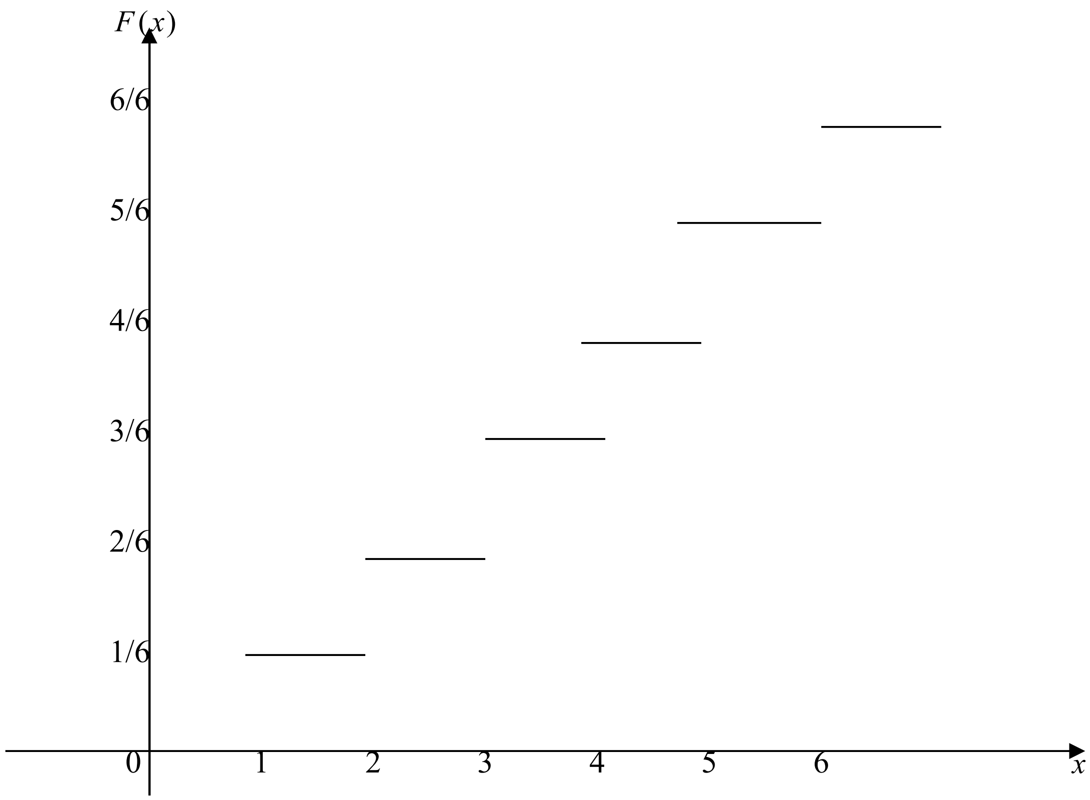
Distribution function:
1) in analytic form:

$$F(x) = \begin{cases} 0 & \text{when} \quad -\infty < x \le 1 \\ \frac{1}{6} & \text{when} \quad 1 < x \le 2 \\ \frac{2}{6} & \text{when} \quad 2 < x \le 3 \\ \frac{3}{6} & \text{when} \quad 3 < x \le 4 \\ \frac{4}{6} & \text{when} \quad 4 < x \le 5 \\ \frac{5}{6} & \text{when} \quad 5 < x \le 6 \\ 1 & \text{when} \quad 6 < x < +\infty \end{cases}$$

2) in tabular form:

x	$F(x)$
$]-\infty,\ 1]$	0
$]1,\ 2]$	$\frac{1}{6}$
$]2,\ 3]$	$\frac{2}{6}$
$]3,\ 4]$	$\frac{3}{6}$
$]4,\ 5]$	$\frac{4}{6}$
$]5,\ 6]$	$\frac{5}{6}$
$]6,\ +\infty[$	$\frac{6}{6}=1$

3) in graphical form:



D. 5. 3. (***Density Function***)
Let $F(x)$ be a differentiable distribution function of a continuous random variable X. The function

$$f(x) := F^{'}(x)$$

is called the (*probability) density function of* X.

T. 5. 3. (***Important Properties of a Density Function***)

1.

$$F(x) = P(X < x)$$

$$= P(-\infty < X < x)$$

$$= \int_{-\infty}^{x} f(t)dt \; .$$

2.

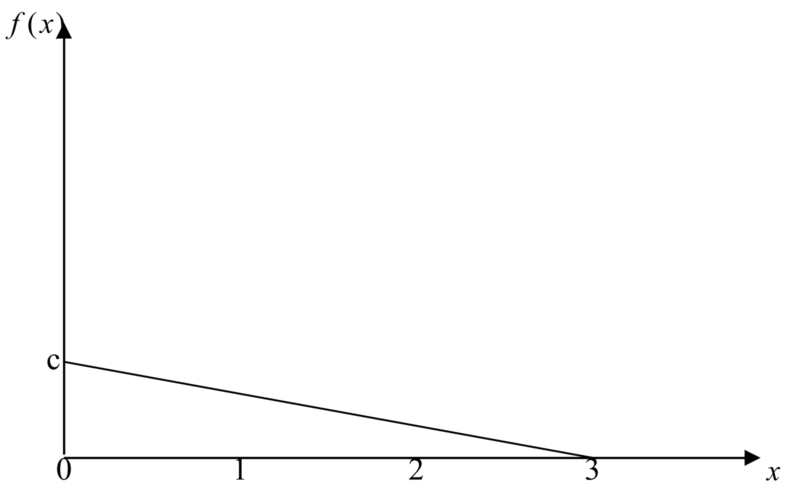
$$P(a \le X < b) = F(b) - F(a)$$

$$= \int_{a}^{b} f(x)dx \; .$$

3.

$$P(-\infty < X < +\infty) = \int_{-\infty}^{+\infty} f(x)dx = 1 \; .$$

Ex. 5. 3.
The following graph represents the probability density function of a random variable X:



1. Determine $f(0) = c$.
2. Find the analytical representation of $f(x)$.
3. Find $F(x)$.
4. Calculate $P(1.5 < X < 3)$.
5. Interpret the above results geometrically.

Solution:

$$y = ax + b$$

$$x = 0 \quad \Rightarrow \quad y = b = c$$

$$y = 0 \quad \Rightarrow \quad x = -\frac{b}{a} = -\frac{c}{a} = 3$$

$$a = -\frac{c}{3} .$$

1.

$$y = f(x) = -\frac{c}{3}x + c$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{0}^{3} f(x)dx = 1$$

$$\int_{0}^{3}\left(-\frac{c}{3}x + c \right)dx = \left[-\frac{c}{3}\cdot\frac{x^2}{2} + cx \right]_0^3$$

$$= -c \div \frac{9}{6} + 3c = -c \cdot \left(\frac{3}{2} - 3 \right) = 1$$

$$c = \frac{2}{3} .$$

2.

$$f(x) = -\frac{\frac{2}{3}}{3}\cdot x + \frac{2}{3}$$

$$f(x) = -\frac{2}{9}\cdot x + \frac{2}{3}, \qquad x \in [0, \quad 3] .$$

3.

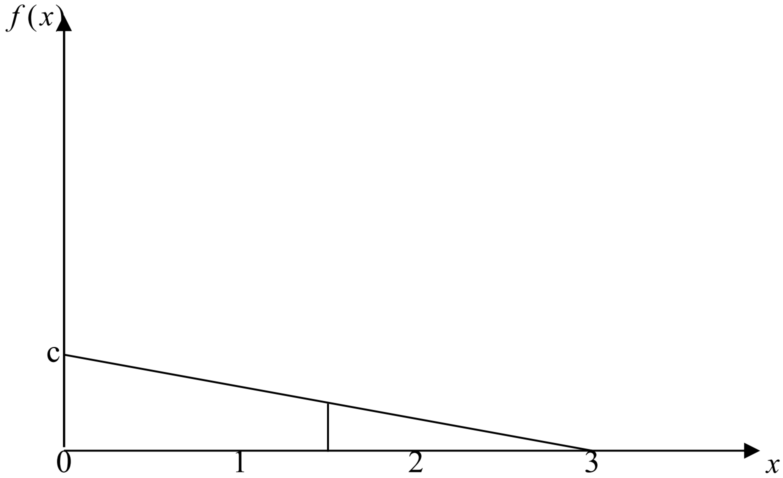
$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{3}\left(-\frac{2}{9}t + \frac{2}{3} \right)dt = \left[-\frac{2}{9}\cdot\frac{t^2}{2} + \frac{2}{3}t \right]_0^x$$

$$F(x) = -\frac{x^2}{9} + \frac{2}{3}x, \quad x \in [0, \quad 3] .$$

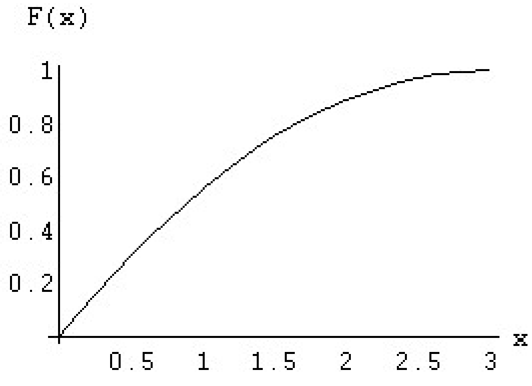
4.

$$P\big(1.5 < X < 3)\big) \approx P(1.5 \le X < 3) = \int_{1.5}^{3} f(x)dx = F(3) - F(1.5) = \frac{1}{4}$$

5.



$f(x)$



(Last revised: 06.06.07)

Chapter VI

Parameters of a Random Variable

<u>D. 6. 1.</u> (*Expected Value*)
The *expected value* of the random variable X , denoted by $E\left(X\right)$, is defined as

$$E(X) := \begin{cases} \sum_{i=1}^{\infty} x_i \cdot p_i & \text{when } X \text{ discrete} \\ \int_{-\infty}^{+\infty} x \cdot f(x)dx & \text{when } X \text{ continuous} \end{cases}$$

under the assumption

$$\sum_{i=1}^{\infty} |x_i| \cdot p_i < \infty$$

and

$$\int_{-\infty}^{+\infty} |x| \cdot f(x)dx < \infty \,.$$

<u>R. 6. 1.</u>
The expected value of a discrete random variable is the weighted mean of all outcomes x_i of X with the probabilities p_i acting as weights.

<u>R. 6. 2.</u>
The notion "expected value" in the probability theory has similarities with the notion "mean value", they are not, however, identical. The following example illustrates this fact:

<u>Ex. 6. 1.</u>
A die will be tossed with the following outcomes:

3, 5, 4, 3, 1

The average mean is equal to

$$\bar{x} = \frac{1}{5} \cdot \left(3+5+4+3++1\right) = 3.2$$

On the other hand, the expected value of the random variable

X : „number of dots facing uppermost“

will be

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

Whereas the mean value can vary from trial to trial, the expected value is an objective number independent of concrete outcomes of trials. The mean value approaches the expected value if the number of trials approaches infinity.

<u>Ex. 6. 2.</u>
Given the probability density function

$$f(x) = -\frac{2}{9}x + \frac{2}{3}, \quad x \in [0,\ 3] \ ,$$

find the expected value $E(X)$.

Solution:

$$E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \int_0^3 x \cdot \left(-\frac{2}{9}x + \frac{2}{3} \right) dx$$

$$= \int_0^3 \left(-\frac{2}{9}x^2 + \frac{2}{3}x \right) dx$$

$$= -\frac{2}{9} \cdot \frac{x^3}{3} + \frac{2}{3} \cdot \frac{x^2}{2} \Bigg|_0^3$$

$$= -\frac{2}{9} \cdot \frac{27}{3} + \frac{9}{3} = 1$$

<u>D. 6. 2.</u> (*Variance or Dispersion, Standard Deviation*)
The *dispersion* or *variance* of the random variable X, denoted by $D^2(X)$, is defined as

$$D^2(X) := E(X - E(X))^2$$

i. e.

$$D^2(X) := \begin{cases} \sum_{i=1}^{\infty} (x_i - E(X))^2 \cdot p_i & \text{when } X \text{ discrete} \\ \int_{.\infty}^{+\infty} (x - E(X))^2 \cdot f(x)dx & \text{when } X \text{ continuous} \end{cases}$$

under the assumption that the expected value exists.

The *standard deviation,* denoted by D, is defined as

$$D(X) := \sqrt{D^2(X)} \quad (>0)$$

R. 6. 3.
The following relations can be easily verified:

$$D^2(X) = E(X^2) - (E(X))^2,$$

i. e.

$$D^2(X) := \begin{cases} \sum\limits_{i=1}^{\infty} x_i^2 \cdot p_i - \left(\sum\limits_{i=1}^{\infty} x_i \cdot p_i \right)^2 & \text{when } X \text{ discrete} \\ \int\limits_{.\infty}^{+\infty} x^2 \cdot f(x) dx - \left(\int\limits_{-\infty}^{+\infty} x \cdot f(x) \right)^2 & \text{when } X \text{ continuous} \end{cases}$$

Ex. 6. 1. (*continued)*
Find the variance and the standard deviation of X .

Solution:

$$D^2(X) = (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + ... + (6-3.5)^2 \cdot \frac{1}{6} \approx 2.92,$$

or

$$D^2(X) = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + ... + 36 \cdot \frac{1}{6} - 3.5^2 \approx 2.92 .$$

$$D(X) \approx 1.71 .$$

Ex. 6. 2. (*continued)*
Find the variance and the standard deviation of X .

Solution:

$$D^2(X) = \int\limits_0^3 (x-1)^2 \cdot \left(\frac{2}{9}x + \frac{2}{3} \right) dx$$

$$= \int\limits_0^3 \left(-\frac{2}{9}x^3 + \frac{10}{9}x^2 - \frac{14}{9}x + \frac{2}{3} \right) dx$$

$$= -\frac{2}{9} \cdot \frac{x^4}{4} + \frac{10}{9} \cdot \frac{x^3}{3} - \frac{14}{9} \cdot \frac{x^2}{2} + \frac{2}{3}x \Big|_0^3$$

$$= \frac{1}{2}$$

or

$$D^2(X) = \int_0^3 x^2 \cdot \left(-\frac{2}{9}x + \frac{2}{3}\right) dx - 1$$

$$= \int_0^3 \left(-\frac{2}{9}x^3 + \frac{2}{3}x^2\right) dx - 1$$

$$= -\frac{2}{9} \cdot \frac{x^4}{4} + \frac{2}{3} \cdot \frac{x^3}{3} \Bigg|_0^3 - 1$$

$$= \frac{1}{2}.$$

R. 6. 4.

Let X and Z be two random variables. Given the transformation

$$Z = g(X),$$

we would like to derive informations about Z assuming we know the distribution of X . This will be illustrated by the following example:

Ex. 6. 3.

Let us consider the function

$$Z = 4X$$

with the following probability and distribution functions for the random variable X :

x_i	2	4
$P(X = x_i)$	$\frac{1}{4}$	$\frac{3}{4}$

$$F(x) := \begin{cases} 0 & \text{when} \quad x \le 2 \\ \frac{1}{4} & \text{when} \quad 2 < x \le 4 \\ 1 & \text{when} \quad x > 4 \end{cases}.$$

We are interested in finding the probability and distribution functions for the random variable Z :

$$P(X = x_i) = P(4X = 4x_i)$$

$$= P(Z = 4x_i)$$

$$= P(Z = z_i), \quad i = 1,\ 2,$$

and

$$F(z) = P(Z < z)$$

$$= P(4X < z)$$

$$= P\left(X < \frac{z}{4}\right)$$

$$= P\left(\frac{z}{4}\right) = F(x).$$

Thus, we have the probability function:

z_i	8	16
$P(Z = z_i)$	$\frac{1}{4}$	$\frac{3}{4}$

and the distribution function:

$$F(z) := \begin{cases} 0 & \text{when} \quad z \le 8 \\ \frac{1}{4} & \text{when} \quad 8 < z \le 16 \\ 1 & \text{when} \quad z > 16 \end{cases}.$$

Let us now calculate the expected value of Z :

$$\begin{aligned} E(Z) &= z_1 \cdot P(Z = z_1) + z_2 \cdot P(Z = z_2) \\ &= 8 \cdot P(Z = 8) + 16 \cdot P(Z = 16) \\ &= 8 \cdot P(4X = 8) + 16 \cdot P(4X = 16) \\ &= 8 \cdot P(X = 2) + 16 \cdot P(X = 4) \\ &= 8 \cdot \frac{1}{4} + 16 \cdot \frac{3}{4} \\ &= 14 \quad . \end{aligned}$$

R. 6. 5.

Generalising the results of the above example and assuming that the corresponding expected values exist, the following relations can be proved:

$$E(Z) = E(g(x)) = \begin{cases} \sum_{i=1}^{\infty} g(x_i) \cdot P(X = x_i) & \text{when } X \text{ discrete} \\ \int_{-\infty}^{+\infty} g(x) \cdot f(x) dx & \text{when } X \text{ continuous} \end{cases}$$

Ex. 6. 4.
Consider the random variable X with the probability function

x_i	x_1	x_2	$\dots$	x_n
$p_i = P(X = x_i) = f(x_i)$	p_1	p_2	$\dots$	p_n

Let

$$Z = aX + b, \quad a, b = const.$$

Then, we have:

$$\begin{aligned} E(Z) &= \sum_{i=1}^{\infty} \left(ax_i + b\right) \cdot p_i \\ &= a \cdot \sum_{i=1}^{\infty} x_i \cdot p_i + b \cdot \sum_{i=1}^{\infty} p_i \\ &= a \cdot \sum_{i=1}^{\infty} x_i \cdot p_i + b . \end{aligned}$$

Hence, assuming that $E(X)$ exists, we have

$$E\left(aX + b\right) = a \cdot E(X) + b .$$

Similarly, it can be proved:

$$D^2\left(aX + b\right) = a^2 \cdot D^2(X),$$

assuming that $D^2\left(X\right)$ exists.

Ex. 6. 5.
Consider the random variable X with

$$E\left(X\right) = \mu, \quad D^2\left(X\right) = \sigma^2 \quad (\sigma^2 \neq 0) .$$

For the function

$$Z = g(X) := \frac{X - \mu}{\sigma} = \frac{1}{\sigma} \cdot X - \frac{\mu}{\sigma} ,$$

we obtain

$$\begin{aligned} E\left(Z\right) &= \frac{1}{\sigma} \cdot E\left(X\right) - \frac{\mu}{\sigma} \\ &= \frac{1}{\sigma} \cdot \mu - \frac{\mu}{\sigma} = 0 \\ &= 0 , \end{aligned}$$

$$D^2(Y) = \frac{1}{\sigma^2} \cdot D^2(X)$$

$$= \frac{\sigma^2(X)}{\sigma^2(X)}$$

$$= 1.$$

<u>D. 6. 3.</u> (*Standardisation, Standardised Random Variable*)
The random variable Z is called a *standardised random variable* if

$$E(Z) = 0, \quad D^2(Z) = 1.$$

The process

$$Z = g(X) := \frac{X - \mu}{\sigma} = \frac{1}{\sigma} \cdot X - \frac{\mu}{\sigma}$$

is called *standardisation*.

(Last revised: 03.06.08)

Chapter VII

Some Special Discrete Distributions

D. 7. 1. (***Hypergeometric Distribution***)
A discrete variable X has a hypergeometric distribution if its probability function is of the form

$$P\left(X = x\right) = p_i = \frac{\binom{M}{x} \cdot \binom{N-M}{n-x}}{\binom{N}{n}},$$

$$x = 0,1,\ldots,n; \quad n \le M \le N .$$

R. 7. 1.
The probability function of the hypergeometric distribution is formally equivalent to the sampling scheme *without replacement*.

R. 7. 2.
The requirements for a hypergeometric experiment are as follows:

1. There must be a fixed number of trials.
2. Trials must be independent.
3. All outcomes of trials must be of two categories.
4. Probabilities must remain constant for each trial.

T. 7. 1.
Let X have a hypergeometric distribution. Then

$$E\left(X\right) = n \cdot \frac{M}{N}$$

$$= n \cdot p .$$

$$D^2\left(X\right) = \frac{N-n}{N-1} n \cdot \frac{M}{N} \cdot \left(1 - \frac{M}{N}\right)$$

$$= \frac{N-n}{N-1} n \cdot p \cdot q .$$

Proof:

$$E\left(X\right) = \sum_{x=0}^{n} \frac{\binom{M}{x} \cdot \binom{N-M}{n-x}}{\binom{N}{n}} \cdot x = n \cdot \frac{M}{N} = n \cdot p ,$$

$$D^2(X) = E(X^2) - (E(X))^2$$

$$= \sum_{x=0}^{n} \frac{\binom{M}{x} \cdot \binom{N-M}{n-x}}{\binom{N}{n}} \cdot x^2 - \left(n \cdot \frac{M}{N}\right)^2$$

$$= n \cdot \frac{M}{N} \cdot \left(1 - \frac{M}{N}\right) \cdot \frac{N-n}{N-1}$$

$$= \frac{N-n}{N-1} \cdot n \cdot p \cdot q \,.$$

Ex. 7. 1.
A box contains 25 items, 10 of which are defective. A sample of two items will be taken, *without replacement:*

1. Find the probability and the distribution function.
2. What is the probability that

   a) none of the items is defective?

   b) at most one of the items is defective?

3. Calculate the expected value, the variance, and the standard deviation of the random variable.

Solution:
Let X denote the number of defective items in the sample. Then, we have

$$N = 25, \quad M = 10, \quad n = 2 \,.$$

1.

$$P(X=0) = \frac{\binom{10}{0} \cdot \binom{25-10}{2-0}}{\binom{25}{2}} = \frac{7}{20} = 0.35 \cdot$$

$$P(X=1) = \frac{\binom{10}{1} \cdot \binom{25-10}{2-1}}{\binom{25}{2}} = \frac{10}{20} = 0.50$$

$$P(X=2)=\frac{\binom{10}{2}\cdot\binom{25-10}{2-2}}{\binom{25}{2}}=\frac{3}{20}=0.15$$

Probability function:

x_i	0	1	2
$P(X=x_i)$	0.35	0.50	0.15

Distribution function:

$$F(x)=\begin{cases}0.00 & \text{when} \quad -\infty<x\le 0\\ 0.35 & \text{when} \quad 0<x\le 1\\ 0.85 & \text{when} \quad 1<x\le 2\\ 1.00 & \text{when} \quad 2<x<+\infty\end{cases}$$

2.

a)

$$P(X=0)=0.35\,.$$

b)

$$P(X<2)=F(2)=0.85.$$

3.

$$p=\frac{M}{N}=\frac{10}{25}=\frac{2}{5},\qquad q=1-\frac{2}{5}=\frac{3}{5},$$
$$E(X)=n\cdot p=2\cdot\frac{2}{5}=0.8\,,$$
$$D^2(X)=\frac{N-n}{N-1}\cdot n\cdot p\cdot q=\frac{25-2}{25-1}\cdot 2\cdot\frac{2}{5}\cdot\frac{3}{5}=0.46\,,$$
$$D(X)=0.678232998\,.$$

<u>**D. 7. 2.**</u> **(*Binomial Distribution*)**

A discrete variable X has a binomial distribution if its probability function is of the form

$$P(X=x)=p_i=\binom{n}{x}\cdot p^x\cdot q^{n-x},\quad x=0,\ 1,\ldots,n\,.$$

R. 7. 3.
The probability function of the binomial distribution is formally equivalent to the sampling scheme with replacement.

R. 7. 4.
The requirements for a binomial experiment are as follows:

1. There must be a fixed number of trials.
2. Trials must be independent.
3. All outcomes of trials must be of two categories.
4. Probabilities must remain constant for each trial.

T. 7. 2
Let X have a binomial distribution. Then

$$E(X) = n \cdot p,$$

$$D^2(X) = n \cdot p \cdot q.$$

Proof:

$$E(X) = \sum_{x=0}^{n} x \cdot \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} = n \cdot p,$$

$$D^2(X) = \sum_{x=0}^{n} x^2 \cdot \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} - n^2 \cdot p^2 = n \cdot p \cdot (1-p) = n \cdot p \cdot q.$$

Ex. 7. 2.
A box contains 25 items, 10 of which are defective. A sample of two items will be taken, *with replacement:*

1. Find the probability and the distribution function.
2. What is the probability that

   a. none of the items is defective?

   b. at most one of the items is defective?

3. Calculate the expected value, the variance, and the standard deviation of the random variable.

Solution
Let X denote the number of defective items in the sample. Then, we have

$$n = 2, \quad p = \frac{M}{N} = \frac{10}{25} = \frac{2}{5}, \quad q = \frac{3}{5}.$$

1.

$$P(X=0)=\binom{2}{0}\cdot\left(\frac{2}{5}\right)^{0}\cdot\left(\frac{3}{5}\right)^{2-0}=\frac{9}{25}=0.36\,.$$

$$P(X=1)=\binom{2}{1}\cdot\left(\frac{2}{5}\right)^{1}\cdot\left(\frac{3}{5}\right)^{2-1}=\frac{12}{25}=0.48$$

$$P(X=2)=\binom{2}{2}\cdot\left(\frac{2}{5}\right)^{2}\cdot\left(\frac{3}{5}\right)^{2-2}=\frac{4}{25}=0.16$$

Probability function:

x_i	0	1	2
$P(X=x_i)$	0.36	0.48	0.16

Distribution function:

$$F(x)=\begin{cases}0.00 & \text{when} \quad -\infty<x\le 0\\ 0.36 & \text{when} \quad 0<x\le 1\\ 0.84 & \text{when} \quad 1<x\le 2\\ 1.00 & \text{when} \quad 2<x<+\infty\end{cases}$$

2.

a)

$$P(X=0)=0.36\,.$$

b)

$$P(X<2)=F(2)=0.84.$$

3.

$$E(X)=n\cdot p=2\cdot\frac{2}{5}=0.8\,,$$
$$D^2(X)=n\cdot p\cdot q=2\cdot\frac{2}{5}\cdot\frac{3}{5}=0.48\,,$$
$$D(X)=0.692820323\,.$$

T. 7. 3.

$$\lim_{N\to+\infty}\frac{\binom{M}{x}\cdot\binom{N-M}{n-x}}{\binom{N}{n}}=\binom{n}{x}\cdot p^x\cdot q^{n-x}.$$

R. 7. 5.
For a "sufficiently" large N, the hypergeometric distribution can be approximated by the binomial distribution. It will be recommended to use the following rule of thumb:

> "If $10\cdot n\le N$, then the hypergeometric distribution can be approximated by the binomial distribution."

Ex. 7. 3.
A manufacturer of car tires reports that among a shipment of 5000 sent to a local distributor, 1000 are slightly blemished. If one purchases 10 of these tires at random from the distributor, what is the probability that exactly 3 will be blemished?

Solution:
We have

$$N=5000,\quad M=1000,\quad n=10.$$

$$P(X=3)=p_3=\frac{\binom{1000}{3}\cdot\binom{5000-1000}{10-3}}{\binom{5000}{10}}=0.201477715\cdot$$

Because of

$$10n=10\cdot 10=100\le 5000=N$$

the probability can also be calculated as follows:

$$n=10,\quad p=\frac{M}{N}=\frac{1000}{5000}=\frac{1}{5},\quad q=\frac{4}{5},$$

$$P(X=3)=p_3=\binom{10}{3}\cdot\left(\frac{1}{5}\right)^3\cdot\left(\frac{4}{5}\right)^{10-3}=0.201326592.$$

R. 7. 6.

$$\lim_{N\to+\infty}\frac{N-n}{N-1}\cdot n\cdot p\cdot q=\lim_{N\to+\infty}\frac{1-\frac{n}{N}}{1-\frac{1}{N}}\cdot n\cdot p\cdot q=n\cdot p\cdot q.$$

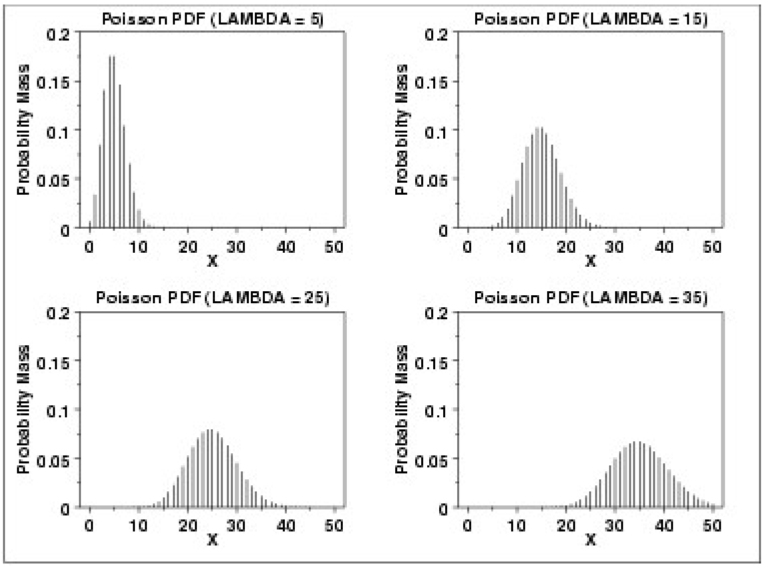
D. 7. 3. (***Poisson Distribution***)
A discrete variable X has a Poisson distribution if its probability function is of the form

$$P\left(X=x\right)=\frac{\lambda^{x}}{x!}\cdot e^{-\lambda},$$
$$x=0,1,...,n\ .$$

T. 7. 4.
Let X have a Poisson distribution. Then

$$E\left(X\right)=D^{2}(X)=n\cdot p=\lambda\ .$$

R. 7. 7.
The following is the plot of the Poisson probability function for four values of λ



Ex. 7. 4.
Assume that the number of defects per square meter of a certain type of cloth manufactured by a mill is measured as no defects, one defect, two defects, and so on. In the average, the number of defects is 0.5.
Compute the probabilities that a square meter will have

a) no defects
b) one defect
c) two defects.

Solution:
We have $\lambda=0.5$.

a)
$$P\left(X=0\right)=\frac{0.5^{0}}{0!}\cdot e^{-0.5}=0.606530659\ .$$

b)

$$P(X=1)=\frac{0.5}{1!}\cdot e^{-0.5}=0.303265329\,.$$

c)

$$P(X=2)=\frac{0.5^2}{2!}\cdot e^{-0.5}=0.075816332\,.$$

T. 7. 5.

$$\lim_{\substack{n\to+\infty\\ n\cdot p=\lambda}}=\binom{n}{x}\cdot p^x\cdot q^{n-x}=\frac{\lambda^x}{x!}\cdot e^{-\lambda}\,.$$

Proof:

$$\binom{n}{x}\cdot p^x\cdot q^{n-x}=\binom{n}{x}\cdot\left(\frac{\lambda}{n}\right)^x\cdot\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$=\frac{\lambda^x}{x!}\cdot\frac{\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\cdot\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}\,.$$

The statement of the theorem follows directly from:

$$\lim_{n\to+\infty}\left(1-\frac{\lambda}{n}\right)^n=e^{-\lambda},\qquad\qquad \lim_{n\to+\infty}\left(1-\frac{\lambda}{n}\right)^x=1\,.$$

R. 7. 8.
The binomial distribution can be approximated by the Poisson distribution for n “sufficiently” large ($n\to\infty$) while $n\cdot p=\lambda$ remaining constant. That is why the Poisson distribution is also known as the “distribution of rare events”.
It will be recommended to use the following rule of thumb:

“If

$n\cdot p\le 10$ and $n\ge 1500p$,

then the binomial distribution can be approximated by the Poisson distribution.”

Ex 7. 5.
80 people work in a factory. The probability that one of them becomes sick in winter is estimated to be 0.05.
What is the probability that at least 11 people will become sick as a result of a flue epidemic?

Solution:
Denote the number of sick people by X. Using the binomial distribution with $n = 80$ and $p = 0.05$, we shall have to calculate

$$P(X \geq 11) = 1 - P(X < 11) = 1 - F(11) = 1 - \sum_{x=0}^{10} \binom{80}{x} \cdot 0.05^x \cdot (1 - 0.05)^{80-x}.$$

A rather „tedious" job. However, this will not be necessary, since because of

$$n \cdot p = 80 \cdot 0.05 = 4 < 10 \quad \text{and} \quad n = 80 \geq 1500 \cdot 0.05 = 75 = 1500 \cdot p$$

the probability can be found by using the Poisson distribution:

$$\lambda = n \cdot p = 80 \cdot 0.05 = 4,$$
$$P(X \geq 11) = 1 - P(X < 11) = 1 - F(11) = 1 - \sum_{x=0}^{10} \frac{4^x}{x!} \cdot e^{-4} = 1 - 0.997158 = 0.002842$$

(Last revised: 30.06.09)

Chapter VIII

Some Special Continuous Distributions

D. 8. 1. (***Normal Distribution***)
A continuous variable X has *normal* (or *Gaussian*) distribution if its probability density function is of the form

$$f(x;\mu,\sigma) = \frac{1}{\sigma}\cdot\varphi\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in R^1$$

T. 8. 1.
The *distribution function* of a normally distributed variable X is given by

$$F(x;\mu,\sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{t^2}{2}}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt, \qquad x \in R^1.$$

Proof: (see: D. 5. 3.)

R. 8. 1.
The old German money 10 Mark notes had Karl F. Gauss printed on the back, and a small bell shaped curve and its formula in the background:

T. 8. 2.

Let X be a normally distributed variable. Then

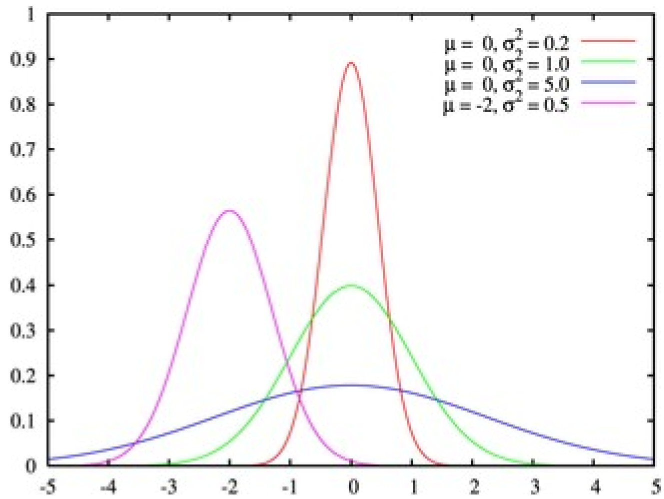
$$E(X) = \mu ,$$

$$D^2(X) = \sigma^2 .$$

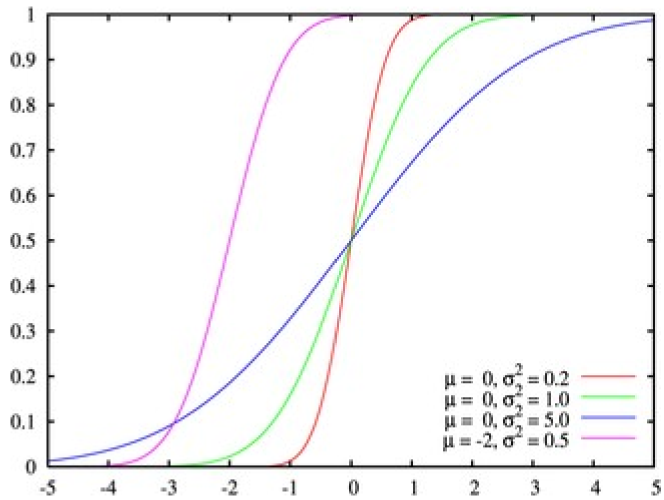
R. 8. 2.

The following charts show the probability density and the distribution functions of a normally distributed variable X for different values of μ and σ^2:

Probability Density Function



Distribution Function



R. 8. 3.(***Some Important Properties of the Normal Density Function)***
The probability density function of a normally distributed random variable has the following properties:

1. The function assumes its maximum at μ .

2. Increasing the mean shifts the distribution to the right without changing the shape, decreasing the mean shifts it to the left.

3. Decreasing the standard deviation "shrinks" it while making the peak higher, increasing the standard deviation makes it flatter.

4. The function is symmetric and centred about the mean. Hence, the area under the curve to the left of the mean equals the area under the curve to the right of the mean. Both of these areas are equal to 0.5.

5. The function has inflection points at $x = \mu \pm \sigma$.

6. Mean = median = mode.

7. $x \to \pm\infty \quad \Rightarrow \quad f(x;\mu,\sigma) \to 0$.

R. 8. 4. (***Standardised Normal Distribution***)
Using the standardisation

$$Z = \frac{X-\mu}{\sigma}$$

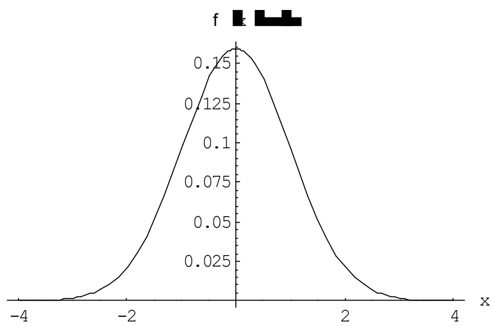
with

$$E(Z) = 0, \quad D^2(Z) = 1$$

(See D. 6. 3.), we obtain:

1. The *probability density function of the standardised normal distribution*:

$$f(x) = \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in R^1 .$$



2. The *standardised normal distribution function*:

$$F(x) = \Phi(x)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \qquad x \in R^1.$$

R. 8. 5.

$$\varphi(-x;\ 0,\ 1) = \varphi(x;\ 0,\ 1),$$

$$\Phi(-x;\ 0,\ 1) = 1 - \Phi(x;\ 0,\ 1).$$

T. 8. 3.

$$P(|X - \mu| < c) = 2 \cdot \Phi\left(\frac{c}{\sigma}\right) - 1, \quad c \in R^1.$$

Ex. 8. 1.
Consider a normally distributed random variable X with the mean 1 and the variance 9.

1. Find the probability that

a. X lies in the interval $]2,\ 5[$,
b. X is at least equal to 2.7,
c. X deviates from the mean by less than 1.5.

2. Determine a number z for which the probability $P(X \geq z)$ is at least 0.1.

Solution:
a.

$$P(2 < X < 5) \approx P(2 \leq X < 5) = F(5) - F(2)$$
$$= \Phi\left(\frac{5-1}{\sqrt{9}}\right) - \Phi\left(\frac{2-1}{\sqrt{9}}\right)$$
$$= \Phi\left(\frac{4}{3}\right) - \Phi\left(\frac{1}{3}\right)$$
$$= 0.908241 - 0.629300 = 0.278941.$$

b.

$$P(X \geq 2.7) = 1 - P(X < 2.7)$$
$$= 1 - F(2.7)$$
$$= 1 - \Phi\left(\frac{2.7-1}{3}\right)$$
$$= 1 - \Phi(0.56666)$$
$$= 1 - 0.715661 = 0.274339.$$

c.

$$P\left(|X-1|<1.5\right)=2\cdot\Phi\left(\frac{1.5}{3}\right)-1$$
$$=2\cdot\Phi(0.5)-1$$
$$=2\cdot 0.691462-1=0.382929\,.$$

2.

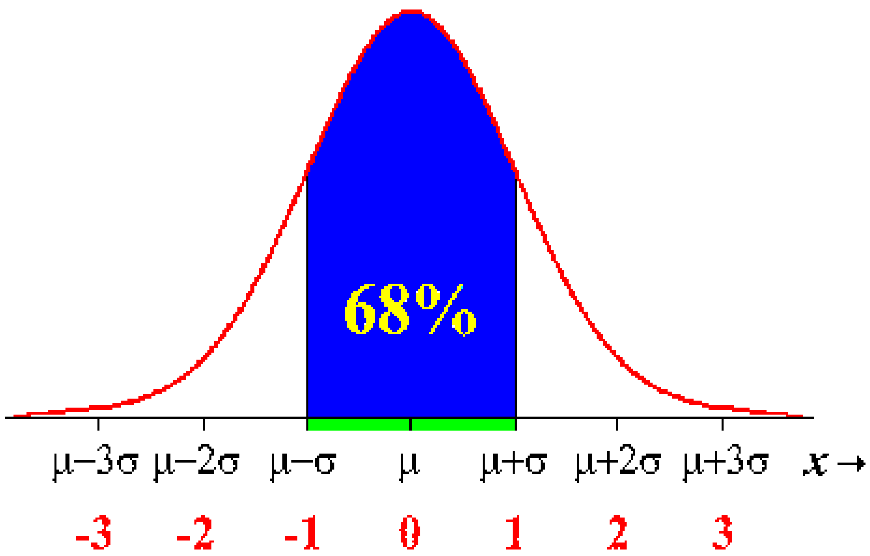
$$P(X\geq z)\geq 0.1$$
$$P(X<z)\leq 0.9$$
$$P(X<z)=F(z)=\phi\left(\frac{z-1}{3}\right)\leq 0.9=\Phi(1.28)$$
$$\Phi\left(\frac{z-1}{3}\right)\leq\Phi(1.28)$$
$$\frac{z-1}{3}\leq 1.28$$
$$z\leq 4.84\,.$$

<u>R. 8. 6.</u>(*"68-95-99.7 Rule" or "Empirical Rule"*)
All normal density curves satisfy the following property which follows from T. 8. 3.:

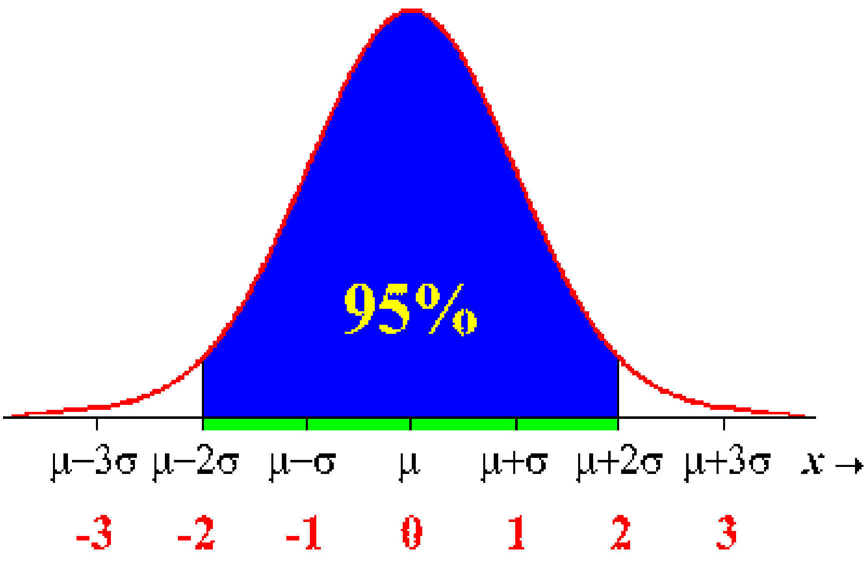
$$P\left(|X-\mu|<\sigma\right)=0.6828$$
$$P\left(|X-\mu|<2\sigma\right)=0.9544$$
$$P\left(|X-\mu|<3\sigma\right)=0.9972\,.$$

This means:

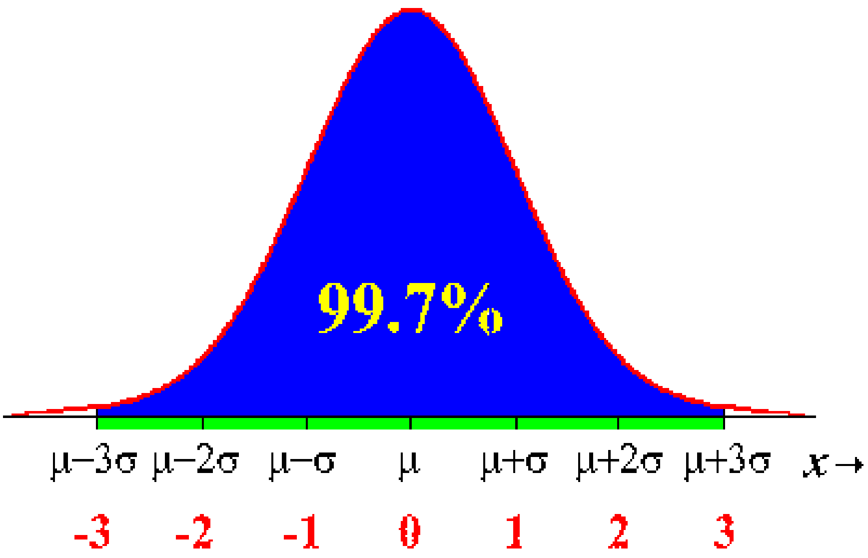
Approximately 68% of the observations fall within 1 standard deviation of the mean:



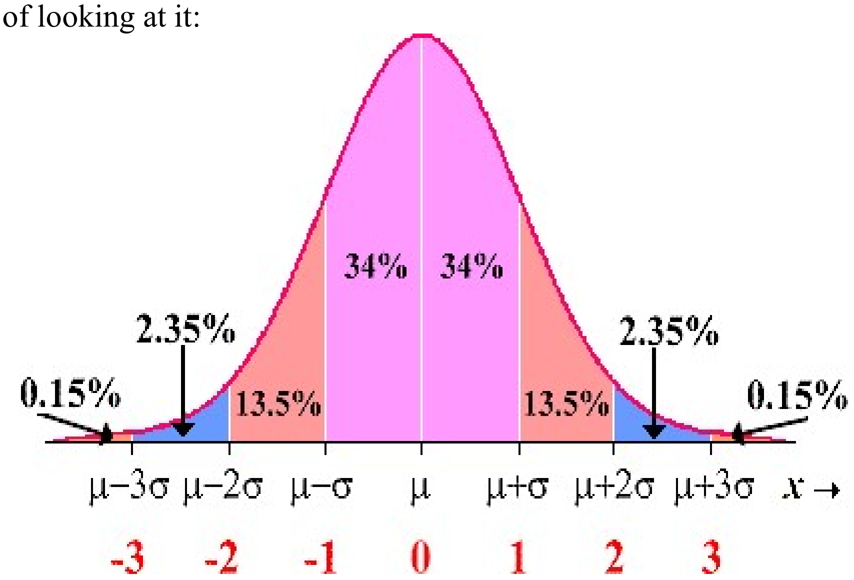
Approximately 95% of the observations fall within 2 standard deviations of the mean:



Approximately 99.7% of the observations fall within 3 standard deviations of the mean:



Another way of looking at it:



<u>Ex. 8. 1.</u> (*Continued*)
Approximately

68% of the observations fall within the interval $[-2,\ 4]$,
95% of the observations fall within the interval $[-5,\ 7]$,
99.7% of the observations fall within the interval $[-8,\ 10]$.

<u>T. 8. 4.</u>
Let $X_i,\ i=1,\ 2,...,$ be binomially distributed random variables with parameters n and p.
Then the sequence of the corresponding standardised random variables

$$\begin{aligned} X_i &:= \frac{X_i-\mu}{\sigma} \\ &= \frac{X_i-n\cdot p}{\sqrt{n\cdot p\cdot q}}, \qquad i=1,\ 2,... \end{aligned}$$

converges against a *standardised normally* distributed random variable.
In particular, we have

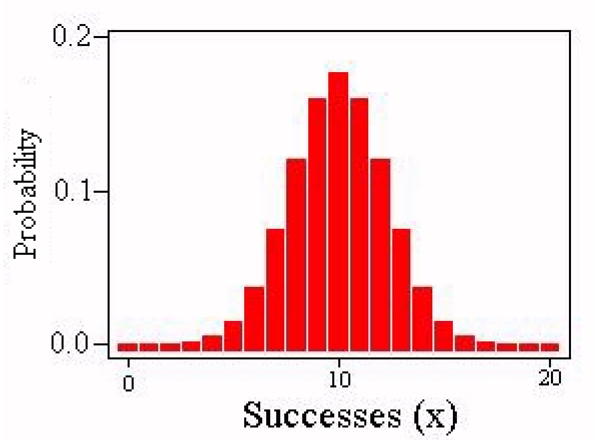
$$P(X_i<x)=\Phi\left(\frac{x-n\cdot p}{\sqrt{n\cdot p\cdot q}}\right).$$

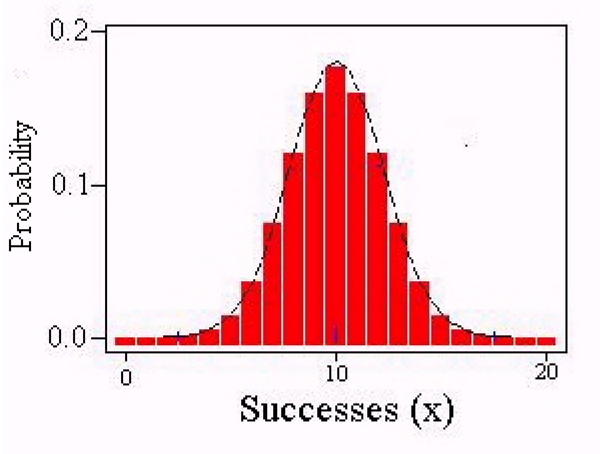
<u>R. 8. 5.</u> (*Normal Approximation to Binomial*)
The above theorem can be used to approximate binomial distribution by normal distribution.
It will be recommended to use the following <u>rule of thumb</u>:

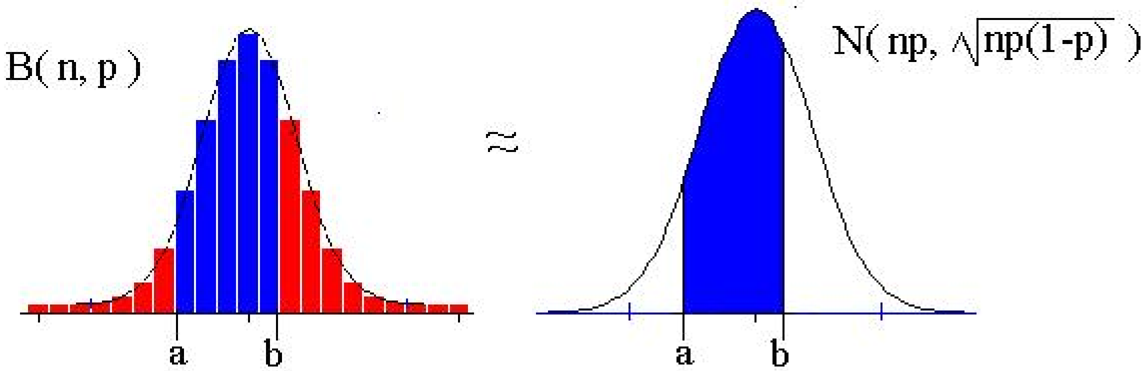
> "If
>
> $n\cdot p>5$ and $n\cdot q>5$,
>
> then the binomial distribution can be approximated by normal distribution."

<u>Ex. 8. 2.</u>
Shown below is the probability distribution of a binomial random variable X with $n=20$ and $q=0.5$:

Successes(x)	Probability
0	0.000001
1	0.000019
2	0.000181
3	0.001087
4	0.004621
5	0.014786
6	0.036964
7	0.073929
8	0.120134
9	0.160179
10	0.176197
11	0.160179
12	0.120134
13	0.073929
14	0.036964
15	0.014786
16	0.004621
17	0.001087
18	0.000181
19	0.000019
20	0.000001







Ex. 8. 3.
Experience shows that 90% of the products of a firm are of highest quality. Find the probability of finding at least 950 such products in a sample of 1000.

Solution:
Let X denote the number of products of highest quality. Then, we have

$$n = 1000, \qquad p = 0.9 .$$

Using the binomial distribution, we find the „ugly“ expression:

$$P\left(X \geq 950\right) = \sum_{x=950}^{1000} \binom{1000}{x} \cdot 0.9^{x} \bullet 0.1^{1000-x} .$$

Let us, therefore, approximate it by the normal distribution (Obviously the conditions for the application of the above rule of thumb are fulfilled!)

$$E\left(X\right) = \mu = n \cdot p = 1000 \cdot 0.9 = 900$$
$$D^{2}\left(X\right) = \sigma^{2} = n \cdot p \cdot q = 900 \cdot 0.1 = 90$$

$$P\left(X \geq 950\right) = 1 - P\left(X < 950\right) = 1 - \Phi\left(\frac{950 - 900}{\sqrt{90}}\right) = 1 - \Phi\left(5.3\right) = 0.0001 .$$

(Last revised: 09.06.08)

Chapter I

Random Events, Events Algebra

Exercises

1. 1.
A plant consists of 4 boilers, 2 turbines and 1 generator. Following events denote:

A: „The generator is working normally. "

B_k $(k = 1,2,3,4)$: „The boiler k is working normally"

C_i $(i = 1,2)$: „The turbine i is working normally".

The plant can only work (denoted by the event D) if the generator, at least one boiler and at least one turbine work normally.

Using the events A, B_k $(k = 1,2,3,4)$ and C_i $(i = 1,2)$ describe the events D and $\bar{D}$.

1. 2.
Consider four machines and the following events

A: „merely one machine falls out",

B: „at least one machine falls out.",

C: „not less than two machines fall out",

D: „only two machines fall out",

E: „only three machines fall out",

F: „all machines fall out",

E_i: „the machine i falls out $(i = 1(1)4)$".

1. Using E_i, $(i = 1(1)4)$ describe the events A-F.
2. Which of the events A- F are equivalent to the following events?

a) $A \cup B$ b) $A \cap B$ c) $B \cup C$ d) $B \cap C$ e) $D \cup E \cup F$

f) $B \cap F$

1. 3.
A die will be thrown. Consider the following events:

A: „the die shows a 6“,
B : „the die shows an odd number“,
C : „the die shows at least a 4“,
D : „the die shows at most a 3“
E : „the die shows 2 or 4“,

1. Which of the above events is complementary to C ?
2. Which events are mutually exclusive to B ?
3. Which events form together with B and E a mutually exclusive and exhaustive system of events?

1. 4.
The three fire engines in a small town operate independently. Let $E_i, i = 1, 2, 3,$ denote the event that the engine i is available when needed.
Describe the following events:

1. Two fire engines are available.
2. At least one fire engine is not available.
3. No fire engine is available.

1. 5.
Three students toss a fair coin. Let $E_i, i = 1, 2, 3,$ denote the event that the student i tosses a "head".
Describe the following events:

1. At most one student tosses a "tail".
2. All three students toss a "head".
3. Student 2 does not toss a "head".

1. 6.
Two coins are tossed. Describe the sample space.

1. 7.
Two students are randomly selected from a statistics class, and it is observed whether or not they suffer from math anxiety.

1. List all the possible outcomes.
2. Describe all the outcomes indicated in each of the following events. Indicate which are simple and which are compound events:

    a) Both students suffer from math anxiety.
    b) Exactly one student suffers from math anxiety.
    c) The first student does not suffer and the second suffers from math anxiety.
    d) None of the students suffers from math anxiety.

(Last revised: 22.06.08)

Chapter I

Random Events, Events Algebra

Solutions

1. 1.

$$D = A \cap (B_1 \cup B_2 \cup B_3 \cup B_4) \cap (C_1 \cup C_2)$$
$$\bar{D} = \bar{A} \cup \left(\bar{B}_1 \cap \bar{B}_2 \cap \bar{B}_3 \cap \bar{B}_4 \right) \cup \left(\bar{C}_1 \cap \bar{C}_2 \right)$$

1. 2.

1.

$$A = (\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3 \cap E_4) \cup (\bar{E}_1 \cap \bar{E}_2 \cap E_3 \cap \bar{E}_4) \cup (\bar{E}_1 \cap E_2 \cap \bar{E}_3 \cap \bar{E}_4)$$
$$\cup (E_1 \cap \bar{E}_2 \cap \bar{E}_3 \cap \bar{E}_4)$$

$$B = E_1 \cup E_2 \cup E_3 \cup E_4$$

$$C = \Omega \setminus (A \cup (\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3 \cap \bar{E}_4))$$

$$D = (E_1 \cap E_2 \cap \bar{E}_3 \cap \bar{E}_4) \cup (E_1 \cap \bar{E}_2 \cap E_3 \cap \bar{E}_4) \cup$$
$$\cup (\bar{E}_1 \cap E_2 \cap E_3 \cap \bar{E}_4) \cup (\bar{E}_1 \cap \bar{E}_2 \cap E_3 \cap E_4)$$

$$E = (\bar{E}_1 \cap E_2 \cap E_3 \cap E_4) \cup (E_1 \cap \bar{E}_2 \cap E_3 \cap E_4) \cup$$
$$(E_1 \cap E_2 \cap \bar{E}_3 \cap E_4) \cup (E_1 \cap E_2 \cap E_3 \cap \bar{E}_4)$$

$$F = E_1 \cap E_2 \cap E_3 \cap E_4$$

2.

a) $A \cup B = B$

b) $A \cap B = A$

c) $B \cup C = B$

d) $B \cap C = C$

e) $D \cup E \cup F = C$

f) $B \cap F = F$

1. 3.

$A = \{6\}$, $B = \{1, 3, 5\}$, $C = \{4, 5, 6\}$, $D = \{1, 2, 3\}$, $E = \{2, 4\}$

1.
D.

2.
A , E

3.
Fehler! Textmarke nicht definiert., since:

$$A, B, E \neq \emptyset$$
$$A \cup B \cup E = \Omega$$
$$A \cap B = \emptyset, \quad A \cap E = \emptyset, \quad B \cap E = \emptyset$$

1. 4.
The following table lists all possible cases:

Cases	Events		
1	E_1	E_2	E_3
2	E_1	E_2	$\bar{E}_3$
3	E_1	$\bar{E}_2$	E_3
4	$\bar{E}_1$	E_2	E_3
5	E_1	$\bar{E}_2$	$\bar{E}_3$
6	$\bar{E}_1$	E_2	$\bar{E}_3$
7	$\bar{E}_1$	$\bar{E}_2$	E_3
8	$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$

1.
Cases to be considered: 1, 2, 3, 4:

$$A := (E_1 \cap E_2 \cap E_3) \cup (E_1 \cap E_2 \cap \bar{E}_3) \cup (E_1 \cap \bar{E}_2 \cap E_3) \cup (\bar{E}_1 \cap E_2 \cap E_3)$$

2.
Cases to be considered: 2, 3, 4, 5, 6, 7, 8

$$B := \Omega \setminus (E_1 \cap E_2 \cap E_3)$$

3.
Case to be considered: 8

$C := (\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3)$

1. 5.
(See the exercise 1. 4. for a listing of all possible cases)
1.
Cases to be considered: 1, 2, 3, 4:

$$A := (E_1 \cap E_2 \cap E_3) \cup (E_1 \cap E_2 \cap \bar{E}_3) \cup (E_1 \cap \bar{E}_2 \cap E_3) \cup (\bar{E}_1 \cap E_2 \cap E_3)$$

2.
Case to be considered: 1

$B := (E_1 \cap E_2 \cap E_3)$

3.
Cases to be considered: 3, 5, 7, 8

$$C := (E_1 \cap \bar{E}_2 \cap E_3) \cup (E_1 \cap \bar{E}_2 \cap \bar{E}_3) \cup (\bar{E}_1 \cap \bar{E}_2 \cap E_3) \cup (\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3)$$

1. 6.
Let H_i $(i = 1, 2)$ and T_i $(i = 1, 2)$ denote the events "head" and "tail" respectively on the i^{th} coin.
Then the sample space will consist of (H_1, H_2), (H_1, T_2), (T_1, H_2), (T_1, T_2).

1. 7.
Denote by

$E_i,\ i = 1, 2$: "Student i suffers from math anxiety."

1.

$\{(E_1, E_2), (E_1, \bar{E}_2), (\bar{E}_1, E_2), (\bar{E}_1, \bar{E}_2)\}$

2.

a)

$\{E_1, E_2\}$: simple.

b)

$\left\{(E_1, \bar{E}_2), (\bar{E}_1, E_2)\right\}$: compound.

c)

$\{\bar{E}_1, E_2\}$: simple.

d)

$\{\bar{E}_1, \bar{E}_2\}$: simple.

(Last revised: 15.05.09)

Chapter II

Introduction to Probability

Exercises

2. 1.
Consider a box with n white and m red balls. Find the probability of selecting a white ball.

2. 2.
What is the probability that at least one “head” will occur in two throws of a coin?

2. 3.
During a given week the probability that a particular common stock will increase, remain unchanged, or decline is estimated to be 0.30, 0.20, and 0.50, respectively.

1. What is the probability that the stock issue will increase in price or remain unchanged?
2. What is the probability that the price of the issue will change during the week?

(Last updated: 22.06.2008)

Chapter II

Introduction to Probability

Solutions

1. 1.
Let W denote the event of selecting a white ball. Then

$$P(W) = \frac{n}{m+n}$$

1. 2.
Let S denote all possible outcomes:

$$S = \{hh, ht, th, tt\}.$$

The probability that at least one "head" will occur will then be equal to

$$P = \frac{3}{4}.$$

1. 3.
Denote by

I: "The stock issue will increase in price."
U: "The stock issue will remain unchanged."
D: "The stock issue will decrease in price."

1.
$$P(I \cup U) = P(I) + P(U) = 0.30 + 0.20 = 0.50.$$

2.
$$P(I \cup D) = P(I) + P(D) = 0.30 + 0.50 = 0.80.$$

(Last updated: 05.05.2009)

Chapter III

Probability Algebra

Exercises

3. 1.
With reference to the following table what is the probability that a randomly chosen family will have household income

1. between at least 20000 € und less than 40000 €?
2. less than 40000 €?
3. at one of the two extremes of being either less than 20000 € or at least 100000 €?

Annual Household Income for 500 Families

Category	Income range	Number of families
1	[0 20000[	60
2	[20000 40000[	100
3	[40000 60000[	160
4	[60000 100000[	140
5	100000 and above	40
		Total 500

3. 2.
A consumer survey reveals that the probability of a computer owner shopping on the Internet was 0.17, while the probability of a computer owner downloading software was 0.33. Further, the probability of a computer owner doing both was 0.14.
Find the probability of the following events:

1. that a computer owner does not shop on the Internet
2. that a computer owner will either shop on the Internet or download software
3. that the computer owner will neither shop on the Internet nor download software.

3. 4.
According to official statistics, 33% of U. S. adults 20 years of age and over are overweight. If two people in the U. S. are selected at random, what is the probability that both are overweight?

3. 5.
In a sample survey, 1800 senior citizens were asked whether or not they have ever been victimised by a dishonest telemarketer. The following table gives the responses by age group:

Age in Years	Victimised	Not Victimised
60- 69 (A)	106	698
70 – 79 (B)	145	447
80 or over (C)	61	343

Suppose one person is randomly selected from these senior citizens.
Find the probability for the following events:

1. A person has been victimised or belongs to group B
2. A person has never been victimised or belongs to group C.

3. 6.
According to The Digest of Education Statistics 1996, 78.0% of U.S. 13-year-olds were able to perform numerical operations and beginning problem solving.
If two 13- year-olds are randomly selected, what is the probability that none of them can perform numerical operations and beginning problem solving?

3. 7.
According to the Labor Force Statistics from the Current Population Survey, in 1996, 52.8% of families in the USA had the husband and wife employed. If 3 families were randomly selected, what would have been the probability that they all had both spouses employed?

3. 8.
A survey found that 47% of teenagers have a part time job. The same survey found that 78% plan to attend college.
If a teenager is chosen at random, what is the probability that the teenager has a part time job and plans to attend college?

3. 9.
Let A and B be two random events with the probabilities:

$$P(A) = 0.3, \quad P(B) = 0.5, \quad P(A \cap B) = 0.2.$$

Find the probabilities for the following events:

a) $\bar{A}$, b) $\bar{B}$, c) $A \cup B$, d) $\bar{A} \cap \bar{B}$.

3. 10.
The machines $M_i, i = 1, 2, 3,$ produce 20%, 45% and 35% respectively of the total production of a certain article in a firm. Experience shows that 2%, 5% and 3% of the articles produced on these machines are defective.
Find the probability that

1. a defective article has <u>not</u> been produced on the second machine .
2. a non-defective article has been produced either on the second or on the third machine.

3. 11.
The machines $M_i, i = 1, 2, 3,$ produce 30%, 45% and 25% respectively of the total production of a certain article in a firm. Experience shows that 95%, 92% and 98% of the articles produced on these machines are of the highest quality.
Find the probability that

1. an article not being of the highest quality has <u>not</u> been produced on the first machine .
2. an article of highest quality has been produced either on the first or on the third machine.

3. 12.

Suppose a survey classified the population as male or female, and as favouring or opposing the death penalty. Suppose, the proportions in each category were:

	Death	not Death
Male	0.459	0.441
Female	0.051	0.049

Find the probability of an individual favouring the death penalty, conditional on being male.

3. 13.

Suppose there is a certain defect randomly found in 0.5% of the products of a firm. A quality control is *positive* (shows the presence of this certain defect) in 99% of all cases. But it yields a *negative* result (indicates the presence of this type of defect where there is actually no such defect) with a probability of 5%.

Find the probability that

1. the defect will not be present in any particular product.
2. the quality control will yield a negative result if the defect is present.
3. the quality control will yield a negative result if the defect is not present.
4. the quality control will yield a positive result, irrespective of whether the defect is present or not.
5. the quality control will yield a negative result, irrespective of whether the defect is present or not.
6. the defect is present if the quality control result is positive.
7. the defect is not present if the quality control result is positive.
8. the defect is absent if the quality control result is negative.
9. the defect is present if the quality control result is negative.

3. 14.

If there is an increase in capital investment next year, the probability that structural steel will increase in price is 0.90. If there is no increase in such investment, the probability of an increase is 0.40. Overall, we estimate that there is a 60 percent chance that capital investment will increase next year.

1. What is the probability that structural steel prices will not increase even though there is an increase in capital investment?
2. What is the overall probability of an increase in structural steel prices next year?
3. Suppose that during the next year structural steel prices in fact increase. What is the probability that there was an increase in capital investment?

3. 15.

A factory has three types of machines producing an item. Probabilities that the item is of high quality if it is produced on i-th machine $(i = 1, 2, 3)$ are given in the following table:

Machine	Probability of High Quality
1	0.8
2	0.7
3	0.9

The total production is done 30% on type 1 machine, 50% on type 2, and 20% on type 3.

One item is selected at random from the production.

1. What is the probability that it is of high quality?
2. What is the probability that it is not of high quality?
3. If it is of high quality, what is the probability that it was produced on machine 1?
4. If it is of high quality, what is the probability that it was produced on machine 3?
5. If it is not of high quality, what is the probability that it was not produced on machine 3?
6. If it is not of high quality, what is the probability that it was not produced on machine 2?

(Last revised: 24.11.14)

Chapter III

Probability Algebra

Solutions

3. 1.

1.

$$P(2)=\frac{100}{500}=0.20\,.$$

2.

$$P(1\text{ or }2)=\frac{60}{500}+\frac{100}{500}=0.32\,.$$

3.

$$P(1\text{ or }5)=\frac{60}{500}+\frac{40}{500}=0.20\,.$$

3. 2.

Denote by

I: "A computer owner shops on the Internet",
D: "A computer owner downloads software",

1.

$$P(\bar{I})=1-0.17=0.83\,.$$

2.

$$P(I\cup D)=P(I)+P(D)-P(I\cap D)=0.17+0.33-0.14=0.36.$$

3.

$$P(\bar{I}\cap\bar{D})=1-(I\cup D)=1-0.36=0.64.$$

3. 4.

$$P=\frac{1}{3}\cdot\frac{1}{3}\approx 0.11$$

3. 5.

Denote by

V: „A senior citizen has been victimised"

1.

$$P\left(V\cup B\right)=P(V)+P(B)-P(V\cap B)$$

$$P\left(V\cup B\right)=\frac{106+145+61}{1800}+\frac{145+447}{1800}-\frac{145}{1800}=\frac{759}{1800}=0.421666666\approx 0.42\,.$$

2.

$$P(\bar{V} \cup C) = P(\bar{V}) + P(C) - P(\bar{V} \cap C)$$

$$P(\bar{V} \cup C) = \frac{698 + 447 + 343}{1800} + \frac{61 + 343}{1800} - \frac{343}{1800} = \frac{1549}{1800} = 0.860555555 \approx 0.86..$$

3. 6.

$$P = (1 - 0.78)^2 = 0.22^2 = 0.0484.$$

3. 7.

$$P = 0.528^3 \approx 0.15$$

3. 8.

Denote by

J: "A teenager has a part time job."
C: "A teenager plans to attend college."

$$P(J \cap C) = P(C) \cdot P(J) = 0.47 \cdot 0.78 = 0.3666$$

(Note: It will be assumed that two events are independent.)

3. 9.

a)

$$P(\bar{A}) = 1 - P(A) = 1 - 0.3 = 0.7$$

b)

$$P(\bar{B}) = 1 - P(B) = 1 - 0.5 = 0.5$$

c)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.5 - 0.2 = 0.6$$

d)

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.6 = 0.4$$

3. 10.

Let

A: „The article is defective."

B_i: "the article has been produced on machine M_i, $i = 1, 2, 3$."

We have:

$P(B_1) = 0.20,$ $P(B_2) = 0.45,$ $P(B_3) = 0.35,$

$$P(A/B_1)=0.02, \qquad P(A/B_2)=0.05, \qquad P(A/B_3)=0.03.$$

1.

$$P(\bar{B}_2/A)=1-P(B_2/A)$$

$$=1-\frac{0.45\cdot 0.05}{0.20\cdot 0.02+0.45\cdot 0.05+0.35\cdot 0.03}$$

$$=1-\frac{0.0225}{0.037}\approx 0.39$$

2.

$$P(\bar{B}_1/\bar{A})=1-P(B_1/\bar{A})$$

$$=1-\frac{0.20\cdot 0.98}{0.20\cdot 0.98+0.45\cdot 0.95+0.35\cdot 0.93}$$

$$=1-\frac{0.1960}{0.949}\approx 0.79.$$

3. 11.
Let

A: „The article is of the highest quality."

B_i: "The article has been produced on machine M_i, $i=1,2,3$."

We have:

$$P(B_1)=0.30, \qquad P(B_2)=0.45, \qquad P(B_3)=0.25,$$

$$P(A/B_1)=0.95, \qquad P(A/B_2)=0.92, \qquad P(A/B_3)=0.98.$$

1.

$$P(\bar{B}_1/\bar{A})=1-P(B_1/\bar{A})$$

$$=1-\frac{0.30\cdot 0.05}{0.30\cdot 0.05+0.45\cdot 0.08+0.25\cdot 0.02}$$

$$=1-\frac{0.015}{0.056}\approx 0.73$$

2.

$$P(\bar{B}_2/A)=1-P(B_2/A)$$

$$=1-\frac{0.45\cdot 0.92}{0.30\cdot 0.95+0.45\cdot 0.92+0.25\cdot 0.98}$$

$$=1-\frac{0.414}{0.944}\approx 0.56$$

3. 12.
Definition of events:

D: „An individual favours death penalty",
M: „An individual is a man. "

$$P(D/M)=\frac{P(D\cap M)}{P(M)}$$
$$=\frac{0.459}{0.459+0.441}=0.51$$

3. 13.
Definition of events:

A: „ The defect will not be present in any particular product"
B: „The quality control will yield a positive result".

Therefore, we have:

$$P(A)=0.005\,,\quad ,\quad P(B/A)=0.99\,,\qquad P(B/\bar{A})=0.05\,.$$

1.

$$P(\bar{A})=1-P(A)=1-0.005=0995\,.$$

2.

$$P(\bar{B}/A)=1-(B/A)=1-0.99=0.01\,.$$

3.

$$P(\bar{B}/\bar{A})=1-P(B/\bar{A})=1-0.05=0.95\,.$$

4.

$$P(B)=P(B/A)\cdot P(A)+P(B/\bar{A})\cdot P(\bar{A})$$

$$=0.99\cdot 0.005+0.05\cdot 0.995=0.0547\,.$$

5.

$$P(\bar{B})=P(\bar{B}/A)\cdot P(A)+P(\bar{B}/\bar{A})\cdot P(\bar{A})$$

$$=0.01\cdot 0.005+0.95\cdot 0.995=0.9453$$

$(= 1 - P(B))$.

6.

$$P(A/B) = \frac{P(B/A) \cdot P(A)}{P(B)} = \frac{0.99 \cdot 0.005}{0.0547} = 0.0905 .$$

7.

$$P(\bar{A}/B) = \frac{P(B/\bar{A}) \cdot P(\bar{A})}{P(B)} = \frac{0.05 \cdot 0.995}{0.0547} = 0.9095 \quad (= 1 - P(A/B))$$

8.

$$P(\bar{A}/\bar{B}) = \frac{P(\bar{B}/\bar{A}) \cdot P(\bar{A})}{P(\bar{B})} = \frac{0.95 \cdot 0.995}{0.9453} = 0.99995$$

9.

$$P(A/\bar{B}) = \frac{P(\bar{B}/A) \cdot P(A)}{P(\bar{B})} = \frac{0.01 \cdot 0.005}{0.9453} = 0.00005 \ (= 1 - P(\bar{A}/\bar{B})).$$

3. 14.

Denote by

I: "Increase of capital investment."
R: "Rise of structural steel prices."

1.

$$P(\bar{R}/I) = 0.10.$$

2.

$$P(R) = P\left(I \cap R\right) + P(\bar{I} \cap R)$$

$$= P(I) \cdot P(R/I) + P(\bar{I}).P(R(R/\bar{I})$$

$$= 0.60 \cdot 0.90 + 0.40 \cdot 0.40 = 0.70.$$

3.

$$P(R/I) = \frac{P(I \cap R)}{P(R)} = \frac{P(I) \cdot P(R/I)}{P(I) \cdot P(R/I) + P(\bar{I}).P(R(R/\bar{I})}$$

$$= \frac{0.60 \cdot 0.90}{0.60 \cdot 0.90 + 0.40 \cdot 0.40} = \frac{0.54}{0.70} \approx 0.77.$$

3. 15.
Denote by

A: „An item is of high quality“,

$B_i \ (i = 1, 2, 3)$: „An item is produced on machine i“.

We have:

$P(B_1) = 0.30$, $P(B_2) = 0.50$, $P(B_3) = 0.20$,

$P(A/B_1) = 0.80$, $P(A/B_2) = 0.70$, $P(A/B_3) = 0.90$.

1.

$$P(A) = \sum_{i=1}^{3} P(B_i) \cdot P(A/B_i) = 0.30 \cdot 0.80 + 0.50 \cdot 0.70 + 0.20 \cdot 0.90 = 0.77 .$$

2.

$$P(\bar{A}) = 1 - P(A) = 1 - 0.77 = 0.23.$$

3.

$$P(B_1 / A) = \frac{P(B_1) \cdot P(A/B_1)}{P(A)} = \frac{0.30 \cdot 0.8}{0.77} = 0.311688311 \approx 0.31.$$

4.

$$P(B_3 / A) = \frac{P(B_3) \cdot P(A/B_3)}{P(A)} = \frac{0.20 \cdot 0.9}{0.77} = 0.233766233 \approx 0.23.$$

5.

$$P(\bar{B}_3/ \bar{A}) = 1 - P(B_3 / \bar{A}) = 1 - \frac{P(\bar{B}_3) \cdot P(\bar{A}/ B_3)}{P(\bar{A})} = 1 - \frac{0.20 \cdot 0.10}{0.23} = 0.913043478 \approx 0.91$$

6.

$$P(\bar{B}_2/ \bar{A}) = 1 - P(B_2 / \bar{A}) = 1 - \frac{P(\bar{B}_2) \cdot P(\bar{A}/ B_2)}{P(\bar{A})} = 1 - \frac{0.50 \cdot 0.30}{0.23} = 0.347826087 \approx 0.35.$$

(Last revised: 24.11.14)

Chapter IV

Random Variables

Exercises

4. 1.

Suppose you randomly select a student attending you university. Classify each of the following random variables as discrete or continuous:

1. Number of credit hours by the student this semester.
2. Current point average of the student.

4. 2.

Classify each of the following random variables as discrete or continuous:

1. The number of babies born in a clinic in a certain week.
2. The height of a student from your university.
3. The weight of a student from your university (in grams).

(Last revised: 05.06.07)

Chapter IV

Random Variables

Solutions

4. 1.
1.
The number of credit hours taken by the student this semester is a discrete random variable because it can assume only a countable number of values (for example 10, 11, 12, and so on).

2.
The grade point average for the student is a continuous random variable because it could theoretically assume any value (for example 3.56, 5.37) corresponding to the points on the interval 1 to 5 of a line.

4. 2.
1. Discrete.
2. Continuous.
3. Discrete.

(Last revised: 05.06.07)

Chapter V

Discrete and Continuous Random Variables

Exercises

5. 1.
A builder orders a shipment of bricks. The random variable X, the number of broken bricks per lot, is estimated by suppliers to have the following probability function:

x_i	0	1	2	3	4	≥ 5
$P(X = x_i)$	0.7	0.1	0.05	0.05	0.03	p_5

1. Find p_5.
2. What is the probability that the number of broken bricks is at most 3?
3. Determine and sketch the distribution function $F(x)$ of the random variable X.
4. Find and interpret

   i) $F(3.8)$
   ii) $F(4.7) - F(1.8)$

5. 2.
The random variable X giving the number of passengers (excluding the driver) per car in rush hour traffic has the following probability function:

x_i	0	1	2	3	4
$P(X = x_i)$	0.7	p_2	0.1	0.05	0.05

1. Find p_2.
2. What is the probability that the number of passengers is at least 2?
3. Determine and sketch the distribution function $F(x)$ of the random variable X.
4. Find and interpret

   a. $F(2.06)$
   b. $F(3.9) - F(0.05)$

5. 3.
After the start of observation on a given summer evening, the time T, in minutes until the first shooting star is observed, follows an exponential distribution for which

$$P(T > t) = e^{-\frac{1}{10}t}$$

where $t > 0$.

1. Determine the probability that it takes between five and ten minutes for the first shooting star to be observed.
2. Determine the probability density function of T.

5. 4. (See Example 5. 2.)
A car pooling study shows that the number of passengers, X, in a car (excluding the driver) is likely to assume the values 0, 1, 2, 3, and 4 with probabilities given by the table

x_i	0	1	2	3	4
$P(X = x_i)$	0.7	p_2	0.1	0.05	0.05

1. Determine $P(X > 2)$.
2. Determine $P(X \geq 2)$.
3. What is the probability that a car will have no passengers?
4. Determine the smallest value of k so that $P(X < k) > 0.85$.
5. Evaluate $P(X \leq k)$ for

$$k = -12,\ 0,\ 0.5,\ 2.4,\ 103.$$

5. 5.
Let the random variable X have the following distribution function:

$$F(x) = \begin{cases} 0 & \text{when} \quad x \leq -1 \\ \frac{3}{4}x + \frac{3}{4} & \text{when} \quad -1 < x \leq \frac{1}{3}. \\ 1 & \text{when} \quad \frac{1}{3} < x. \end{cases}$$

What is the probability that that X lies in the interval $\left]0,\ \frac{1}{3}\right[$?

5. 6.
Consider the function

$$f(x) = \begin{cases} a(3+x) & \text{when} \quad -3 \leq x \leq 0 \\ a(3-x) & \text{when} \quad 0 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

a) For what value of a will $f(x)$ be the density function of the random variable X ?
b) Determine the distribution function of X.
c) Find the probability that X lies in the interval $\left[\frac{1}{2},\ 1\right]$.

(Last revised: 07.03.2018)

Chapter V

Discrete and Continuous Random Variables

Solutions

5. 1.

1.

$$p_3 = 1 - (0.70 + 0.10 + 0.05 + 0.05 + 0.03) = 1 - 0.93 = 0.07$$

2.

$$P(X \le 3) = 0.70 + 0.10 + 0.05 + 0.05 = 0.90$$

3.

$$F(x) = \begin{cases} 0.00 & \text{when} \quad -\infty < x \le 0 \\ 0.70 & \text{when} \quad 0 < x \le 1 \\ 0.80 & \text{when} \quad 1 < x \le 2 \\ 0.85 & \text{when} \quad 2 < x \le 3 \\ 0.90 & \text{when} \quad 3 < x \le 4 \\ 0.93 & \text{when} \quad 4 < x \le 5 \\ 1.00 & \text{when} \quad 5 < x < +\infty \end{cases}$$

4.

i)

$$F(3.8) = P(X < 3.8) = 0.9$$

The probability that the number of broken bricks is less than 3.8 is equal to 90%.

ii)

$$F(4.7) - F(1.8) = 0.93 - 0.80 = 0.13 = P(1.8 \le X < 1.8)$$

The probability that the number of broken bricks is at least 1.8 and less than 4.7 is equal to 13%.

5. 2.

1.

$$p_2 = 1 - (0.70 + 0.10 + 0.05 + 0.05) = 1 - 0.90 = 0.10$$

2.

$$P(X \geq 2) = 0.10 + 0.05 + 0.05 = 0.20$$

3.

$$F(x) = \begin{cases} 0.00 & \text{when} \quad -\infty < x \leq 0 \\ 0.70 & \text{when} \quad 0 < x \leq 1 \\ 0.80 & \text{when} \quad 1 < x \leq 2 \\ 0.90 & \text{when} \quad 2 < x \leq 3 \\ 0.95 & \text{when} \quad 3 < x \leq 4 \\ 1.00 & \text{when} \quad 4 < x < +\infty \end{cases}$$

4.
i)

$$F(2.06) = 0.9$$

The probability that the number of passengers is less than 2.06 is equal to 90%.

ii)

$$F(3.9) - F(0.05) = 0.95 - 0.70 = 0.25 = P(0.05 \leq X < 3.9)$$

The probability that the number of passengers is at least 0.05 and less than 3.9 is equal to 25 %.

5. 3.
1.

$$F(t) = P(T < t) \approx P(T \leq t) = 1 - P(T > t) = 1 - e^{-\frac{t}{10}}$$

$$P(5 \leq T < 10) = \left(1 - e^{-1}\right) - (1 - e^{-0.5}) = 0.23865122$$

2.
1.

$$P(X > 2) = 0.05 + 0.05 = 0.10$$

2.

$$P\left(X \geq 2\right) = 0.10 + 0.05 + 0.05 = 0.20\ .$$

3.

$$P(X = 0) = 0.7$$

4.

$$k = 3.$$

5.

0, 0.7, 0.7, 0.9, 1.

5. 5.

$$P\left(0 < X < \frac{1}{3}\right) = F\left(\frac{1}{3}\right) - F(0) = \left(\frac{3}{4}\cdot\frac{1}{3} + \frac{3}{4}\right) - \frac{3}{4} = \frac{1}{4}.$$

5. 6.

a)

$$\int_{-\infty}^{+\infty} f(x)dx = 1;$$

$$\int_{-3}^{0} a\cdot(3+x)dx + \int_{0}^{3} a\cdot(3-x)dx = \left[3ax + a\cdot\frac{x^2}{2}\right]_{-3}^{0} + \left[3ax - a\cdot\frac{x^2}{2}\right] = 1;$$

$$-\left(3a\cdot(-3) + a\cdot\frac{9}{2}\right) + \left(9a - 9\cdot\frac{a}{2}\right) = 1;$$

$$a = \frac{1}{9}.$$

b)

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$F(x) = \begin{cases} 0 & -\infty < x \le -3 \\ \frac{1}{18}x^2 + \frac{1}{3}x + \frac{1}{2} & -3 < x \le 0 \\ -\frac{1}{18}x^2 + \frac{1}{3}x + \frac{1}{2} & 0 < x \le 3 \\ 1 & 3 < x < +\infty \end{cases};$$

c)

$$P\left(\frac{1}{2} \le X \le 1\right) = F(1) - F(0.5) = \frac{1}{9}\cdot\int_{0.5}^{1}(3-x)dx = \frac{1}{8}.$$

(Last revised: 18.11.2013)

Chapter VI

Parameters of a Random Variable

Exercises

6. 1.
Let the random variable X have the probability function

x_i	-3	0	1	2	3
$P(X = x_i)$	0.1	0.15	0.1	0.25	0.4

Find
a) the distribution function of X.
b) $P(X > 0)$.
c) the expected value and the dispersion of X.

6. 2.
An apparatus comprises 3 sensitive elements. Let p_i, $(i = 1,\ 2,\ 3)$, denote the probability that the ith element falls out.
Find the expected value of the number of elements that will fall out.

6. 3.
Consider the function f with

$$f(x) = \begin{cases} \alpha x^2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

a) For what value of α will f be the density function of a random variable X?
b) Find the distribution function, the expected value and the dispersion of X.
c) Determine $P(X < \frac{1}{2})$ and $P(X < E(X))$.

6. 4.
The following table identifies the probability that a computer network will be inoperative for the indicated number of periods per week during the initial installation phase for the network:

Number of Inoperative Periods per Week for a New Computer Network

Number of periods (X)	4	5	6	7	8	9
Probability ($P(X)$)	0.01	0.08	0.29	0.42	0.14	0.06

Calculate

1. the expected number of times per week that the network is inoperative
2. the variance
3. the standard deviation

for this variable.

(Last revised: 0703.2018)

Chapter VI

Parameters of a Random Variable

Solutions

6. 1.
a)

$$F(x)=\begin{cases} 0 & -\infty < x \le -3 \\ 0.10 & -3 < x \le 0 \\ 0.25 & 0 < x \le 1 \\ 0.35 & 1 < x \le 2 \\ 0.60 & 2 < x \le 3 \\ 1 & 3 < x < +\infty \end{cases}$$

b)

$$P(X>0)=P(X=1)+P(X=2)+P(X=3)$$

$$=0.10+0.25+0.40=0.75$$

c)

$$E(X)=-3\cdot 0.1+0\cdot 0.15+1\cdot 0.1+2\cdot 0.25+3\cdot 0.4=1.5$$

$$D^2(X)=(-3)^2\cdot 0.1+0^2\cdot 0.15+1^2\cdot 0.1+2^2\cdot 0.25+3^2\cdot 0.4-(1.5)^2=3.35\,.$$

6. 2.
Let

X **: „**Number of elements that will fall out“.

$$E(X)=\sum_{i=1}^{3} p_i\cdot x_i=0\cdot(1-p_1)\cdot(1-p_2)\cdot(1-p_3)$$

$$+1\cdot[p_1\cdot(1-p_2)\cdot(1-p_3)+p_2\cdot(1-p_1)\cdot(1-p_3)+p_3\cdot(1-p_1)\cdot(1-p_2)]$$

$$+2\cdot[p_1\cdot p_2\cdot(1-p_3)+p_1\cdot p_3\cdot(1-p_2)+p_2\cdot p_3\cdot(1-p_1)]+3\cdot p_1\cdot p_2\cdot p_3$$

$$=p_1+p_2+p_3\,.$$

6. 3.

a)

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^1 \alpha \cdot x^2 \cdot (1-x)\,dx = 1,$$

$$\alpha \cdot \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1, \qquad \alpha \cdot \left(\frac{1}{3} - \frac{1}{4}\right) = 1, \qquad \alpha = 12.$$

b)

$$F(x) = \begin{cases} 0 & x \le 0 \\ 12(\frac{x^3}{3} - \frac{x^4}{4}) & 0 < x \le 1, \\ 1 & 1 < x \end{cases}$$

$$E(X) = \int_0^1 x \cdot f(x)dx = \int_0^1 12 \cdot (x^3 - x^4)dx = 12 \cdot \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 12 \cdot \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{3}{5}.$$

$$D^2(X) = \int_0^1 x^2 \cdot f(x)dx - (E(X))^2 = 12 \cdot \int_0^1 (x^4 - x^5)dx - \frac{9}{25} = 12 \cdot \left[\frac{x^5}{5} - \frac{x^6}{6}\right]_0^1 - \frac{9}{25} = \frac{1}{25}.$$

c)

$$P\left(X < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) = 12 \cdot \left[\frac{\frac{1}{8}}{3} - \frac{\frac{1}{16}}{4}\right] = 0.3125.$$

$$P(X < E(X)) = P\left(X < \frac{3}{5}\right) = F\left(\frac{3}{5}\right) = 12 \cdot \left[\frac{\left(\frac{3}{5}\right)^3}{3} - \frac{\left(\frac{3}{5}\right)^4}{4}\right] = 0.4752.$$

6. 5.

1.

$E(X) = 4 \cdot 0.01 + 5 \cdot 0.08 + 6 \cdot 0.29 + 7 \cdot 0.42 + 8 \cdot 0.14 + 9 \cdot 0.06 = 6.78$ periods.

2.

$D^2(X) = 4^2 \cdot 0.01 + 5^2 \cdot 0.08 + 6^2 \cdot 0.29 + 7^2 \cdot 0.42 + 8^2 \cdot 0.14 + 9^2 \cdot 0.06 - (6.78)^2 = 1.0316.$

3.

$D(X) = \sqrt{1.0316} \approx 1.02$ periods.

(Last revised: 07.03.2018)

Chapter VII

Some Special Discrete Distribution Functions

Exercises

7. 1.
A test for impurities commonly found in drinking water from private wells showed that 30% of all wells in a particular country have impurity A.
If a random sample of 5 wells is selected from the large number of wells in the country, what is the probability that

1. exactly 3 will have impurity A?
2. at least 3?
3. fewer than 3?

7. 2.
Fifty items are submitted for acceptance. It is known that there are 4 defective items in the lot

1. What is the probability of finding exactly 1 defective item in a sample of 5?
2. What is the probability of finding less than 2 defective items in a sample of 5?

7. 3.
A certain drug cures 90% of cases of hookworm in children. Suppose that 20 children suffering from hookworm are to be treated. and that the children can be regarded as a random sample from the population. Find the probability that

a) all 20 will be cured
b) all but one will be cured
c) exactly 90% will be cured.

7. 4.
The sex ratio of newborn human infants is about 105 males:100 females. If 4 infants are chosen at random, what is the probability that

a) 2 are male and 2 are female?
b) all four all male?
c) all four are the same sex?

7. 5.
There are 50 misprints in a book which has 250 pages. Find the probability that page 100 has no misprints.

7. 6.
A department store sells biros in packets of 10. It is known that 5% of biros are defective on average.
Find the probability that there are

1. less than 3
2. at least 3

defective biros in a packet.

7. 7.

The Secret Service monitors the number of death threats that are made on the President of the United States. A report suggests that the number of death threats made on the President averages four per week.
Identify the type of variable that Secret Service must monitor and the specific distribution that could most likely be used to model the number of death threats received.

7. 8.

Suppose that we are investigating the safety of a dangerous intersection. Past police records indicate a mean of 5 accidents per month at this intersection. Suppose the number of accidents is distributed according to a Poisson distribution.
Calculate the probability in any month of exactly 0, 1, ... or 12 accidents.

7. 9.

A firm employs 50 people in the Assembly Department. Forty of the employees belong to a trade union and ten do not. Five employees are selected at random to form a committee to meet with management regarding shift starting times.
What is the probability that four of the five selected for the committee belong to a trade union?

7. 10.

There are five flights daily from Berlin to Leipzig. Suppose the probability that any flight arrives late in Leipzig is 0.20.

1. What is the probability that none of the flights are late today?
2. What is the average number of late flights?
3. What is the variance of the number of late flights?

7. 11.

Suppose a bank knows that on average 60 customers arrive in a certain service hour. Using *a time interval of 1 minute*, calculate the probability of

1. exactly one customer
2. no customers
3. exactly three customers
4. more than three customers

arriving in a given one minute interval within that hour.

7. 12.

On average, a ship arrives at a certain dock every second day. What is the probability that two or more shops will arrive on a randomly selected day?

7. 13.

An Internal Revenue Service inspector is to select 3 corporations from a list of 15 for tax audit purposes. Of the 15 corporations, 6 earned profits and 9 incurred losses during the year for which the tax returns are to be audited.
If the IRS inspector selects 3 corporations randomly, find the probability that the number of corporations in these 3 that incurred losses during the year for which the tax returns are to be audited is

1. exactly 2
2. none
3. at most 1.

7. 14.

A survey reported that 70% of all U.S. households have cellular phones. If you randomly selected 11 households, what is the probability that

1. each of the 11 households has a cellular phone?
2. more than four households have a cellular phone?
3. fewer than five households have a cellular phone?
4. more than seven households do not have a cellular phone?

(Last revised: 11.08.2010)

Chapter VII

Some Special Discrete Distribution Functions

Solutions

7. 1.
First we confirm that this experiment possesses the characteristics of a binomial experiment. This experiment consists of 5 trials, one corresponding to each random selected well. Each trail results in an S (the well contains impurity A) or an F (the well does not contain impurity A). Since the total number of wells in the country is large, the probability of drawing a single well and finding that it contains impurity A is equal to 0.30 and this probability will remain the same for each of the 5 selected wells. Further, since the sampling is random, we assume that the outcome on any one well is unaffected by the outcome of any other and that the trials are independent. Finally we are interested in the number of wells in the sample of 5 that contain impurity A.
Therefore, the sampling process represents a binomial experiment with $n = 5$ and $p = 0.30$.
Let X denote the number of wells containing impurity A.

1.

$$P(X = 3) = \binom{5}{3} \cdot 0.30^3 \cdot 0.70^2 = 0.1323 .$$

2.

$$P(X \geq 3) = \sum_{x=3}^{5} \binom{5}{x} \cdot 0.30^x \cdot 0.70^{5-x} = 0.13230 + 0.02835 + 0.00243 = 0.16308 .$$

3.

$$P(X < 3) = 1 - P(X \geq 3) = 1 - 0.16380 = 0.83692 .$$

7. 2.
We have

$$N = 50, \quad M = 4, \quad n = 5 .$$

1.

$$P(X = 1) = p_1 = \frac{\binom{4}{1} \cdot \binom{50-4}{5-1}}{\binom{50}{5}} = 0.308076422 \cdot$$

2.

$$F(2) = P\left(X < 2\right) = \sum_{x=0}^{1} \frac{\binom{4}{x}\cdot\binom{50-4}{5-x}}{\binom{50}{5}} = 0.646960486 + 0.308076422 = 0955036908\cdot$$

7. 3.
We have

$$n = 20, \quad p = 0.90\,.$$

a)

$$P(X = 20) = \binom{20}{20}\cdot 0.90^{20}\cdot 0.10^{0} = 0.121576654\,.$$

b)

$$P(X = 19) = \binom{20}{19}\cdot 0.90^{19}\cdot 0.10^{1} = 0.270170343\,.$$

c)

$$P(X = 18) = \binom{20}{18}\cdot 0.90^{18}\cdot 0.10^{2} = 0.285179807\,.$$

7. 4.
Let the random variable X be the number of female. We have

$$n = 4, \quad p = 0.4878 \quad (100\cdot 100 : 205 \approx 48.78\%)\,.$$

a)

$$P(X = 2) = \binom{4}{2}\cdot 0.5122^{2}\cdot 0.4878^{2}\cdot = 0.374553612\,.$$

b)

$$P(X = 4) = \binom{4}{0}\cdot 0.4878^{0}\cdot 0.5122^{4} = 0.068826913\,.$$

c)

$$P(X = 4) + P(X = 0) = \binom{4}{4}\cdot 0.4878^{4}\cdot 0.5122^{0} + \binom{4}{0}\cdot 0.4878^{0}\cdot 0.5122^{4}$$

$$= 0.068826913 + 0.05661965 = 0.137653826\,.$$

7. 5.

Let the random variable X be the number of misprints on a page. We have $\lambda = \dfrac{50}{250} = 0.2$.

$$P(X = 0) = p_0 = \frac{0.2^{0}}{0!}\cdot e^{-0.2} = 0.818730753\,.$$

7. 6.

Let the random variable X be the number of defective biros in a packet. X is binomially distributed with

$$n = 10, \quad p = 0.05 .$$

1.

$$P(X < 3) = \sum_{x=0}^{2} \binom{10}{x} \cdot 0.05^{x} \cdot 0.95^{10-x} .$$

$$= 0.598736939 + 0.315124704 + 0.074634798 = 0.988496441 .$$

2.

$$P(X \geq 3) = 1 - P(X < 3)$$

$$= 1 - P(X < 3) = 1 - \sum_{x=0}^{2} \binom{10}{x} \cdot 0.05^{x} \cdot 0.95^{10-x} = 0.011503558 .$$

7. 7.

Discrete, Poisson distribution.

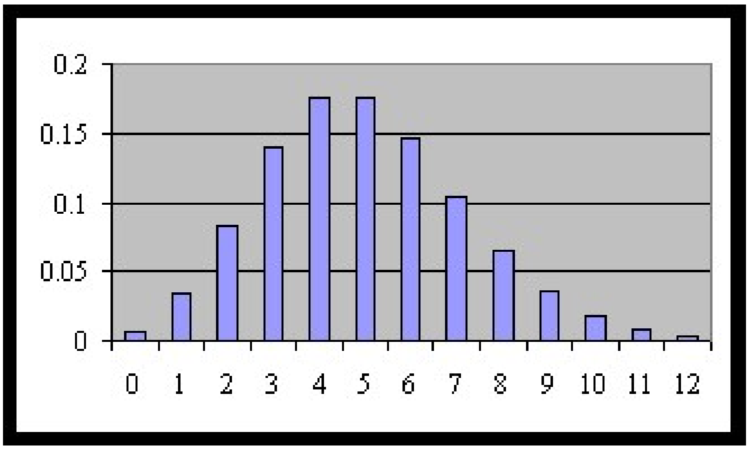
7. 8.

Let X denote the number of accidents per month. We have to calculate

$$P(X = x) = \frac{5^{x}}{x!} \cdot e^{-5}, \quad x = 0,\ 1,\ \ldots,\ 12 .$$

The probability distribution of X is presented below:

Number of Accidents	Probability
0	0.006738
1	0.03369
2	0.084224
3	0.140374
4	0.175467
5	0.175467
6	0.146223
7	0.104445
8	0.065278
9	0.036266
10	0.018133
11	0.008242
12	0.003434
Poisson probability distribution of the number of accidents per month	



The Poisson probability distribution of the number of accidents

7. 9.

Let X denote the number employees belonging to a trade union. We have a hypergeometrically distributed random variable with

$$N = 50, \quad M = 40, \quad n = 5.$$

$$P(X = 4) = \frac{\binom{40}{4} \cdot \binom{50-40}{5-4}}{\binom{50}{5}} \approx 0.431. \cdot$$

7. 10.

Let X denote the number of late flights. We have a binomially distributed random variable with

$$n = 5, \quad p = 0.2.$$

1.

$$P(X = 0) = \binom{5}{0} \cdot 0.2^0 \cdot 0.8^5 = 0.32768..$$

2.

$$E(X) = 5 \cdot 0.2 = 1, \qquad D^2(X) = 0.8.$$

7. 11.

Let X denote the number of customers arriving within an interval of one minute. We have the Poisson distribution with $\lambda = 1$.

1. $P(X=0)=0.367879$.
2. $P(X=1)=0.367879$.
3. $P(X=3)=0.0.061313$.
4. $P(X>3)=1-P(X\le 3)=1-(0.367879+0.367879+0.183840+0.061313)=0.018989$.

7. 12.
Let X denote the number of ships arriving at the dock. We have the Poisson distribution with $\lambda=0.5$.

$$P(X\ge 2)=1-P(X<1)=1-(0.606531+0.303265)=0.090204.$$

7. 13.
Denote by

X: "The number of corporations which incurred losses."

X is Hypergeometrically distributed with

$$N=15,\quad M=9,\quad n=3.$$

1.

$$P(X=2)=\frac{\binom{9}{2}\cdot\binom{15-9}{3-2}}{\binom{15}{3}}=\frac{36\cdot 6}{455}\approx 0.47.47$$

2.

$$P(X=0)=\frac{\binom{9}{0}\cdot\binom{15-9}{3-0}}{\binom{15}{3}}=\frac{1\cdot 20}{455}\approx 0.0440.$$

3.

$$P(X=0)+P(X=1)=0.0440+\frac{\binom{9}{1}\cdot\binom{15-9}{3-1}}{\binom{15}{3}}=0.0440+\frac{9\cdot 15}{455}$$

$$=0.0440+0.0297\approx 0.3407$$

7. 14.
Let

X: "the number of cellular phones in a U.S. household."

X is binomially distributed.

1.

$$n = 11, \quad p = 0.70$$

$$P(X = 11) = \binom{11}{11} 0.70^{11} \cdot 0.30^{11-11} \approx 0.01977326743.$$

2.

$$n = 11, \quad p = 0.70$$

$$P(X > 4) = 1 - P(X \le 4) = 1 - \sum_{x=0}^{4} P(X = x)$$

$$= 1 - \sum_{x=0}^{4} \binom{11}{x} \cdot 0.70^{x} \cdot 0.30^{11-x}$$

$$\approx 1 - \left(0.00000 + 0.00005 + 0.00053 + 0.00371 + 0.01733\right) = 0.97838.$$

3.

$$n = 11, \quad p = 0.70$$

$$P(X < 5) = P(X \le 4) = \sum_{x=0}^{4} P(X = x)$$

$$= \sum_{x=0}^{4} \binom{11}{x} \cdot 0.70^{x} \cdot 0.30^{11-x}$$

$$\approx 0.00000 + 0.00005 + 0.00053 + 0.00371 + 0.01733 = 0.02162$$

4.

$$n = 11, \quad p = 0.30$$

$$P(X > 7) = \sum_{x=8}^{11} P(X = x) = \sum_{x=8}^{11} \binom{11}{x} \cdot 0.30^{x} \cdot 0.70^{11-x}$$

$$\approx 0.00371 + 0.00053 + 0.00005 + 0.00000 = 0.00429.$$

(Last revised: 11.08.2010)

Chapter VIII

Some Special Continuous Distribution Functions

Exercises

8. 1.
Assume that the average weekly income of 10000 workers in an enterprise is 500 € and the standard deviation is 100 €. If the distribution is normal, find the number of workers having a weekly income

a) below 500 €,
b) above 500 € but below 600 €
c) above 600 €.

8. 2.
The distribution of heights of American women aged 18 to 24 is approximately normally distributed with mean 65.5 inches and standard deviation 2.4 inches.
Determine the intervals in which the heights of

1. 68%
2. 95%

of these American women lie.

8. 3.
Show graphically that the binomial distribution with $n = 10$ and $p = 0.5$ is well approximated by the normal distribution.

8. 4.
The life of a particular type of light bulb is normally distributed with a mean of 1100 hours and a standard deviation of 100 hours.
What is the probability that a light bulb of this type will last between 1000 and 1200 hours?

8. 5.
The target value for the thickness of a machined cylinder is 8 cm. The upper specification limit is 8.2 and the lower specification limit is 7.9. A machine produces cylinders that have a mean thickness of 8.1 cm and a standard deviation of 0.1 cm.
Find the probability that the thickness of a cylinder

1. lies within the specification limits.
2. deviates from the mean by 0.5 cm?
3. is at least 8 cm?

8. 6.
The mean and the standard deviation of all cash sale amounts in a small shop in the last year are 60 € and 10 €, respectively. Assume that the amounts are distributed normally.

1. What proportion of the amounts is

a) between 55.00 € and 72.50 €?
b) above 72.50 €
c) below 55.00 €

2. What is the amount above which the proportion of the amounts is 20%?

8. 7.
A set of exam marks has a mean of 45 and a standard deviation of 15. You should assume that the marks are integers and that they can be approximated by a normal distribution.

1. If the pass mark is 40, what percentage of candidates passes?
2. What mark should be used to ensure that 75% of the candidates pass?
3. Suppose that the pass mark is 40 and that ten candidates are selected at random. Use the probability that you found in part1 to find the probability that at least six of them pass.

8. 8.
More and more households in the United States have at least one computer. The computer is used for office work at home, research, communication, personal finances, education, entertainment, and a myriad of other things.
Suppose the average number of hours a household personal computer is used for entertainment is 2 hours per day. Assume the times for entertainment are normally distributed and the standard deviation for the times is half an hour.

1. Find the probability that a household personal computer is used for entertainment between 1.8 and 2.75 hours per day.
2. Find the probability that a household personal computer is used for entertainment at least 1.4 hours per day.
3. Find the maximum number of hours per day that the bottom quartile of households uses a personal computer for entertainment.

8. 9.
An airline determines that there is a 95% chance that a passenger with a ticket will show up for a given flight. Suppose that the airline has sold tickets to 330 passengers for a flight with 320 seats.

1. Find the mean and standard deviation for the number of passengers that will show up for the flight.
2. Explain why the normal approximation to the binomial distribution can be used in the situation.
3. Use the normal distribution to the binomial distribution to compute the probability that 320 or less of the 330 people holding the ticket will show up for the flight.

(Last revised: 16.08.10)

Chapter VIII

Some Special Continuous Distribution Functions

Solutions

8. 1.
Let X denote the weekly income of workers.

$$\mu = 500 \text{ €}, \quad \sigma = 100 \text{ €}$$

a)

$$P(X < 500) = F(500) = \Phi\left(\frac{500-500}{100}\right) = \Phi(0) = 0.5 .$$

The number of workers having a weekly income below 500 is: $10000 \cdot 0.5 = 5000$.

b)

$$P(500 < X < 600) \approx P(500 \le X < 600)$$

$$= \Phi\left(\frac{600-500}{100}\right) - \Phi\left(\frac{600-500}{100}\right)$$

$$= \Phi(1) - \Phi(0) = 0.84134 - 0.50000 = 0.34134 .$$

The number of workers having a weekly income above 500 € but below 600 € is:
$10000 \cdot 0.34134 = 3413$.

c)

$$P(X > 600) = 1 - P(X \le 600) \approx 1 - P(X < 600)$$

$$= 1 - P(X < 600) = 1 - F(600) = 1 - \Phi\left(\frac{600-500}{100}\right)$$

$$= 1 - \Phi(1) = 1 - 0.84134 = 0.15866 .$$

The number of workers having a weekly income above 600 € is : $10000 \cdot 0.15866 = 1587$.

8. 2.
1.

$$[65.5 - 2.4,\ 65.5 + 2.4] = [63.1,\ 67.9].$$

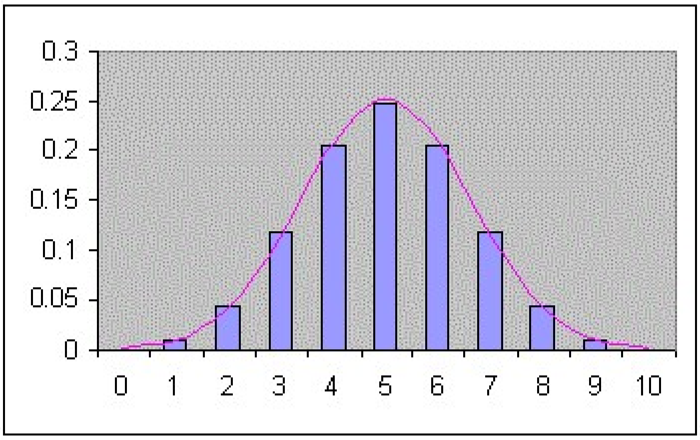
2.

$$[65.5 - 2 \cdot 2.4,\ 65.5 + 2 \cdot 2.4] = [60.7,\ 70.3].$$

8. 3.

$$E(X) = \mu = n \cdot p = 10 \cdot 0.5 = 5$$
$$D(X) = \sigma = \sqrt{n \cdot p \cdot q} = \sqrt{10 \cdot 0.5 \cdot 0.5} \approx 1.58$$

x	Binomial Distribution	Normal Distribution
0	0.000977	0.001700
1	0.009766	0.010285
2	0.043945	0.041707
3	0.117188	0.113372
4	0.205078	0.206577
5	0.246094	0.252313
6	0.205078	0.206577
7	0.117188	0.113372
8	0.043945	0.041707
9	0.009766	0.010285
10	0.000977	0.001700



8. 4.
Applying the "empirical rule", we obtain a probability of approximately 68%.

8. 5.
Let X denote the thickness of a machined cylinder.

$$\mu = 8.1 \text{ cm}, \quad \sigma = 0.1 \text{ cm}$$

1.
$$\begin{aligned} P(7.9 \le X < 8.2) &= F(8.2) - F(7.9) \\ &= \Phi\left(\frac{8.2-8.1}{0.1}\right) - \Phi\left(\frac{7.9-8.1}{0.1}\right) = \Phi(1) - \Phi(-2) \\ &= \Phi(1) - (-\Phi(2)) \end{aligned}$$

$$= 0.841345 - 1 + 0.977250 = 0.818595.$$

2.

$$P\left(\left|X - 8.1\right| < 0.5\right) = 2 \cdot \Phi\left(\frac{0.5}{0.1}\right) - 1 = 2 - 1 = 1.$$

3.

$$P\left(X \geq 8\right) = 1 - P(X < 8) = 1 - F(8)$$

$$= 1 - \Phi\left(\frac{8 - 8.1}{0.1}\right) = 1 - \Phi\left(-1\right)$$

$$= 1 - \left(1 - \Phi\left(1\right)\right) = 0.841345.$$

8. 6.
Let X denote the cash sale amounts.

1.
a)

$$P(55.00 \leq X < 72.50) = F(72.50) - F(55.00)$$

$$= \Phi\left(\frac{72.50 - 60.00}{10}\right) - \Phi\left(\frac{55.00 - 60.00}{10}\right)$$

$$= \Phi\left(1.25\right) - \Phi\left(-0.50\right) = \Phi\left(1.25\right) - \left(1 - \Phi\left(0.50\right)\right)$$

$$= 0.894350 - 1 + 0.691462 = 0.585812.$$

b)

$$P(X \geq 72.50) = 1 - P(X < 72.50)$$

$$= 1 - F(72.50) =$$

$$= 1 - \Phi\left(1.25\right) = 1 - 0.844350 = 0.105650.$$

c)

$$P(X < 55.00) = F(55.00)$$

$$= \Phi(-0.5) = 1 - \Phi(0.5) = 1 - 0.691462 = 0.308538.$$

2.

$$P(X > c) \approx P(X \geq c) = 0.20,$$

$$1 - P(X < c) = 0.2 \ ; \qquad P(X < c) = 0.8,$$

$F(c)=0.8;\qquad \Phi\left(\frac{c-60}{10}\right)=0.8=\Phi(0.84),$

$$\frac{c-60}{10}=0.84 \quad \Rightarrow \quad c=68.4\,.$$

8. 7.
Let X denote the exam mark.

$$\mu=45, \quad \sigma=15\,.$$

1.

$$P(X\geq 40)=1-P(X<40)=1-F(40)$$
$$=1-\Phi\left(\frac{40-45}{15}\right)=1-\Phi(-0.33)=1-\left(1-\Phi(0.33)\right)=0.629300\,,$$

i.e. nearly 63%

2.

$$P(X\geq c)=1-P(X<c)=0.75\,,$$
$$1-F(c)=1-\Phi\left(\frac{c-45}{15}\right)=0.75\,,$$
$$\Phi\left(\frac{c-45}{15}\right)=0.25=\Phi(-0.68)$$
$$\frac{c-45}{15}=-0.68 \quad \Rightarrow \quad c=34.8\approx 35.$$

3.

$$P(X\geq 6)=1-\sum_{x=6}^{10}\binom{10}{x}\cdot 0.63^{x}\cdot 0.37^{10-x}$$
$$=1-\left(0.246076+0.239425+0.152876+0.057845+0.009849\right)$$

$=1-0.706071=0.293929$, i.e. nearly 29%.

8. 8.
Let

X: “The amount of time [hours] a household personal computer is used for entertainment

X is normally distributed with

$$\mu=2, \quad \sigma=0.5\,.$$

1.

$$P(1.80 \le X < 2.75) = F(2.75) - F(1.80)$$

$$= \Phi\left(\frac{2.75-2.00}{0.5}\right) - \Phi\left(\frac{1.80-2.00}{0.5}\right) = \Phi(1.5) - \Phi(-0.4)$$

$$= \Phi(1.5) - (1-\Phi(0.4)) = \Phi(1.5) - 1 + \Phi(0.4)$$

$$= 0.9332 - 1 + 0.6554 = 0.5886.$$

2.

$$P(X \ge 1.4) = 1 - P(X < 1.4) = 1 - F(1.4)$$

$$= 1 - \Phi\left(\frac{1.4-2}{0.5}\right) = 1 - \Phi(-1.2)$$

$$= 1 - (1-\Phi(1.2)) = 0.8849$$

3.

$$P(X < x) = 0.25$$

$$F(x) = 0.25$$

$$\Phi\left(\frac{x-2}{0.5}\right) = 0.25 = \Phi(-0.67)$$

$$\frac{x-2}{0.5} = -0.67$$

$$x = 1.67$$

8. 9.

Let

X: "the number of passengers with a ticket who show up for a given flight."

X is binomially distributed with

$$n = 330, \quad p = 0.95.$$

1.

$$\mu = n \cdot p = 330 \cdot 0.95 = 313.5$$

$$\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{330 \cdot 0.95 \cdot 0.05} \approx 3.959.$$

2.

$$n \cdot p = 313.5 > 5 \quad \wedge \quad n \cdot q \approx 313.5 > 5.$$

3.

$$P(X \le 320) \approx P(X < 320) = F(320)$$

$$= \Phi\left(\frac{320-313.5}{3.959}\right) \approx \Phi(1.64) = 0.9495.$$

(Last revised: 16.08.2010)

Probability Theory Formulary

I. Events Algebra

Random Trial
A trial whose outcome cannot be predicted in advance is called a *random trial*.

Random Event
The outcome of a random trial is called a *random event*.

Impossible and Certain Events
An event that in all repetitions of a certain random trial will never happen is called an *impossible event*. It will be denoted by $\varnothing$.
An event that in all repetitions of a certain event will always occur is called a *certain event*. It will be denoted by Ω.

Subevent
The event E_1 is called a *subevent of the event* E_2 if it always accompanies the event E_2.
We write

$$E_1 \subseteq E_2.$$

Sum of Events
E is said to be the *sum* of the events $E_i, i = 1, 2, ..., n$, if <u>at least</u> one of the events E_i occurs:

$$E := \bigcup_{i=1}^{n} E_i$$

Product of Events
E is said to be the *product* of the events $E_i, i = 1, 2, ..., n$, if the events E_i occurs <u>at the same time:</u>

$$E := \bigcap_{i=1}^{n} E_i .$$

Mutually exclusive events
The events E_1 and E_2 are said to be *mutually exclusive* if

$$E_1 \cap E_2 = \varnothing$$

Complementary Events

The events E and $\bar{E}$ are said to be *complementary* if

$$E \cup \bar{E} = \Omega \quad \wedge \quad E \cap \bar{E} = \varnothing$$

Difference

The *difference* of the events E_1 and E_2, denoted by $E_1 \setminus E_2$ is defined as the case in which E_1 occurs while E_2 does not occur.

System of Mutually Exclusive and Exhaustive Events

The events E_i, $i = 1, 2, ..., n$, form a *mutually exclusive and exhaustive* system if the following conditions are fulfilled:

1. $E_i \neq \emptyset, \qquad i = 1, 2, ..., n$
2. $\bigcup_{i=1}^{n} E_i = \Omega$
3. $E_i \cap E_j = \emptyset, \quad i \neq j, \quad i, j = 1, 2, ..., n$

Elementary or Atomic Event

An *elementary* or *atomic event* is an event which is not further genuinely decomposable.

II. Probability Algebra

***Addition rule*:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional probability

$$P(A/B) := \begin{cases} \dfrac{P(A \cap B)}{P(B)} & \text{if} \quad P(B) > 0 \\ 0 & \text{if} \quad P(B) = 0 \end{cases}$$

Multiplication rule

$$P(A \cap B) = P(B) \cdot P(A/B)$$

$$= P(A) \cdot P(B/A)$$

Dependent and independent events

The event A is said to be *independent* of B if

$$P(A) = P(A/B).$$

Otherwise, A is said to be *dependent* of B.

Multiplication rule for two independent events
Let $A, B \in S$ be two *independent* events. Then

$$P(A \cap B) = P(A) \cdot P(B)$$

Multiplication rule for n independent events
Let E_i, $i = 1, 2, ..., n$, be *independent* events. Then

$$P\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} P(E_i).$$

Addition rule for n independent events
Let E_i, $i = 1, 2, ..., n$, be *independent* events. Then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = 1 - \prod_{i=1}^{n} \left(1 - P(E_i)\right)$$

Total probability
Let B_i, $i = 1, 2, ..., n$, form a group of mutually exclusive and exhaustive events. For an arbitrary event A we have

$$P(A) = \sum_{i=1}^{n} P(A / B_i) \cdot P(B_i).$$

Bayes-Theorem
Let B_i, $i = 1, 2, ..., n$, form a group of mutually exclusive and exhaustive events. For an arbitrary event $A \neq \emptyset$ we have:

$$P(B_i / A) = \frac{P(A / B_i) \cdot P(B_i)}{\sum_{i=1}^{n} P(A / B_i) \cdot P(B_i)}, \qquad i = 1, 2, ..., n.$$

Sampling schemes
Consider a finite set of N elements, $M \leq N$ elements of which have a certain property. Let us choose a sample of $n \leq N$ elements.
The probability that the sample contains $m \leq n$ elements with the above-mentioned property is in case of

1. *Nonreplacement:*

$$P_{noplacement} = \frac{\binom{M}{m} \cdot \binom{N-M}{n-m}}{\binom{N}{n}}$$

2. *Replacement*

$$P_{nreplacet} = \binom{n}{m} \cdot p^m \cdot q^{n-m}, \qquad \frac{M}{N} =: p, \qquad q := 1 - p$$

III. Random Variables

Distribution function

A *distribution function* is defined as:

$$F(X) = P(X < x), \quad x \in R^1$$

Important properties of the distribution function

1.

$$0 \le F(x) \le 1, \quad \forall x \in R^1.$$

2.

$$\forall x_1, x_2 : x_1 < x_2 \quad \Rightarrow \quad F(x_1) \le F(x_2).$$

3.

$$\forall x_1, x_2 : x_1 < x_2 \quad \Rightarrow \quad P(x_1 \le X < x_2) = F(x_2) - F(x_1).$$

4.

$$x \to -\infty \quad \Rightarrow \quad F(x) \to 0$$

$$x \to +\infty \quad \Rightarrow \quad F(x) \to 1.$$

5.

$F(x)$ is at least left-sided continuous and has at most a finite number of jump discontinuities.

Probability function

If X is a discrete random variable, then the function

$$p(x) = P(X = x)$$

defined on the outcomes of X is called the *probability function* of the discrete random variable X.

If X has the outcomes x_i, $i = 1, 2, ..., n$, then we can write:

x_i	x_1	x_2	$\dots$	x_n
$p_i = P(X = x_i) = f(x_i)$	p_1	p_2	$\dots$	p_n

Distribution function

$$F(x) = P(X < x) = \begin{cases} 0 & \text{for} \quad x \le x_1 \\ \sum_{i=1}^{k} p_i & \text{for} \quad x_k < x \le x_{k+1}, \quad k = 1,\ 2,\ \dots, n-1 \\ 1 & \text{for} \quad x > x_n \end{cases}$$

Density Function

Let $F(x)$ be a differentiable distribution function of a continuous random variable X. The function

$$f(x) := F'(x)$$

is called the (*probability) density function of* X.

Important properties of a density function:

1.

$$F(x) = P(X < x)$$

$$= P(-\infty < X < x)$$

$$= \int_{-\infty}^{x} f(t)dt \ .$$

2.

$$P(a \le X < b) = F(b) - F(a)$$

$$= \int_{a}^{b} f(x)dx \ .$$

3.

$$P(-\infty < X < +\infty) = \int_{-\infty}^{+\infty} f(x)dx = 1 \ .$$

Expected value

The *expected value* of the random variable X , denoted by $E(X)$, is defined as

$$E(X) := \begin{cases} \sum_{i=1}^{\infty} x_i \cdot p_i & \text{when } X \text{ discrete} \\ \int_{-\infty}^{+\infty} x \cdot f(x)dx & \text{when } X \text{ continuous} \end{cases}$$

Variance or dispersion, standard deviation

The *dispersion* or *variance* of the random variable X , denoted by $D^2(X)$, is defined as

$$D^2(X) := E\left(X - E(X)\right)^2$$

i. e.

$$D^2(X) := \begin{cases} \sum_{i=1}^{\infty} \left(x_i - E(X)\right)^2 \cdot p_i & \text{when } X \text{ discrete} \\ \int_{.\infty}^{+\infty} \left(x - E(X)\right)^2 \cdot f(x)dx & \text{when } X \text{ continuous} \end{cases}$$

under the assumption that the expected value exists.

The standard deviation of the random variable X , denoted by $D(X)$, is defined as

$$D(X) := \sqrt{D^2(X)} \quad (>0)$$

$$D^2(X) := \begin{cases} \sum_{i=1}^{\infty} x_i^2 \cdot p_i - \left(\sum_{i=1}^{\infty} x_i \cdot p_i\right)^2 & \text{when } X \text{ discrete} \\ \int_{.\infty}^{+\infty} x^2 \cdot f(x)dx - \left(\int_{-\infty}^{+\infty} x \cdot f(x)\right)^2 & \text{when } X \text{ continuous} \end{cases}$$

Standardisation:
Let

$$Z = aX + b, \quad a, b = const.$$

Then

$$E\left(aX + b\right) = a \cdot E(X) + b\,.$$

$$D^2\left(aX + b\right) = a^2 \cdot D^2(X).$$

The random variable Z is called a *standardised random variable* if

$$E\left(Z\right) = 0, \qquad D^2\left(Z\right) = 1.$$

The process

$$Z = g(X) := \frac{X - \mu}{\sigma} = \frac{1}{\sigma} \cdot X - \frac{\mu}{\sigma}$$

is called *standardisation*.

IV. Special Discrete Distribution Functions

Hypergeometric Distribution:
A discrete variable X has a hypergeometric distribution if its probability function is of the form

$$P\left(X = x\right) = p_i = \frac{\binom{M}{x} \cdot \binom{N - M}{n - x}}{\binom{N}{n}},$$
$$x = 0, 1, \ldots, n; \quad n \le M \le N\,.$$

The probability function of the hypergeometric distribution is formally equivalent to the sampling scheme *without replacement*.

Let X have a hypergeometric distribution. Then

$$E\left(X\right) = n \cdot \frac{M}{N} = n \cdot p,$$

$$D^2\left(X\right) = \frac{N - n}{N - 1} n \cdot p \cdot q$$

Binomial Distribution
A discrete variable X has a binomial distribution if its probability function is of the form

$$P\left(X = x\right) = p_i = \binom{n}{x} \cdot p^x \cdot q^{n-x}, \quad x = 0,\ 1, \ldots, n\,.$$

The probability function of the binomial distribution is formally equivalent to the sampling scheme with replacement.

Let X have a binomial distribution. Then

$$E(X) = n \cdot p,$$

$$D^2(X) = n \cdot p \cdot q.$$

Rule of thumb:
For a "sufficiently" large N, the hypergeometric distribution can be approximated by the binomial distribution. It will be recommended to use the following rule of thumb:
"If $10 \cdot n \le N$, then the hypergeometric distribution can be approximated by the binomial distribution."

Poisson Distribution:
A discrete variable X has a Poisson distribution if its probability function is of the form

$$P(X = x) = \frac{\lambda^x}{x!} \cdot e^{-\lambda},$$
$$x = 0, 1, \ldots, n.$$

Let X have a Poisson distribution. Then

$$E(X) = D^2(X) = n \cdot p = \lambda.$$

Rule of thumb:
The binomial distribution can be approximated by the Poisson distribution for n "sufficiently" large ($n \to \infty$) while $n \cdot p = \lambda$ remaining constant. That is why the Poisson distribution is also known as the "distribution of rare events".
It will be recommended to use the following rule of thumb:

"If

$$n \cdot p \le 10 \quad \text{and} \quad n \ge 1500p,$$

the binomial distribution can be approximated by the Poisson distribution."

V. Special Continuous Distribution Functions

A continuous variable X has *normal* (or *Gaussian*) distribution if its probability density function is of the form

$$f(x; \mu, \sigma) = \frac{1}{\sigma} \cdot \varphi\left(\frac{x - \mu}{\sigma}\right)$$

$$= \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in R^1$$

Let X be a normally distributed variable. Then

$$E(X) = \mu,$$

$$D^2(X) = \sigma^2.$$

The probability density function of the standardised normal distribution:

$$f(x) = \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad x \in R^1.$$

The *standardised normal distribution function*:

$$\begin{aligned} F(x) &= \Phi(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \qquad x \in R^1. \end{aligned}$$

Some important formulas:

$$\varphi(-x;\ 0,\ 1) = \varphi(x;\ 0,\ 1),$$

$$\Phi(-x;\ 0,\ 1) = 1 - \Phi(x;\ 0,\ 1).$$

$$P(|X - \mu| < c) = 2 \cdot \Phi\left(\frac{c}{\sigma}\right) - 1, \quad c \in R^1.$$

"68-95-99.7 Rule" or "Empirical Rule":
All normal density curves satisfy the following property which follows from T. 8. 3.:

$$\begin{aligned} P(|X - \mu| < \sigma) &= 0.6828 \\ P(|X - \mu| < 2\sigma) &= 0.9544 \\ P(|X - \mu| < 3\sigma) &= 0.9972. \end{aligned}$$

Normal approximation to binomial:
The above theorem can be used to approximate binomial distribution by normal distribution. It will be recommended to use the following rule of thumb:

"If

$$n \cdot p > 5 \quad \text{and} \quad n \cdot q > 5,$$

then the binomial distribution can be approximated by normal distribution."

(Last revised: 24.10.2017)