Operations Research

Chapter 1

Introduction
to
Operations Research

<u>R. 1. 1.</u> (*Historical Background*)
Early origins:

- 300 BC Euclid: How to find the shortest distance between a point and a line.
- 1629 AD Fermat: He developed a method for finding maxima and minima and thus discovered differential calculus.

1917: First optimisation book published: Harris Handcock: *Theory of Maxima and Minima*.

World War II: Modern Operations Research developed during the war:

- Multidisciplinary teams of scientists were assembled as special units within the armed forces. For example, such teams existed in Britain by 1941 in each of the three wings of the Armed Forces.
- Typical projects: Radar deployment policies, aircraft fire control, fleet convoy sizing, and detection of enemy submarines.
- Because of military nature of the applications, the science became known as "Operational Research" in the United Kingdom and as "Operations Research" elsewhere.

After the War:

- Many of the war scientists turned their attention toward the application of the OR techniques to civilian problems.
- Large companies were the first to use OR methods, notably the petroleum industry (linear optimisation for production scheduling).
- Initially, only large companies could afford the research. Later, researchers were able to categorise and standardise types of problems (inventory, allocation, replacement, scheduling, etc.) and smaller companies gained access to OR techniques.
- Important factor in the success of OR has been the current development in electronic computing technology.
- In 1952 the Operations Research Society of America (ORSA) was founded in the United States giving OR a unique disciplinary identity.
  In 1953 The Institute for Management Sciences (TIMS) was founded in the United States. The two groups merged in mid-90's to form the Institute for Operations Research and Management Science (INFORMS).

R. 1. 2. (***A Definition of OR)***

There are various definitions of OR and its scope. We confine ourselves to the following:

> "Operational research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government, and defence. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies and controls. The purpose is to help management determine its policy and actions scientifically."
>
> *The Operational Research Society of Great Britain*

R. 1. 3. (***OR Methodology***)

A possible procedure:

Step 1: Define the problem

Step 2: Observe the system.

Step 3: Formulate a mathematical model.

Step 4: Verify the model.

Step 5: Select an alternative.

Step 6: Present the results and conclusions.

Step 7: Implement and evaluate.

R. 1. 4. (***Some Applications***)

- Stock and bond portfolio selection.
- Manufacturing.
- Chemical plant (blending)
- Fuel selection for fossil power plant
- Scheduling and queuing
- Inventory management
- Site location
- Forecasting.

Chapter 2

Linear Optimization Introduction

Ex. 2. 1. (***Profit Maximisation***)
A firm produces two products P_1 and P_2 using the raw materials R_1, R_2, R_3. The availability of the raw materials, and the amount of each raw material used to produce one unit of each product (the so-called "technical coefficients") and the profit to the firm of producing one unit of these products are given in the following table:

Raw Material	P_1	P_2	Availability
R_1	5	3	45
R_2	2	3	36
R_3	1	-	6
Profit (€)	50	20	

The firm would like to maximize its profit.

Ex. 2. 2. (***The Diet Problem***)
A dietician has to construct meals from two foods: type F_1 and type F_2. Each 100 gm of type F_1 costs 3 € and has 3 energy units, 1 unit of vitamins and 1 unit of minerals. Each type of F_2 costs 5 € and has 4 energy units, 1 unit of vitamins and 5 units of minerals. For a balanced diet, a patient needs at least 340 energy units, 100 units of vitamins and 100 units of minerals. What is the least expensive balanced meal?

Ex. 2. 3. (***Transportation Problem***)
A certain commodity is to be transported from the sources S_1, S_2 and S_3 to the destinations D_1, D_2, D_3 and D_4. The following table shows the availabilities of the commodity, the requirements of the destinations and the costs per unit for the transportation of the commodity from the three sources to the four destinations:

Destination $\rightarrow$ Sources $\downarrow$	D_1	D_2	D_3	D_4	Availabilities
S_1	5	2	4	3	30
S_2	6	4	9	5	40
S_3	2	3	8	1	55
Requirements $\rightarrow$	15	20	40	50	

Determine a transportation plan that minimises the total transportation cost.

R. 2. 1
The problems formulated in Ex. 1. – Ex. 1. 3. represent the first civil application of linear optimisation.

Ex. 2. 4.
Formulate the problems in Ex. 2. 1. – Ex. 2. 3. as linear optimisation models.

1.
Let

$x_i,\ i=1,2$: output of P_i

$\pi(x_1,x_2)$: profit function.

The Model:

$$
\begin{aligned}
\pi(x_1,x_2) = 50x_1 + 20x_2 &\to \max \\
5x_1 + 3x_2 &\le 45 \\
2x_1 + 3x_2 &\le 36 \\
x_1 \qquad &\le 6 \\
x_1, x_2 &\ge 0 .
\end{aligned}
$$

2.
Let

$x_i,\ i=1,2$: amount of F_i

$C(x_1,x_2)$: cost function.

The Model:

$$
\begin{aligned}
C(x_1,x_2) = 3x_1 + 5x_2 &\to \min \\
3x_1 + 4x_2 &\ge 340 \\
x_1 + x_2 &\ge 100 \\
x_1 + 5x_2 &\ge 100 \\
x_1, x_2 &\ge 0 .
\end{aligned}
$$

3.
Let

$x_{ij},\ i=1,2,3;\ j=1,2,3,4$: amount of commodity transported from S_i to D_j function.

$C(...)$: cost function.

$$C = 5x_{11} + 2x_{12} + ... + 8x_{33} + x_{34} \to \min$$

$$\sum_{j=1}^{4} x_{1j} = 30,\quad \sum_{j=1}^{4} x_{2j} = 40,\quad \sum_{j=1}^{4} x_{3j} = 55,$$

$$\sum_{i=1}^{3} x_{i1} = 15,\quad \sum_{i=1}^{3} x_{i2} = 20,\quad \sum_{i=1}^{3} x_{i3} = 40,\quad \sum_{i=1}^{3} x_{i4} = 50$$

$$x_{ij} \ge 0,\ \forall i, j.$$

D. 2. 1. (***Optimisation Problem***)
The *Optimisation Problem* (*OP*) is defined as:

(2. 1.) $$f(x) \to opt$$

subject to (s.t.)

(2. 2.) $$g_i(x) \leq,\ =,\ \geq 0,\ i = 1,2,...,m,$$

(2. 3.) $$x \geq 0.$$

Here is: $x := (x_1, x_2, ..., x_n)^T$.

Now, we define:

(1) The function $f(x)$ to be optimised is termed as *objective function*;
(2) The relations in (2. 2.) are *constraints*;
(3) The conditions in (2. 3.) are *nonnegative restrictions*;
(4) Variables $x_1, x_2, ..., x_n$ are *decision variables*;
(5) The terminology *optimise* stands for *minimise* or *maximise*.

The symbol $\geq,\ =,\ \leq$ means that one and only one of these is involved in each constraint.

The problem (OP) is further classified into two classes:

1. *Linear Optimisation Problem* (*LOP*)
   If the objective function $f(x)$ and all the constraints $g_i(x)$ are linear in an optimisation problem, we call the problem a *linear optimisation problem* (*LOP*).

2. *Nonlinear Optimisation Problem*
   If the objective function $f(x)$ or at least one of the constraints $g_i(x)$ or both are nonlinear functions in an optimisation problem, then the problem is termed as *nonlinear optimisation problem* (*NLOP*).

D. 2. 2. (***Linear Optimisation Problem***)
A *linear optimisation problem* (*LOP*) has the general form:

$$z = \sum_{j=1}^{n} c_j x_j \to opt$$

s.t. $$\sum_{j=1}^{n} a_{ij} x_j \leq,\ =,\ \geq b_i,\quad i = 1,2,...,m$$

$$x_1, x_2, ..., x_n \geq 0,$$

where $c_j,\ a_{ij},\ b_i \in R^1,\ i = 1,2,...,m;\ j = 1,2,...,n.$

D. 2. 3. (***Standard Form of a Linear Optimisation Problem***)
The *standard form* of a LOP is defined as

$$z = \sum_{j=1}^{n} c_j x_j \to \max!$$

s.t.
$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, ..., m$$

$$x_1, x_2, ..., x_n \geq 0,$$

or, in the matrix form

$$\begin{aligned} z = c^T x &\to \max! \\ Ax &= b \\ x &\geq 0, \end{aligned}$$

where

$$c = (c_1, c_2, ..., c_n)^T,$$
$$c = (x_1, x_2, ..., x_n)^T,$$
$$A = (a_{ij}),\ i = 1, 2, ..., m;\ j = 1, 2, ..., n,$$
$$b = (b_1, b_2, ..., b_m)^T.$$

Ex. 2. 4.
Write the linear optimisation problem in Ex. 2. 1. into the standard form.

Solution:

$$\begin{aligned} \pi(x_1, x_2) = 50x_1 + 20x_2 &\to \max \\ 5x_1 + 3x_2 + x_3 &= 45 \\ 2x_1 + 3x_2 + x_4 &= 36 \\ x_1 + x_5 &= 6 \\ x_i \geq 0,\ i &= 1, 2, ..., 5. \end{aligned}$$

The variables x_3, x_4, x_5 are termed as the *slack variables*.

Ex. 2 5.
Write the linear optimisation problem in Ex. 2. 2. into the standard form.

Solution:

$$\begin{aligned} -C(x_1, x_2) = -x_1 - 5x_2 &\to \max \\ 3x_1 + 4x_2 - x_3 &= 340 \\ x_1 + x_2 - x_4 &= 100 \\ x_1 + 5x_2 - x_5 &= 100 \\ x_i \geq 0,\ i &= 1, 2, ..., 5. \end{aligned}$$

The variables x_3, x_4, x_5 are termed as the *surplus variables*.

Ex. 2. 6.

Write the following linear optimisation problem into the standard form

$$\begin{aligned} z = x_1 + 2x_2 &\to \max \\ -3 \le x_1 &\le +3 \\ -2 \le x_2 &\le +2 . \end{aligned}$$

Solution:

$$\begin{aligned} z = x_1 + 2x_2 &\to \max \\ x_1 &\ge -3 \\ x_1 &\le 3 \\ x_2 &\ge -2 \\ x_2 &\le 2 . \end{aligned}$$

$$\begin{array}{l} z = x_1 + 2x_2 \to \max \\ x_1 - x_3 = -3 \\ x_1 + x_4 = 3 \\ x_2 - x_5 = -2 \\ x_2 + x_6 = 2 . \end{array}$$

Let

$$x_1 := x_1^+ - x_1^- , \qquad x_2 := x_2^+ - x_2^- .$$

$$\begin{array}{l} z = x_1^+ - x_1^- + 2x_2^+ - 2x_2^- \to \max \\ x_1^+ - x_1^- - x_3 = -3 \\ x_1^+ - x_1^- + x_4 = 3 \\ x_2^+ - x_2^- - x_5 = -2 \\ x_2^+ - x_2^- + x_6 = 2 \end{array}$$

$$x_1^+, x_1^-, x_2^+, x_2^-, x_3, x_4, x_5, x_6 \ge 0 .$$

Chapter 2

Linear Optimization (Introduction)

Exercises

2. 1.

A firm can produce a product using three production processes. Each process uses labour (L), capital (C) and materials and supplies (M).
The variable capital stands for various types of machinery and equipment. The use of labour and capital is measured in hours. M is an index number for the quantities of materials and supplies used.
The availability of inputs of labour, capital, and materials and supplies, and the amount of each input used to produce one unit of product by each production activity, are categorised by the coefficients of the following table:

	Production Process			
	1	2	3	Availability
L	25	32	43	850
C	35	25	20	800
M	30	36	40	980

The first column of the table means that 25 hours of labour (L), 35 hours of capital (C), and 30 units of materials and supplies (M) are used by process 1 to produce one unit of the product produced by the activity 1. The first row of the table means that 25, 32, and 43 units of labour are used to produce one unit of the product produced by the activities 1, 2 and 3, respectively. Furthermore, the total hours of labour used can not exceed 850 hours. The other columns and rows read similarly.

The unit costs of L, C, and M are:

L: 14, C: 34, M: 12.

Consumers' tastes are reflected in the prices of one unit of the product produced by the activities 1, 2 and 3:

1: 2610, 2: 2400, M: 2250.

The firm would like to maximise its profit.
Formulate the problem as a linear optimisation model.

2. 2.

Suppose we have been asked to commence the preliminary design of a small complex of apartments. Our studies show that 1-bedroom and 2-bedroom apartments are the most desirable in this area. Our client has pointed out that there is, at most, demand for only three 3-bedroom and six 2-bedroom apartments and, consequently, has requested that no more than

this number be built. Our objective is to find the number of 1-bedroom, 2-bedroom, and 3-bedroom apartments that will maximise the financial return.
The following costs and profits have been ascertained:

Apartment Type	Capital Cost (in $1,000s)	Profit (in $1,000s)
1-bedroom	90	20
2-bedroom	180	24
3-bedroom	220	27

The available capital is $1,800,000. If maximising profit were the only criterion, clearly, it would be most advantageous if we built all 1-bedroom apartments. However, the local planning authority has a planning code that discourages small apartments by placing penalties against them. For this site, the maximum number of penalty points a development can accrue is 960. The following penalties are incurred for each unit:

Apartment Type	Penalty Points
1-bedroom	120
2-bedroom	60
3-bedroom	20

Formulate the problem as a model of linear optimisation.

2. 3.
The liquid part of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Suppose that dietary drink X costs 0.12 € per cup and drink Y costs 0.15 € per cup.
How many cups of each drink should be consumed each day to minimise the cost and still meet the stated daily requirements?
Formulate the problem as a model of linear optimisation.

2. 4.

A fruit grower has 150 acres of land available to raise two crops, A and B. It takes one day to trim an acre of crop A and two days to trim an acre of crop B, and there are 240 days per year available for trimming. It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking.
The fruit grower would like to know the number of acres of each fruit that should be planted in order to maximise his profit, assuming that the profit is 140 € per acre of crop A and 235 € per acre of crop B.
Formulate the problem as a model of linear optimisation.

2. 5. (*Product Mix Problem*)

A manufacturing process requires three different input viz., A, B and C. A sandal soap of the first type requires 30 gm of A, 20 gm of B and 6 gm of C, while this data for the second type of soap is 25, 5 and 15, respectively. The maximum availability of A, B and C are 6000, 3000 and 3000 gm, respectively. The selling price of the sandal soap of the first and second type are 14 € and 15, respectively. The profit is proportional to the amounts of soaps manufactured. The manufacturer would like to know how many soaps of the first and second kinds should be produced to maximise his profit.
Formulate the problem as a model of linear optimisation.

2. 6. (*Bus Scheduling Problem*)

A company runs buses during the time period 5 AM to 1 AM. Each bus can operate for 8 hours successively, and then it is directed to workshop for maintenance and fuel. The minimum number of buses required fluctuates with the time interval. The desired numbers of buses during different time intervals are given in the following table:

Time Intervals	Minimum No. of Buses Required
5 AM – 9 AM	5
9 AM – 1 PM	13
1 PM – 5 PM	11
5 PM – 9 PM	14
9 PM – 1 AM	4

The company keeps in view the reduction of air pollution and smog problem. It is required to determine the number of buses to operate during different shifts that will meet the minimum requirement while minimising the total number of daily buses in operation.
Formulate the problem as a model of linear optimisation.

2. 7. (*The Warehousing Problem*)

A warehouse has a capacity of 2000 units. The manager of the warehouse buys and sells the stock of potatoes over a period of 6 weeks to make profit. Assume that in the j – th week the same unit price p_j holds for both purchase and sale. In addition, there is unit cost 15 € as weekly expense for holding stock. The warehouse is empty at the beginning and is required to be empty after the sixth week. The question is: How should the manager operate to maximise his profit?
Formulate the problem as a model of linear optimisation.

2. 8. (*Caterer Problem*)
A festival organiser has to organise its annual cultural festival continuously for next five years. There is an arrangement of dinner for every invited team. The requirement of napkins during these five days is

Days	1	2	3	4	5
Napkins required	80	50	100	80	150

Accordingly, a caterer has been requested to supply the napkins according to the above schedule. After the festival is over the caterer has no use of napkins. A new napkin costs 2 €. The washing charges for a used napkin is 0.50 € by ordinary services and 1 €, if express service is used. A napkin given for washing by ordinary service is returned third day, while under express service is returned next day.
The question is: How should the caterer meet the requirement of the festival organiser so that the total cost is minimised?
Formulate the problem as a model of linear optimisation.

2. 9. (*Trim-Loss Problem*)
Paper cutting machines are available to cut standard news print rolls into the subrolls. Each standard roll is of 180 cm width and a number of them must be cut to produce smaller subrolls at the current orders for 30 of width 70 cm, 60 of width 50 cm and 40 of width 30 cm. Formulate the problem as to minimise the amount of wastes. Ignoring the recycling or other use for the trim, assume that the length of each required subroll is the same as that of the standard roll.

2. 10.
Two alloys, A and B are made from four different metals, I, II, II, and IV, according to the following specifications:

Alloy	Specifications	Selling price (€)/ton
A	at most 80% of I	200
	at least 30% of II	
	at least 50% of IV	
B	between 40% and 60% of II	300
	at least 30% of II	
	at most 70% of IV	

The four metals, in turn are extracted from three different ores with the following data:

Ore	Max. Quantity (tons)	Constituents (%) I	II	III	IV	others	Purchase Price (€)/ton
1	1000	20	10	30	30	10	30
2	2000	10	20	30	30	10	40
3	3000	5	5	70	20	0	50

How much of each alloy should be produced to maximise the profit? Formulate the problem as a linear optimisation model.

2. 11. (*An Agricultural Allocation Model*)
A farmer owns 1000 acres of more or less homogeneous farmland. His options are to breed cattle, or plant wheat, corn, or tomatoes. It takes four acres to support one head of cattle. Annually, 12000 hours of labour are available. (For simplicity, we assume here that these 12000 hours could be used at any time during the year, i.e., through hiring casual labour during seasons of high need, e.g., for harvesting).
The following table provides information regarding the profit, yield, and labour needs for the four economic activities:

	Cattle	Wheat	Corn	Tomatoes
Profit	$ 1600/head	$5/bushel	$6/bushel	50¢/lb
Yield per Acre	¼ heads/acre	50 bushels	80 bushels	1000 lbs
Annual Labour Requirement	40 hrs/head	10 hrs/acre	12 hrs/acre	25 hrs/acre

Furthermore, it is required that at least 20%of the farmland that is cultivated in the process must be used for the purpose of cattle breeding, at most 30% of the available farmland can be use for growing tomatoes, and the ratio between the amount of farmland assigned to growing wheat and that left uncultivated should not exceed 20 to 1.
The farmer would like to maximise his profit.
Formulate the problem as a linear optimisation model.

2. 12. (*A Portfolio Selection Problem*)
An investor has $1 million to invest in any combination of bonds, stocks, term deposits, a saving account, real estate, and gold. The anticipated (or known) interest rates, the risk factors (where a high number indicates a high risk) and the expected increase in the value of the investments are shown in the following table:

Types of investment	Interest (in % annually)	Risk factor	Expected annual Increase in value [%]
Bonds	5	3	0
Stocks	2	10	7
Term deposits	4	2	0
Saving accounts	3	1	0
Real estate	0	5	7
Gold	0	20	11

The objective is to maximise the amount that is expected to be available in a year's time subject to the following restrictions:

- Of the total amount of money invested, at least 30% must be invested in bonds, not more than 10% in stocks, and at least 10% in term deposits and/or saving accounts.
- Up to 50% of the total money invested in real estate may be borrowed against in the form of a mortgage at an interest rate of 6%. The amount borrowed cannot exceed $150000.
- The average risk factor of the investment cannot exceed 4.5.
- The average annual interest should be at least 2.5%.
- The amount of money invested in gold cannot exceed $100000 or 8% of the total money available, whichever is smaller.

Formulate the problem as a linear optimisation model.

2. 13. (*An Inventory Problem*)

A company wants to plan its production for one of its products for the next four months. The following table shows the anticipated demand, the production capacities, and the unit production costs for the individual months, as well as the inventory holding costs that are incurred carrying over one unit from one month to the next.

	Periods			
	Month 1	Month 2	Month 3	Month 4
Demand	50	120	150	160
Production capacity	100	100	160	150
Unit production cost	$1	$1.1	$1.2	$1.2
Inventory cost	$0.3	$0.2	$0.2	

At present, no units are in stock and after the four months, it is not desired to have any stock left.

Total costs are to be minimised.

Formulate the problem as a linear optimisation model.

Chapter 2

Linear Optimization (Introduction)

Solutions

2. 1.
Denote by

$x_i, \quad i = 1,\ 2,\ 3:$ Amount of the product produced by the activity i,

The costs to produce one unit of $x_i,\ i = 1,2,3$ are:

1: $14 \cdot 25 + 34 \cdot 35 + 12 \cdot 30 = 1900$
2: $14 \cdot 32 + 34 \cdot 25 + 12 \cdot 36 = 1730$
3: $14 \cdot 43 + 34 \cdot 20 + 12 \cdot 40 = 1762$

The profits to the firm of producing one unit of $x_i,\ i = 1,2,3$ are given by:

Profit = price – cost.

Therefore, we have the following profit figures:

1: $2610 - 1900 = 710$
2: $2400 - 1730 = 670$
3: $2250 - 1762 = 488$

The Model:

$$\pi(x_1, x_2, x_3) = 710x_1 + 670x_2 + 488x_3 -> \text{max!}$$

$$\begin{aligned} 25x_1 + 32x_2 + 43x_3 &\le 850 \\ 35x_1 + 25x_2 + 20x_3 &\le 800 \\ 30x_1 + 36x_2 + 40x_3 &\le 980 \\ x_1,\ x_2, x_3 &\ge 0 \end{aligned}$$

2. 2.
The decision variables here are the number of 1-bedroom, 2-bedroom, and 3-bedroom apartments we should design. Let us call them x_1, x_2 and x_3 respectively. We can write the objective (in thousands of dollars) in terms of these variables as maximising:

$$p = 20x_1 + 24x_2 + 27x_3$$

Subject to the following constraints:

$$
\begin{aligned}
90x_1 + 180x_2 + 220x_3 &\le 1800 \\
120x_1 + \ \ 60x_2 + \ \ 20x_3 &\le \ 960 \\
x_2 \qquad\quad &\le 6 \\
x_3 &\le 3 \\
x_1,\ x_2, x_3 \ge 0 :&\quad \text{integer}
\end{aligned}
$$

2. 3.

Denote by

x_1: the number of cups of dietary drink X,

x_2: the number of cups of dietary drink Y,

C: the cost.

<u>*The model*</u>:

$$C(x_1, x_2) = 0.12x_1 + 0.15x_2 \to \text{min!}$$

$$
\begin{aligned}
60x_1 + 60x_2 &\ge 300 \\
12x_1 + \ \ 6x_2 &\ge 36 \\
10x_1 + 30x_2 &\ge 90 \\
x_1, \quad x_2 &\ge 0
\end{aligned}
$$

2. 4.

Denote by

x_1: the number of acres of fruit A,

x_2: the number of acres of fruit B,

P: the profit.

<u>*The model*</u>:

$$P(x_1, x_2) = 140x_1 + 235x_2 \to \text{max!}$$

$$
\begin{aligned}
x_1 + \quad x_2 &\le 150 \\
x_1 + \ \ 2x_2 &\le 240 \\
0.3x_1 + 0.1x_2 &\le 30 \\
x_1, \quad x_2 &\ge 0
\end{aligned}
$$

2. 5.

Let us put the data in the tabular form:

Type	Inputs/unit			Selling price/unit
	A	B	C	
I	30	20	6	14
II	25	5	15	15
Availability	6000	3000	3000	

Denote by

x_i : the number of the $i-$th type of soap $(i=1,2)$.

The model:

$$z = 14x_1 + 15x_2 \to \max!$$
$$30x_1 + 25x_2 \le 6000$$
$$20x_1 + \ \ 5x_2 \le 3000$$
$$6x_1 + \ 15x_2 \le 3000$$
$$x_1, x_2 \ge 0, \text{ integer.}$$

2. 6.

Denote by

x_i : the number of buses starting at the beginning of the $i-$th period, $i = 1,2,...,5.$

Note that each bus operates during two consecutive shifts. Buses which join the crew at 5 AM and 9 AM will be in the operation between 9 AM and 1 PM. As the minimum number of buses required in this interval is 13, we have $x_1 + x_2 \ge 13$, and similarly others.

The model:

$$z = x_1 + x_2 + x_3 + x_4 + x_5 \to \min!$$
$$x_1 + x_2 \ge 13$$
$$x_2 + x_3 \ge 11$$
$$x_3 + x_4 \ge 14$$
$$x_4 + x_5 \ge 4$$
$$x_1 + x_5 \ge 5$$
$$x_i \ge 0, \ i = 1,2,...,5; \text{ integer}.$$

2. 7.

Denote by

x_j : the level of the stock at the the beginning of the $j-$th week;

y_j : the amount bought during the $j-$th week;

z_j : the amount sold during the $j-$th week.

The model:

$$z=\sum_{j=1}^{6} p_j \cdot \left(z_j - y_j\right) - 15x_j \to \max!$$

$$x_{j+1} = x_j + y_j - z_j, \quad j = 1,2,\dots,5$$

$$x_j \le 2000, \quad j = 1,2,\dots,6$$

$$x_1 = 0, \quad x_6 + y_6 - z_6 = 0$$

$$x_j \ge 0, \; y_j \ge 0, \quad z_j \ge 0, \quad j = 1,2,\dots,6.$$

2. 8.

Denote by

x_i : number of napkins purchased on the i – th day, $i = 1,2,\dots,5.$

y_j : number of napkins given for washing on j – th day under express service, $j = 1,2,3,4.$

z_k : number of napkins given for washing on the k – th day under ordinary service, $k = 1,2,3.$

v_l : number of napkins left in the stock on k – th day after the napkins have been given for washing, $l = 1,2,\dots,5.$

The data is tabulated as

Type	Number of napkins required on days 1	2	3	4	5
New napkins	x_1	x_2	x_3	x_4	x_5
Express service	-	y_1	y_2	y_3	y_4
Ordinary service	-	-	z_1	z_2	z_3
Napkins required	80	50	100	80	150

We have to minimise

$$2\left(x_1 + x_2 + x_3 + x_4 + x_5\right) + y_1 + y_2 + y_3 + y_4 + 0.5\left(z_1 + z_2 + z_3\right).$$

From the table:

$$x_1 = 80,\; x_2 + y_1 = 50,\; x_3 + y_2 + z_1 = 100,\; x_4 + y_3 + z_2 = 80,\; x_5 + y_4 + z_3 = 150.$$

Also, there is another set of constraints, which shows the total number of napkins which may be given for washing and some napkins which were not given for washing just on the day these have been used. These constraints are:

$$y_1 + z_1 + v_1 = 80,$$

$$y_2 + z_2 + v_2 = 50 + v_1,$$

$$y_3 + z_3 + v_3 = 100 + v_2,$$

$$y_4 + v_4 = 80 + v_3,$$

$$v_5 = 150 + v_4.$$

The model:

$$z = 160 + 2(x_1 + x_2 + x_3 + x_4 + x_5) + y_1 + y_2 + y_3 + y_4 + 0.5(z_1 + z_2 + z_3) -> \text{min!}$$

$$x_2 + y_1 = 50$$

$$x_3 + y_2 + z_1 = 100$$

$$x_4 + y_3 + z_2 = 80$$

$$x_5 + y_4 + z_3 = 150$$

$$y_1 + z_1 + v_1 = 80$$

$$y_2 + z_2 + v_2 - v_1 = 50$$

$$y_3 + z_3 + v_3 - v_2 = 100$$

$$y_4 + v_4 - v_3 = 80$$

$$v_5 - v_4 = 150$$

$$x_i, y_j, z_k, v_l \geq 0,\ \forall i, j, k, l.$$

2. 9.
A standard roll may be cut according to the following patterns:

Widths ordered in cm	Number of sub rolls cut on different patterns							
	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
30	6	4	3	2	2	1	1	0
50	0	1	0	1	2	0	3	2
70	0	0	1	1	0	2	0	1
Trim loss	0	10	20	0	20	10	0	10

Denote by
x_i : number of the standard print rolls piece to cut on the patterns p_i, $i = 1, 2, ..., 8$.

The model:

$$z = 10x_2 + 20x_3 + 20x_5 + 10x_6 + 10x_7 \rightarrow \text{min!}$$

$$6x_1 + 4x_2 + 3x_3 + 2x_4 + 2x_5 + x_6 + x_7 = 40$$

$$x_2 + x_4 + 2x_5 + 3x_7 + 2x_8 = 60$$

$$3x_3 + x_4 + 2x_6 + x_8 = 30$$

$$x_i \geq 0,\ i = 1, 2, ..., 8: \text{ integer.}$$

Here, in the constraints the equality is desired due to the fact that any left thing is of no use.

2. 10.
Denote by

x_{ij} : tons of ore i allocated to alloy j; $i = 1,2,3$; $j = A, B$,
w_j : tons of alloy j produced, $j = A, B$.

<u>*The model*</u>:

$$z = 200w_A + 300w_B - 30(x_{1A} + x_{1B}) - 40(x_{2A} + x_{2B}) - 50(x_{3A} + x_{3B}) \to \text{max!}$$

Specification constraints:

$$0.2x_{1A} + 0.1x_{2A} + 0.5x_{3A} \le 0.8w_A$$
$$0.1x_{1A} + 0.2x_{2A} + 0.05x_{3A} \ge 0.3w_A$$
$$0.3x_{1A} + 0.3x_{2A} + 0.2x_{3A} \ge 0.5w_A$$
$$0.1x_{1B} + 0.2x_{2B} + 0.05x_{3B} \ge 0.4w_B$$
$$0.1x_{1B} + 0.2x_{2B} + 0.05x_{3B} \le 0.6w_B$$
$$0.3x_{1B} + 0.3x_{2B} + 0.7x_{3B} \ge 0.3w_B$$
$$0.3x_{1B} + 0.3x_{2B} + 0.2x_{3B} \le 0.7w_B$$

Ore constraints:

$$x_{1A} + x_{1B} \le 1000$$
$$x_{2A} + x_{2B} \le 2000$$
$$x_{3A} + x_{3B} \le 3000$$

Alloy constraints:

$$x_{1A} + x_{2A} + x_{3A} \ge w_A$$
$$x_{1B} + x_{2B} + x_{3B} \ge w_B$$
$$x_{ij},\ w_j \ge 0,\ \ i = 1,2,3;\ j = A, B.$$

2. 11.
Denote by

x_1 : the number of cattle,
x_2 : the amount of wheat,
x_3 : the amount of corn,
x_4 : the amount of tomatoes.

The model:

$$z = 1600(1/4)x_1 + 5(50)x_2 + 6(80)x_3 + 1/2(1000)x_4 \to Max$$

whose units are:

[$] = [$/head][heads/acre][acres] + [$/bushel][bushel/acre][acres]+

+ [$/bushel][bushel/acre][acres] + [$/lbs][lbs/acre][acres]

s.t.

$$x_1 + x_2 + x_3 + x_4 \le 1000$$
$$40(1/4)x_1 + 10x_2 + 12x_3 + 25x_4 \le 12000$$
$$x_1 \ge 0.2(x_1 + x_2 + x_3 + x_4)$$
$$x_4 \le 0.3(1000)$$
$$x_2 / (1000 - x_1 - x_2 - x_3 - x_4) \le 2/1$$
$$x_1, x_2, x_3, x_4 \ge 0, \qquad x_1 : \text{ integer.}$$

A few routine transformations yield:

$$z = 400x_1 + 250x_2 + 480x_3 + 500x_4 \to Max$$

s.t.

$$x_1 + x_2 + x_3 + x_4 \le 1000$$
$$10x_1 + 10x_2 + 12x_3 + 25x_4 \le 12000$$
$$0.8x_1 - 0.2x_2 - 0.2x_3 - 0.2x_4 \ge 0$$
$$x_4 \le 300$$
$$2x_1 + 3x_2 + 2x_3 + 2x_4 \le 2000$$
$$x_1, x_2, x_3, x_4 \ge 0, \qquad x_1 : \text{ integer.}$$

(The optimal solution is: $x_1^* = 200,\ x_2^* = 0,\ x_3^* = 769.2308,\ x_4^* = 30.7692,\ z^* = 464615.40.$*)*

2. 12.

Denote by

x_1 : investment in bonds,
x_2 : investment in stocks,
x_3 : investment in term deposits,
x_4 : investment in saving accounts,
x_5 : investment in real estates,
x_6 : investment in gold,
x_7 : loan.

The model:

$$z = 1.05x_1 + 1.09x_2 + 1.04x_3 + 1.03x_4 + 1.07x_5 + 1.11x_6 - 1.06x_7 \to Max!$$

s. t.

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 1000000 + x_7$$
$$x_1 \ge 0.3(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)$$
$$x_2 \le 0.1(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)$$
$$x_3 + x_4 \ge 0.1(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)$$
$$x_7 \le 0.5x_5$$
$$x_7 \le 150000$$

$$\frac{3x_1 + 10x_2 + 2x_3 + x_4 + 5x_5 + 20x_6}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} \le 4.5$$

$$\frac{5x_1 + 2x_2 + 4x_3 + 3x_4}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} \ge 2.5$$

$$x_6 \le 100000$$
$$x_6 \le 0.08(1000000 + x_7)$$
$$x_i \ge 0, \; i = 1, 2, ..., 7.$$

(The optimal solution is: $x_1^* = 437000,\; x_2^* = 115000,\; x_3^* = 115000,\; x_4^* = 0,$
$x_5^* = 478400,\; x_6^* = 4600,\; x_7^* = 150000,\; z^* = 1061794$ *)*

2. 13.

Denote by

x_i : production in month $i = 1, 2, 3, 4$

I_j : inventory in at the beginning of the months $j = 1, 2, 3, 4, 5$.

The Model:

$$z = x_1 + 1.1x_2 + 1.2x_3 + 1.2x_4 + 0.3I_1 + 0.2I_3 + 0.2I_4 \to Min!$$

s.t.

$$x_1 \le 100$$
$$x_2 \le 100$$
$$x_3 \le 160$$
$$x_4 \le 150$$
$$I_1 = 0$$
$$I_2 = 0 + x_1 - 50$$

$$I_3 = I_2 + x_2 - 120$$
$$I_4 = I_3 + x_3 - 150$$
$$I_5 = 0 = I_4 + x_4 - 160$$
$$x_i \geq 0,\ I_j \geq 0,\ \forall i, j.$$

Chapter 3

Linear Optimization
Some Fundamental Definitions and Theorems

D. 3. 1. (***Feasible Solution***)
Consider a linear optimisation problem in the standard form:

$$z = c^T x \to \max!$$
$$Ax = b$$
$$x \geq 0 .$$

A vector x is called a *feasible solution* of the linear optimisation problem if it satisfies the constraints and the non-negativity condition.

Ex. 3. 1.
Consider the problem

$$\begin{aligned} \pi(x_1, x_2) &= 50x_1 + 20x_2 \to \max \\ 5x_1 + 3x_2 + x_3 &= 45 \\ 2x_1 + 3x_2 + x_4 &= 36 \\ x_1 + x_5 &= 6 \\ x_i \geq 0,\ i &= 1,2,...,5. \end{aligned}$$

Show that the vector $x = (5,\ 4,\ 8,\ 14,\ 1)^T$ is a feasible solution of the above problem.

Solution:
We have

$$5 \cdot 5 + 3 \cdot 4 + 8 = 45 ,$$
$$2 \cdot 5 + 3 \cdot 4 + 14 = 36 ,$$
$$5 + 1 = 6 ,$$
$$x = (5,\ 4,\ 8,\ 14,\ 1)^T \geq 0$$

D. 3. 2. (***Set of Feasible Solution***)
Define

$$M := \left\{ x \in R^n \mid Ax = b,\ x \geq 0 \right\} .$$

The set M is called the *set of feasible solutions*.

<u>D. 3. 3.</u> (*Set of Optimal Solutions*)
The set
$$M_{opt} := \left\{ x \in R^n \mid c^T x^* \geq c^T x,\ \forall x \in M \right\}$$
is called the *set of optimal solutions*.

<u>D. 3. 4.</u> (*Basic Solution, Basic and Non-Basic Variables*)
Consider the system of equations
$$Ax = b$$

with $n > m$ .Let B be any non singular $m \cdot m$ submatrix which is formed by appropriate columns of the matrix A . Further let x be a solution to $Ax = b$. Suppose all its $n - m$ components which are not associated with columns of B are zero.
Then, x is called a *basic solution with respect to the basis* B .
The components of x which are associated with columns of B are called *basic variables*, and the remaining components are called *non-basic variables*.

<u>D. 3. 5.</u> (*Degenerate and Non-Degenerate Basic Solutions*)
If one or more of the basic variables in a basic solution are zero, then the basic solution is called a *degenerate basic solution*. Otherwise, this solution is called a *non-degenerate basic solution*.

<u>D. 3. 6.</u> (*Basic Feasible Solution*)
Let x be a basic solution of $Ax = b$. If $x \geq 0$, then it is called a *basic feasible solution* of the linear optimisation problem.

<u>Ex. 3. 2.</u>
$$\begin{aligned} 5x_1 + 3x_2 + x_3 \qquad\quad &= 45 \\ 2x_1 + 3x_2 \qquad + x_4 \quad &= 36 \\ x_1 \qquad\qquad\qquad + x_5 &= 6 . \end{aligned}$$

Let
$$A := \begin{pmatrix} 5 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$x = (6,\ 0,\ 15,\ 24,\ 0)^T$ is a basic solution with the corresponding basis matrix
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

<u>Ex. 3. 3.</u>
$$\begin{aligned} 3x_1 + 3x_2 \quad + x_3 + x_4 \qquad &= 2 \\ x_1 + 3x_2 + 2x_3 + x_4 + x_5 &= 2 . \end{aligned}$$

Let

$$A := \begin{pmatrix} 3 & 3 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 \end{pmatrix}.$$

We see that $x = (0,\ 0, 0, 2, 0)^T$ is a degenerate basic solution with the corresponding basis matrix

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

R. 3. 1.

Maximum possible number of basic solutions for an $m \cdot n$ matrix A with rank m is determined by the number of ways of choosing m columns out of n :

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}.$$

Th. 3. 1.

Consider a linear optimisation problem. We have:

1. If there is a feasible solution, then there exists a basic feasible solution.
2. If there is an optimal solution, then there exits an optimal basic solution.

Proof:

1.

Let the columns of the matrix A be denoted by $a^1, a^2, ..., a^n$. Furthermore, let $x = (x_1, x_2, ..., x_n)^T$ be a feasible solution. Then

$$x_1 a^1 + x_2 a^2 + ... + x_n a^n = b, \quad x_i \geq 0, \quad i = 1, 2, ..., n.$$

Let r bet he number of components of x which are positive. For convenience, we can assume that

$$x_i > 0, \quad i = 1, 2, ..., r,$$

and

$$x_i = 0, \quad i = r+1 ..., n.$$

This implies that

$$x_1 a^1 + x_2 a^2 + ... + x_r a^r = b.$$

There are two cases to be considered:

Case 1: The vectors $a^1, a^2, ... a^r$ are linearly independent, and $r \leq m$. (Note that if $r > m$, then $a^1, a^2, ... a^r$ must be linearly independent.)

a) If $r = m$, then the solution is basic.
b) If $r < m$, then we have a degenerate case.

Case 2: The vectors $a^1, a^2, \ldots a^r$ are linearly dependent. In this case, there exist $y_1, y_2, \ldots, y_r$, and at least one of them is positive, such that

$$y_1 a^1 + y_2 a^2 + \ldots + y_r a^r = 0.$$

Multiplying the above equation by α, we obtain:

$$\alpha y_1 a^1 + \alpha y_2 a^2 + \ldots + \alpha y_r a^r = 0.$$

Subtracting the last equation from the one before,

$$(x_1 - \alpha y_1) a^1 + (x_2 - \alpha y_2) a^2 + \ldots + (x_r - \alpha y_r) a^r = b.$$

Define $y := (y_1, y_2, \ldots, y_r, 0, \ldots, 0)^T$. Then,

$$x - \alpha y = (x_1 - \alpha y_1, \ldots, x_r - \alpha y_r, 0, \ldots, 0)^T$$

is a solution of $Ax = b$, where $x = (x_1, \ldots, x_r, 0, \ldots, 0)^T$. If $\alpha = 0$, then $x - \alpha y$ reduces to x, and hence is a feasible solution. If $\alpha > 0$, we have:

$$\begin{cases} x_i - \alpha y_i & \text{decreases} & \text{if } y_i > 0 \\ x_i - \alpha y_i & \text{remains constant} & \text{if } y_i = 0 \\ x_i - \alpha y_i & \text{increases} & \text{if } y_i < 0 \end{cases}.$$

Since there exists at least one $y_i > 0$, it is easy to see that $\alpha > 0$ cannot increase indefinitely. In fact the maximum value that it can take is

$$\alpha_m = \min\{x_i / y_i \mid \text{those } i \text{ such that } y_i > 0\}.$$

By taking $\alpha = \alpha_m$, the corresponding $x - \alpha y$ is a feasible solution with the number of positive components $\leq (r-1)$.
We can repeat this process until the remaining columns are linearly independent, and so we have a basic feasible solution.

2.
Let $x^* = (x_1^*, \ldots, x_n^*)$ be an optimal (feasible) solution. Thus

$$x_1^* a^1 + x_2^* a^2 + \ldots + x_n^* a^n = b, \;\; x^* \geq 0,$$

and

$$c^T x^* \geq c^T x$$

for all x satisfying $Ax = b$ and $x \geq 0$. We assume that the first r components of x^* are positive, and the rest of the components are zero, i.e.,

$$x_i^* > 0, \text{ for } i = 1, 2, ..., r; \quad x_i^* = 0, \text{ for } i = r+1, ..., n.$$

Hence, we have

(3.1.) $$x_1^* a^1 + x_2^* a^2 + ... + x_r^* a^r = b.$$

There are two cases to be considered:

Case 1:

The vectors $a^1, a^2, ..., a^r$ are linearly independent. In this case, $r \leq m$.

(a) If $r = m$, then the solution is basic.
(b) If $r < m$, then we have a degenerate case.

Case 2:

The vectors $a^1, a^2, ..., a^r$ are linearly dependent. Then, there exist $y_1, ..., y_r$ and at least one of them is positive, such that

$$y_1 a^1 + y_2 a^2 + ... + y_r a^r = 0.$$

Multiplying by α, we obtain

$$\alpha y_1 a^1 + \alpha y_2 a^2 + ... + \alpha y_r a^r = 0.$$

Subtracting the last equation from (2. 1.),

$$\left(x_1^* - \alpha y_1\right) a^1 + \left(x_2^* - \alpha y_2\right) a^2 + ... + \left(x_r^* - \alpha y_r\right) a^r = b.$$

Define $y := \left(y_1, y_2, ..., y_r, 0, ..., 0\right)^T$. Then,

$$x^* - \alpha y = \left(x_1^* - \alpha y_1, ..., x_r^* - \alpha y_r, 0, ..., 0\right)^T$$

is a solution of $Ax = b$, where $x = (x_1^*, ..., x_r^*, 0, ..., 0)^T$. If $\alpha = 0$, then $x^* - \alpha y$ reduces to x^*, and hence is a feasible solution. If $\alpha > 0$, we have:

$$\begin{cases} x_i^* - \alpha y_i & \text{decreases} & \text{if } y_i > 0 \\ x_i^* - \alpha y_i & \text{remains constant} & \text{if } y_i = 0\,. \\ x_i^* - \alpha y_i & \text{increases} & \text{if } y_i < 0 \end{cases}$$

Since there exists at least one $y_i > 0$, it is easy to see that $\alpha > 0$ cannot increase indefinitely. In fact the maximum value that it can take is

$$\alpha_m = \min\left\{x_i^* / y_i \mid \text{those } i \text{ such that } y_i > 0\right\}.$$

Thus, $x^* - \alpha y$ is a feasible solution for all $\alpha \in [0, \ \alpha_m]$. Let

$$\alpha_l = \min\left\{x_i^* / y_i \mid \text{those } i \text{ such that } y_i < 0\right\}.$$

Clearly, if there exist i such that $y_i < 0,$ then $\alpha_l < 0.$ If there does not exist i such that $y_i < 0,$ we let $\alpha_l = 0.$ We can easily prove that $x^* - \alpha y$ is also a feasible solution for all $\alpha \in [0, \ \alpha_l]$.

Our remaining task is to show that $x^* - \alpha y, \ \alpha_l \le \alpha \le \alpha_m$, is optimal. For this, we note that

$$c^T\left(x^* - \alpha y\right) = c^T x^* - \alpha c^T.$$

Now we prove $c^T y$ must be 0. Let us look at the following two situations.

(i) $c^T y > 0.$ In this case, we may choose $\alpha > 0$ such that

$$c^T\left(x^* - \alpha y\right) > c^T x^*.$$

This implies that $c^T x^*$ is not the maximum, and hence x^* is not an optimal solution. This is a contradiction. Thus, we cannot have $c^T y > 0.$

(ii) $c^T y < 0.$ In this case, we may choose $\alpha < 0$ such that

$$c^T\left(x^* - \alpha y\right) > c^T x^*.$$

This also implies that $c^T x^*$ is not the maximum, and hence x^* is not an optimal solution. This is again a contradiction. This, we also cannot have $c^T y < 0.$

Combining these two situations together, we see that if x^* is an optimal situation, then we have

$$c^T y = 0.$$

Therefore,

$$c^T\left(x^* - \alpha y\right) = c^T x^* - \alpha c^T y = c^T x^*,$$

and hence, $x^* - \alpha_{\max} y$ is an optimal solution with positive components x^* whose number is less than $r - 1$.

We can repeat this process until the remaining columns of A are linearly independent, and so we have an optimal basic solution.
This completes the proof.

Th. 3. 2.
The set of feasible solutions

$$M := \left\{ x \in R^n \mid Ax = b,\ x \geq 0 \right\}$$

is convex.

Proof:
Let $x^1, x^2 \in M$, and $\alpha \in [0,\ 1]$ be arbitrary. Then,

$$Ax^1 = b \ \Rightarrow\ A\alpha x^1 = \alpha b$$

$$x^1 \geq 0 \ \Rightarrow\ \alpha x^1 \geq 0$$

and

$$Ax^2 = b \ \Rightarrow\ A\left(1-\alpha\right)x^2 = (1-\alpha)b$$

$$x^2 \geq 0 \ \Rightarrow\ (1\text{-}\alpha)x^2 \geq 0.$$

Thus,

$$A(\alpha x^1 + (1-\alpha)x^2) = A\alpha x^1 + A(1-\alpha)x^2 = \alpha b + (1-\alpha)b = b$$

$$\alpha x^1 + (1-\alpha)x^2 \geq 0.$$

Therefore,

$$\alpha x^1 + (1-\alpha)x^2 \in M,$$

and hence M is convex.

Th. 3. 3.

$$\left\langle x \text{ is an extreme point of } M \right\rangle \quad \Leftrightarrow \quad \left\langle x \text{ is a basic feasible solution} \right\rangle$$

Proof:
$(\Leftarrow)$:
Let $x = \left(x_1, \dots, x_m, 0, \dots, 0\right)^T$ be a basic feasible solution, where $x_i,\ i = 1, \dots, m,$ are basic variables, and $x_i = 0,\ i = m+1, \dots, n,$ are nonbasic variables. Then,

(3. 2.) $\quad x_1 a^1 + \dots + x_m a^m = b$

and $a^1,...,a^m$ are linearly independent.

Assume that x is not an extreme point. Then, there exit $y, z \in M$ with

(3. 3.) $$y \neq z,$$

and $\alpha \in \left]0,\ 1\right[$ such that

(3. 4.) $$x = \alpha y + (1-\alpha) z .$$

Note that all components of the vectors $x, y,$ and z are nonnegative, and $\alpha \in \left]0,\ 1\right[$. Since $x_j = 0,\ i = m+1,...,n,$ we have that

(3. 5.) $$y_j = 0,\ i = m+1,...,n$$

and

(3. 6.) $$z_j = 0,\ i = m+1,...,n.$$

Let

$$y = \left(y_1,...,y_m,0,...,0\right)^T$$

and

$$z = \left(z_1,...,z_m,0,...,0\right)^T .$$

Recall that y is chosen such that

$$y \in M.$$

Thus,

(3. 7.) $$y_1 a^1 + ... + y_m a^m = b .$$

Subtracting (3. 7.) from (3. 2.), we get

$$(x_1 - y_1) a^1 + ... + (x_m - y_m) a^m = 0.$$

Since $a^1,...,a^m$ are linearly independent, it follows that

$$x_i = y_i,\ \ \forall i = 1,...,m.$$

This, in turn, implies that

$$x = y.$$

Repeating this we also have

$$x = z.$$

Thus

$$x = y = z.$$

This is a contradiction, and hence x is an extreme point.

$(\Rightarrow)$:

Let x be an extreme point of M . We assume that the first k components of x are non-zero. Then, we have

(3. 8.) $$x_1 a^1 + x_1 a^2 + ... + y_k a^k = b$$

with

(3. 9.) $$x_i > 0, \ \ i = 1,...,k; \ x_i = 0, \ \ i = k+1,...,n.$$

If we can show that $a^1,...,a^k$ are linearly independent, then x is a basic feasible solution. We suppose $a^1,...,a^k$ are not linearly independent. Then, there exist $y_1,...y_k$, not all zero, such that

$$y_1 a^1 + y_1 a^2 + ... + x_k a^k = 0.$$

Thus, for any $\alpha > 0$, we have

(3. 10.) $$\alpha y_1 a^1 + \alpha y_1 a^2 + ... + \alpha x_k a^k = 0.$$

Define

$$y = \left(y_1,..., y_k, 0,...,0 \right)^T .$$

Since $x_i > 0, \ \ i = 1,...,k,$ we can find an $\alpha > 0$ such that

$$x + \alpha y \geq 0, \quad x - \alpha y \geq 0.$$

Thus, we have

$$A(x + \alpha y) = (x_1 + \alpha y_1) a^1 + (x_2 + \alpha y_2) a^2 + ... + (x_k + \alpha y_k) a^k = b$$

and

$$A(x-\alpha y)=(x_1-\alpha y_1)a^1+(x_2-\alpha y_2)a^2+\ldots+(x_k-\alpha y_k)a^k=b.$$

Thus,

$$x+\alpha y,\; x-\alpha y\in M$$

and hence

$$x=\frac{1}{2}(x+\alpha y)+\frac{1}{2}(x-\alpha y).$$

Since

$$\frac{1}{2}(x+\alpha y)\neq\frac{1}{2}(x-\alpha y),$$

x is not an extreme point. This is a contradiction. Thus, $a^1,\ldots,a^k$ are linearly independent, and hence x is a basic feasible solution if $k=m$, and it is a degenerate basic feasible solution if $k<m$.
This completes the proof.

C. 3. 1.
If the constraint set M is non-empty, then, it contains at least one extreme point.

Proof:
Since M is non-empty, then there exists a basic feasible solution. Thus, by theorem T. 3. 1., there exists a basic feasible solution. However, from theorem T. 3. 3., basic feasible solutions are extreme points. Therefore, the set M contains at least one extreme point.

C. 3. 2.
Consider the linear optimisation problem in standard form. If it has a finite optimal solution, then there exists a finite optimal solution which occurs at an extreme point of the constraint set M .

Proof:
The conclusion follows from theorems T. 3. 1. and T. 3. 3.

C. 3. 3.
There are only a finite number of extreme points in the constraint set M .

Proof:
The maximum possible number of basic solutions is

$$\binom{n}{m}=\frac{n!}{m!(n-m)!}<\infty.$$

Since the number of basic feasible solutions cannot be larger than that of basic solutions, the conclusion follows readily.

Th. 3. 4.
If the constraint set M is bounded, then it is a convex polytope.

Th. 3. 5.
If the constraint set M is bounded, then at least one optimal feasible solution is an extreme point of M .

Proof:
By theorem T. 3. 4., the constraint set M is a convex polytope. Let $x^1,...,x^k$ be extreme points of M . Then, any point $x = \left(x_1^0,...,x_n^0\right)^T \in M$ can be expressed as a convex combination of these points as follows:

$$x^0 = \alpha_1 x^1 + ... + \alpha_k x^k$$

where

$$\alpha_i \geq 0,\ i = 1,...,k;\quad \sum_{i=1}^{k} \alpha_i = 1.$$

This, in turns, implies that

(3. 11.) $$c^T x^0 = \sum_{i=1}^{k} \alpha_i c^T x^i .$$

Let

$$z_{\max} := \max\left\{c^T x^i \mid i = 1,...,k\ \right\}.$$

Then it follows from (2. 11.) that

$$c^T x^0 = \sum_{i=1}^{k} \alpha_i x^i \leq \sum_{i=1}^{k} \alpha_i \max\left\{c^T x^i \mid i = 1,...,k\right\}$$

$$= \sum_{i=1}^{k} \alpha_i z_{\max} = z_{\max} \cdot \sum_{i=1}^{k} \alpha_i = z_{\max}.$$

This is true for any $x^0 \in M$. Thus, $z_{\max}$ is the maximum value over M , and this maximum value is occurred at one of the extreme points $x^i,\ i = 1,...,n,$ of M .
This completes the proof.

Th. 3. 6. (***Fundamental Theorem of Linear Optimisation***)
A linear programming problem satisfies exactly one of the following:

1. The linear optimisation has no feasible solution.
2. The linear optimisation problem has an optimal solution.
3. The objective function is unbounded over the set of feasible solutions
4. If there is a feasible solution then there is a basic feasible solution and if there is an optimal solution then there is a basic feasible solution which is optimal (*Chvátal*).

Chapter 4

Linear Optimization

Graphical Method

A linear optimization problem can be solved graphically if it contains at most 3 variables. We describe the most important steps of such a solution method in the case of two variables:

- Plot the (linear) constraints.
- Determine the region of feasible solutions.
- Find the coordinates of the vertices of the set of feasible solutions.
- Calculate the values of the objective function at each vertex.
- Select the vertex (vertices) yielding the highest (maximum) or the lowest (minimum) of the objective function. Those coordinates determine the optimal solution (s).

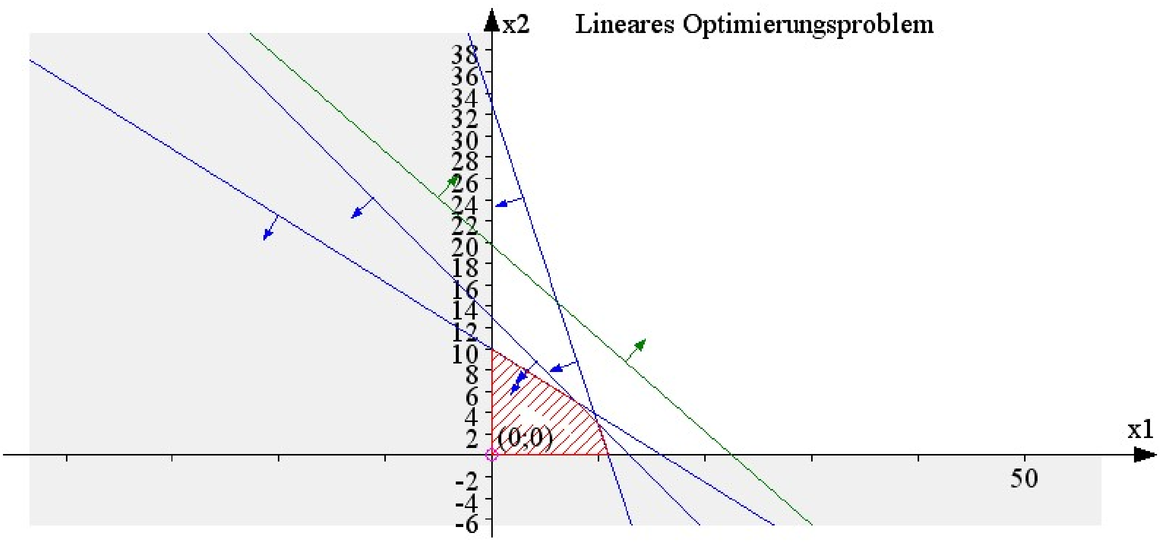
The following examples represent the most important cases occurring in solving a linear optimization problem:

Case 1: (A unique optimum)

$$z = 21x_1 + 24x_2 -> \max$$

$$\begin{aligned} 3x_1 + x_2 &\le 33 \\ x_1 + x_2 &\le 13 \\ 5x_1 + 8x_2 &\le 80 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Solution:



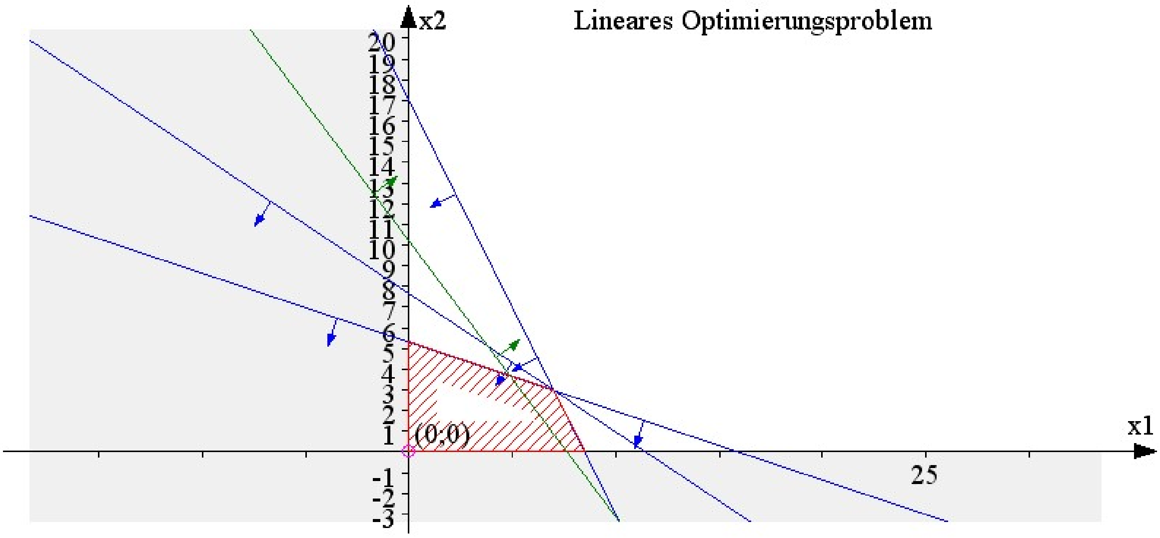
$$x^* = (8,\ 5)^T, \qquad z^* = 288$$

Case 2: (A degenerate optimum)

$$z = 40x_1 + 30x_2 -> \max$$

$$x_1 + 3x_2 \le 16$$
$$2x_1 + x_2 \le 17$$
$$2x_1 + 3x_2 \le 23$$
$$x_1,\ x_2 \ge 0$$

Solution:



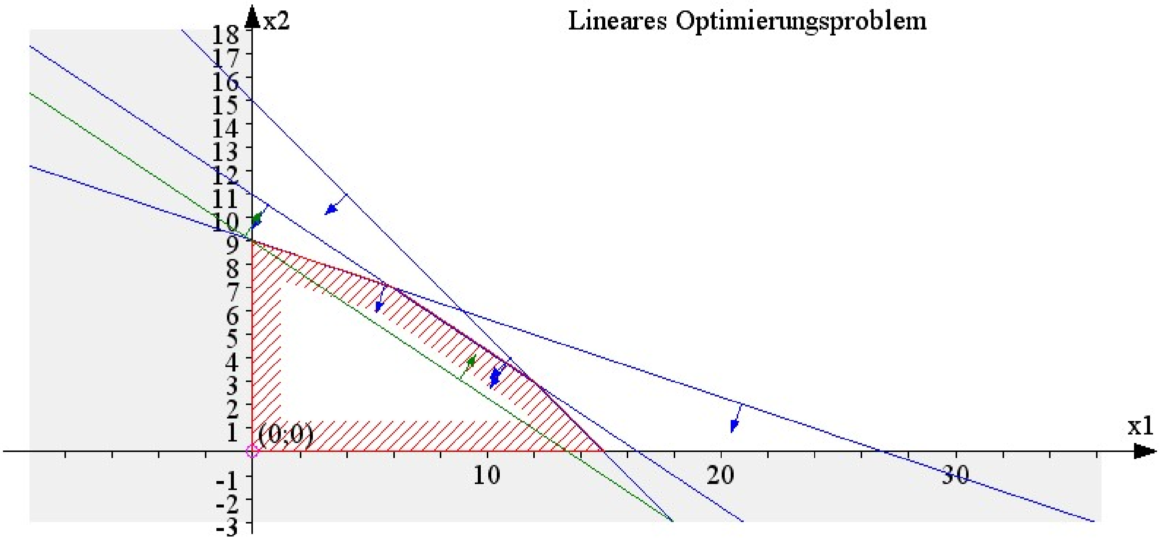
$$x^* = (7,\ 3)^T, \qquad z^* = 370$$

Case 3: (Multiple optima)

$$z = 12x_1 + 18x_2 -> \max$$

$$2x_1 + 3x_2 \le 33$$
$$x_1 + x_2 \le 15$$
$$x_1 + 3x_2 \le 27$$
$$x_1,\ x_2 \ge 0$$

Solution:



Multiple optimal solutions:

$$x^* = \alpha x^{*1} + (1-\alpha)x^{*2}, \quad 0 \le \alpha \le 1, \qquad z^* = 198,$$

with

$$x^{*1} = \begin{pmatrix} 6 & 7 \end{pmatrix}^T, \quad x^{*2} = \begin{pmatrix} 12 & 3 \end{pmatrix}^T$$

Case 4: (Objective function unbounded over the set of feasible solutions)

$$z = 18x_1 + 6x_2 -> \max$$

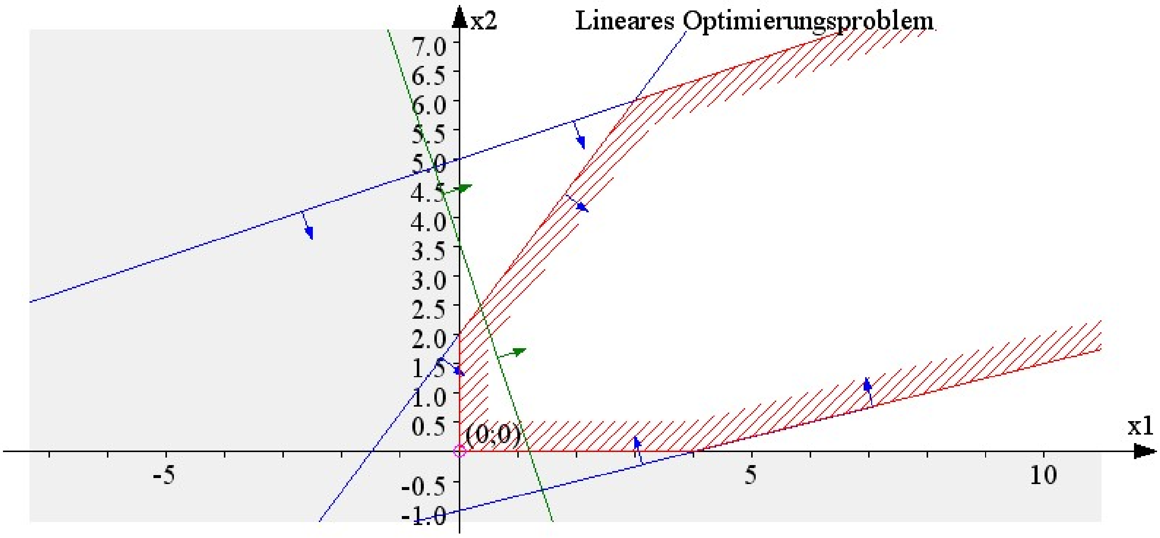
$$-4x_1 + 3x_2 \le 6$$

$$-x_1 + 3x_2 \le 15$$

$$x_1 - 4x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Solution:



The objective function is not bounded on the set of feasible solutions. Therefore, there is no optimal solution.

Case 5: (No feasible solution)

$$z = 12x_1 + 18x_2 -> \max$$

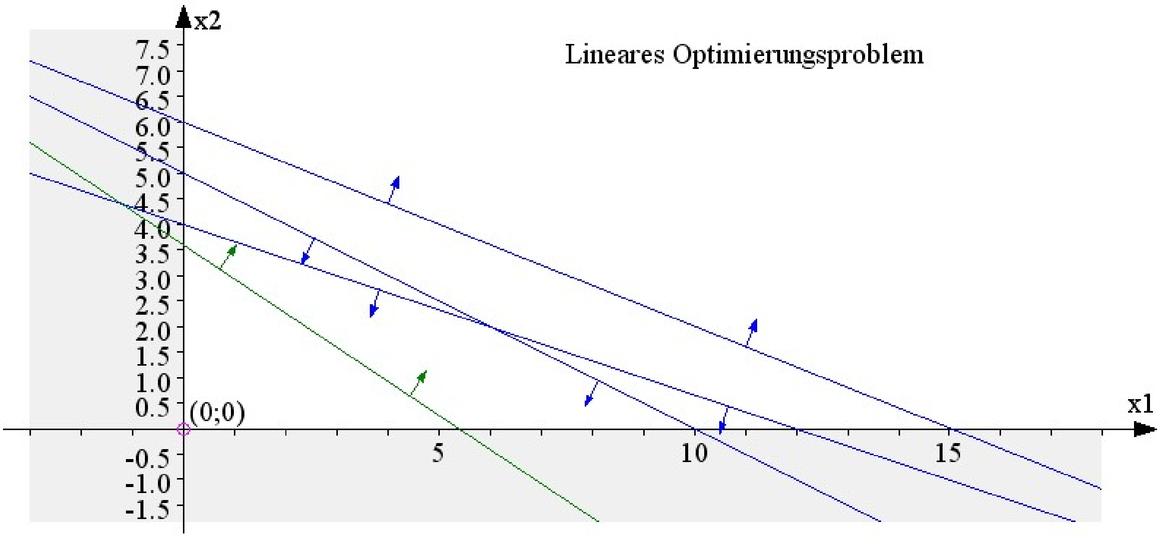
$$x_1 + 3x_2 \leq 12$$

$$x_1 + 2x_2 \leq 10$$

$$2x_1 + 5x_2 \geq 30$$

$$x_1,\ x_2 \geq 0$$

Solution:



There set of feasible solutions is empty. Therefore, there is no optimal solution.

Case 6: (A minimization problem)

$$z = 20x_1 + 40x_2 -> \min$$

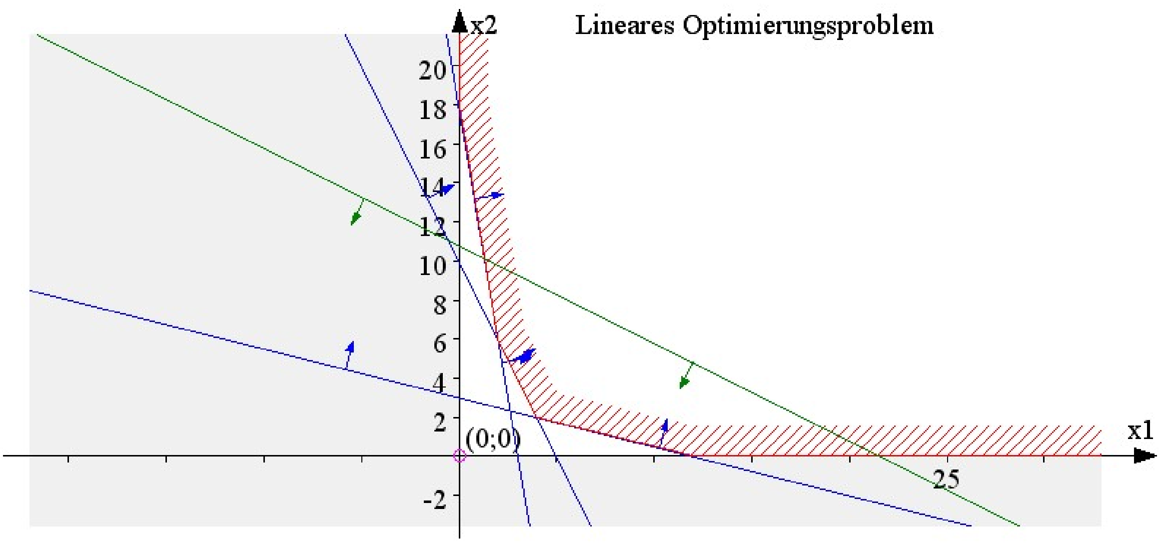
$$6x_1 + x_2 \geq 18$$

$$x_1 + 4x_2 \geq 12$$

$$2x_1 + x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

Solution:



$$x^* = (4, 2)^T, \qquad z^* = 160$$

Chapter 4

Linear Optimization (Graphical Method)

Exercises

4. 1.

A pension fund has $30 million to invest. The money is to be divided among Treasury notes, bonds and stocks. The rules for administration of the fund require that at least $3 million be invested in each type of investment, at least half of the money be invested in Treasury notes and bonds, and the amount invested in bonds not exceed twice the amount invested in Treasury notes. The annual yields for the various investments are 7% for Treasury notes, 8% for bonds, and 9% for stocks.
How should the money be allocated among the various investments to produce the largest return?
Solve the problem graphically.

4. 2.

Suppose the only foods available in your local store are potatoes and meat. The decision about how much of each food to buy is made entirely on dietary and economic considerations. The nutritional and cost informations are given in the following table:

	Per unit of potatoes	Per unit of Steak	Minimum requirements
Units of carbohydrates	3	1	8
Units of vitamins	4	3	19
Units of proteins	1	3	7
Unit cost	25	50	

The problem is to find a diet that meets all minimum nutritional requirements at minimal costs

1. Formulate the problem as a linear optimisation model
2. Solve the model graphically.

Chapter 4

Linear Optimization (Graphical Method)

Solutions

4. 1.
Denote by

x_1 : the amount in Treasury note [in millions of dollars]

x_2 : the amount in bonds [in millions of dollars].

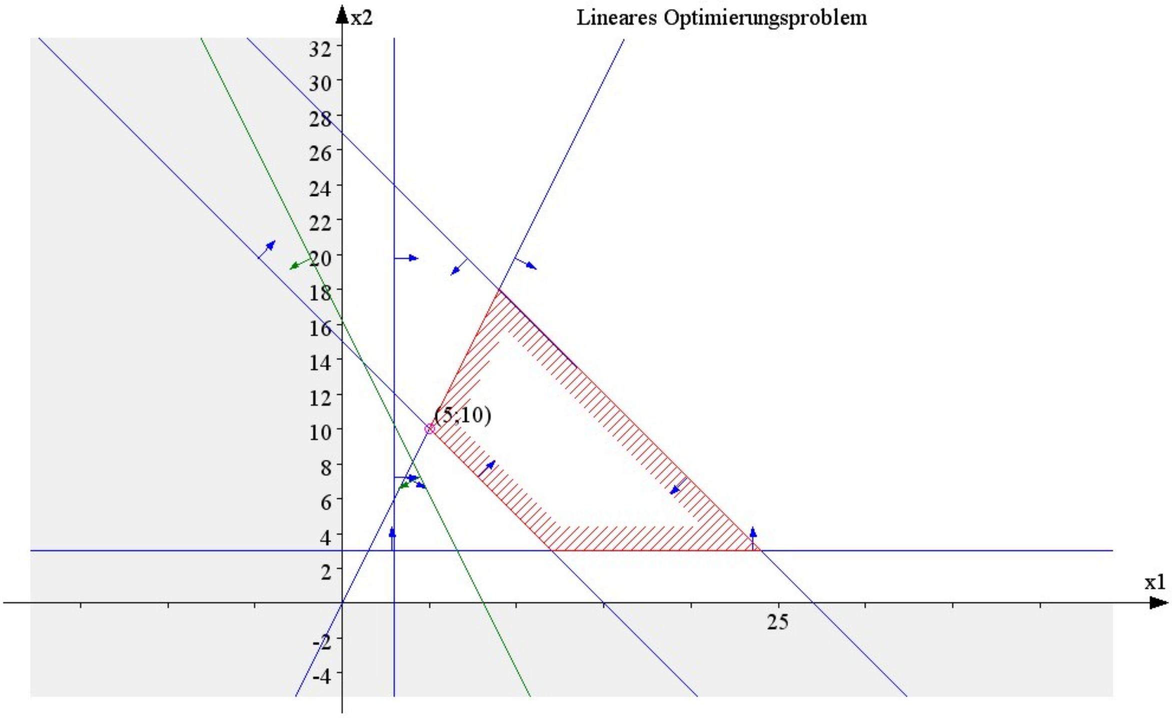
Hence, the amount in stocks [in millions of dollar] will be: $30-(x_1+x_2)$.

The model:

$$
\begin{aligned}
&z = 0.07x_1 + 0.08x_2 + 0.09\cdot(30-(x_1+x_2)) \to \max!\\
&x_1, x_2 \geq 3\\
&30-(x_1+x_2) \geq 3\\
&x_1 + x_2 \geq 15\\
&x_2 \leq 2x_1 .
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
&z = 2.7 - 0.02x_1 - 0.01x_2 \to \max!\\
&x_1 \geq 3\\
&x_2 \geq 3\\
&x_1 + x_2 \leq 27\\
&x_1 + x_2 \geq 15\\
&2x_1 - x_2 \geq 0 .
\end{aligned}
$$



$$x^* = \begin{pmatrix} 5 & 10 & 15 \end{pmatrix}^T, \qquad z^* = 2.5$$

4. 2.

1.

Let

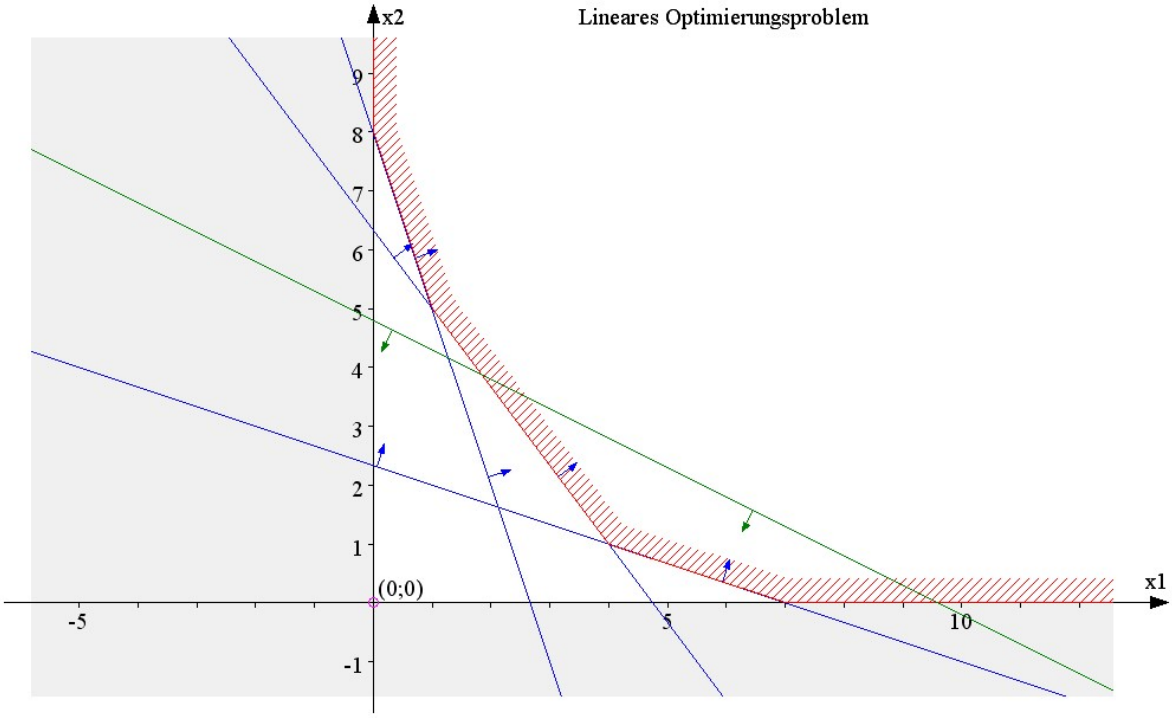
x_1 : amount of potatoes

x_2 : amount of steak

The model

$$z = 25x_1 + 50x_2 \rightarrow \min!$$
$$3x_1 \quad + x_2 \geq 8$$
$$4x_1 + 3x_2 \geq 19$$
$$x_1 + 3x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

2.



$x^* = \begin{pmatrix} 4 & 1 \end{pmatrix}^T, \quad z^* = 150\,.$

Chapter 5

Linear Optimization

The Simplex Method

R. 5. 1.
The simplex method provides a way of moving one basic feasible solution (extreme point) to another basic feasible solution so that the value of the objective function (in standard form) will systematically increase.
The *simplex algorithm* consists of the following steps:

Step 1:
Find an initial basic feasible solution.

Step 2:
Check if it is an optimal solution. If YES, stop.

Step 3:
If NO, exchange a basic variable with a non-basic variable so that a new basic feasible is obtained and the value of the objective function increased.

Step 4:
Repeat Steps 2 and 3 until no further improvement is possible.

R. 5. 2.
Let us assume that a linear optimisation problem has the following form:

$$f(x) = c_1x_1 + c_2x_2 + ... + c_nx_n \to \text{max!}$$

s.t.
$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1,n-m}x_{n-m} + x_{n-m+1} &&= b_1 \\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2,n-m}x_{n-m} + x_{n-m+2} &&= b_2 \\
&. \\
&. \\
&. \\
&a_{m1}x_1 + a_{m2}x_2 + ... + a_{m}x_{n-m} + x_n &&= b_m \\
&x_1, x_2, ..., x_n \ge 0; \quad b_1, b_2, ..., b_m \ge 0 .
\end{aligned}$$

In matrix notation

$$f = c^T x \to \text{max!}$$

s.t. $$Ax_N + x_B = b$$

$x \ge 0;\ b \ge 0,$

where

$$x = (x_1, \ldots, x_n)^T,$$
$$x_N = (x_1, \ldots, x_{n-m})^T,$$
$$x_B = (x_{n-m+1}, \ldots, x_n)^T$$
$$c = (x_1, \ldots, c_n)^T,$$
$$b = (b_1, \ldots, b_m)^T,$$

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1,n-m} \\ a_{21} & a_{22} & . & . & . & a_{2,n-m} \\ & & & . & & \\ & & & . & & \\ & & & . & & \\ a_{m1} & a_{m2} & & & & a_{m,n-m} \end{pmatrix}.$$

In this case, $x_B = b \geq 0,\ x_N = 0$ is a basic feasible solution of the LOP.

Now we can write the problem as follows:

(5. 1.) $\quad f = c_N^T x_N + c_B^T x_B \to \max!$

(5. 2.) $\quad$ s.t. $\quad x_B + Ax_N = b$

(5. 3.) $\quad x_B, x_N \geq 0.$

From (5. 2.) we have

$$x_B = b - Ax_N.$$

Substituting this into (5. 1.) we have

$$\begin{aligned} f &= c_B^T (b - Ax_N) + c_N^T x_N \\ &= c_B^T b - c_B^T A x_N + c_N^T x_N \\ &= c_B^T b - \left(c_B^T A + c_N^T\right) x_N. \end{aligned}$$

Let

$$z := c_B^T A - c_N^T = \left(c_B^T a^1 - c_1,\ c_B^T a^2 - c_2, \ldots,\ c_B^T a^{n-m} - c_{n-m}\right),$$

where a^j is the $j-$th column of A, $j = 1, 2, \ldots, n-m.$

Th. 5. 1. (***Optimality***)
The basic feasible solution $x_B = b \geq 0,\ x_N = 0$ is optimal if

$$z_i = c_B^T a^j - c_j \geq 0 \text{ for } j = 1,2,\dots,n-m.$$

Proof:
The proof will be given later.

R. 5. 2. (***Simplex Tableau***)
The corresponding initial simplex tableau is as follows:

x_B	x_1	. . .	x_{n-m}	x_{n-m+1}	x_{n-m+2}	. . .	x_n	RHS
x_{n-m+1}	a_{11}		$a_{1,n-m}$	1	0	. . .	0	b_1
x_{n-m+2}	a_{21}		$a_{2,n-m}$	0	1	. . .	0	b_2
.	.			.	.	.	.	.
.	.			.	.	.	.	.
.	.			.	.	.	.	.
x_n	a_{m1}		$a_{m,n-m}$	0	0	. . .	1	b_m
z	z_1	. . .	z_{n-m}	0	0	. . .	0	$f = c_B^T b$

We can also write the above initial simplex tableau in matrix form as follows:

x_B	x_N	x_B	RHS
x_B	A	I	b
z	$c_B^T A - c_N^T$	0	$c_B^T b$

Al. 5. 1. (***Simplex Algorithm***)
Step 1: Choose a non-basic variable to become a basic variable.
The non-basic variable x_r is chosen according to the following ("can") rule:

$$z_r = \min_{j \in J} \left\{ z_j = c_B^T a^j - c_j \right\}$$

and $z_r < 0$, where J is the set of indices of non-basic variables. This x_r will become a basic variable.
If all $z_j \geq 0,\ j \in J$, then this basic feasible solution is an optimal solution.

Step 2: Find a basic solution to become a non-basic variable.
Let x_B^i be the i^{th} component of x_B. Then, x_B^p is chosen to become a non-basic variable if

$$\varepsilon = \frac{x_B^p}{a_{pr}} = \min\left\{ \frac{x_B^i}{a_{ir}} \mid a_{ir} > 0,\ i = 1,\dots,m \right\}.$$

If $a_{ir} < 0$ for all $i = 1,2,\dots,m$, then the objective function is unbounded on the set of feasible solutions.

Step 3: Pivot on the element a_{pr}. Let the coefficient of the old tableau be denoted by a_{ij}, then the coefficient $a_{ij}^{'}$, of the old tableau is given by

$$a_{pj}^{'} = \frac{a_{pj}}{a_{pr}}, \quad b_p^{'} = \frac{b_p}{a_{pr}}$$

$$a_{ij}^{'} = a_{ij} - a_{ir}\frac{a_{pj}}{a_{pr}}, \quad b_i^{'} = b_i - a_{ir}\frac{b_p}{a_{pr}}, \quad i \neq p$$

$$z_j^{'} = z_j - z_r\frac{a_{pj}}{a_{pr}}, \quad f' = f - z_r\frac{b_p}{a_{pr}}.$$

We can find a solution by repeating Step 1-Step 3.

Ex. 5. 1.

$$f(x_1,x_2) = 21x_1 + 24x_2 -> \max$$
$$3x_1 + x_2 \le 33$$
$$x_1 + x_2 \le 13$$
$$5x_1 + 8x_2 \le 80$$
$$x_1,\ x_2 \ge 0$$

Solution:
Introducing slack variables $x_3, x_4,$ and x_5, we have

$$f(x_1,x_2) = 21x_1 + 24x_2 -> \max$$
$$3x_1 + x_2 + x_3 = 33$$
$$x_1 + x_2 + x_4 = 13$$
$$5x_1 + 8x_2 + x_5 = 80$$
$$x_i \ge 0,\ i = 1,2,\dots,5$$

Obviously,

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 5 & 8 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 33 \\ 13 \\ 80 \end{pmatrix}.$$

We choose $x_B = (x_3, x_4, x_5)^T$, $x_N = (x_1, x_2)^T$ so that

$$x_B = \begin{pmatrix} 33 \\ 13 \\ 80 \end{pmatrix}, \quad x_N = 0$$

is a basic feasible solution and $c_N^T = (21,\ 24),\ z = c_B^T b - c_N^T = (-21,\ \text{-}24)$. The simplex tableaux will now follow:

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_0
x_3	3	1	1	0	0	33
x_4	1	1	0	1	0	13
x_5	5	8	0	0	1	80
z	-21	-24	0	0	0	0
x_3	$\frac{19}{8}$	0	1	0	$-\frac{1}{8}$	23
x_4	$\frac{3}{8}$	0	0	1	$-\frac{1}{8}$	3
x_2	$\frac{5}{8}$	1	0	0	$\frac{1}{8}$	10
z	-6	0	0	0	3	240
x_3	0	0	1	0	$\frac{2}{3}$	4
x_1	1	0	0	$\frac{8}{3}$	$-\frac{1}{3}$	8
x_2	0	1	0	1	$\frac{1}{3}$	5
z	0	0	0	16	1	288

$$x^* = \begin{pmatrix} 8 & 5 & 4 & 0 & 0 \end{pmatrix}^T, \qquad z^* = 288$$

Ex. 5. 2.

$$z = 40x_1 + 30x_2 -> \max$$

$$\begin{aligned} x_1 \ + 3x_2 &\le 16 \\ 2x_1 \ + x_2 &\le 17 \\ 2x_1 + 3\,x_2 &\le 23 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Solution:

Standard form:

$$z = 40x_1 + 30x_2 -> \max$$

$$\begin{aligned} x_1 \ + 3x_2 + x_3 \qquad\qquad &= 16 \\ 2x_1 \ + x_2 \qquad + x_4 \qquad &= 17 \\ 2x_1 + 3\,x_2 \qquad\qquad + x_5 &= 23 \\ x_i \ge 0,\ i = 1,2,...,5 & \end{aligned}$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_0
x_3	1	3	1	0	0	16
x_4	2	1	0	1	0	17
x_5	2	3	0	0	1	23
z	-40	-30	0	0	0	0
x_3	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	0	$\frac{15}{2}$
x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{17}{2}$
x_5	0	2	0	-1	1	6
z	0	-10	0	20	0	340
x_2	0	1	$\frac{2}{5}$	$-\frac{1}{5}$	0	3
x_1	1	0	$-\frac{1}{5}$	$\frac{3}{5}$	0	7
x_5	0	0	$-\frac{4}{5}$	$-\frac{3}{5}$	1	0
z	0	0	4	18	0	370

$$x^* = \begin{pmatrix} 7 & 3 & 0 & 0 & 0 \end{pmatrix}^T, \qquad z^* = 370$$

The optimal solution is degenerate.

<u>Ex. 5. 3.</u>

$$z = 12x_1 + 18x_2 -> \max$$

$$2x_1 + 3x_2 \le 33$$
$$x_1 + x_2 \le 15$$
$$x_1 + 3x_2 \le 27$$
$$x_1,\ x_2 \ge 0$$

Solution:

Standard form:

$$z = 12x_1 + 18x_2 -> \max$$

$$\begin{aligned} 2x_1 + 3x_2 + x_3 \qquad &= 33 \\ x_1 + x_2 \quad + x_4 \quad &= 15 \\ x_1 + 3x_2 \qquad + x_5 &= 27 \\ x_i \geq 0,\ i = 1,2,\ldots,5. \end{aligned}$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_0
x_3	2	3	1	0	0	33
x_4	1	1	0	1	0	15
x_5	1	3	0	0	1	27
z	-12	-18	0	0	0	0
x_3	1	0	1	0	-1	6
x_4	$\frac{2}{3}$	0	0	1	$-\frac{1}{3}$	6
x_2	$\frac{1}{3}$	1	0	0	$\frac{1}{3}$	9
z	-6	0	0	0	6	162
x_1	1	0	1	0	-1	6
x_4	0	0	$-\frac{2}{3}$	1	$\frac{1}{3}$	2
x_2	0	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	7
z	0	0	6	0	**0**	198
x_1	1	0	-1	3	0	12
x_5	0	0	-2	3	1	6
x_2	0	1	1	-2	0	3
z	0	0	6	**0**	0	198

The problem has the multiple solution:

$$x^* = \alpha\begin{pmatrix}6 & 7 & 0 & 2 & 0\end{pmatrix}^T + (1-\alpha)\begin{pmatrix}12 & 3 & 0 & 0 & 6\end{pmatrix}^T, \quad 0 \le \alpha \le 1$$

$$z^* = 198.$$

Ex. 5. 4.

$$z = 18x_1 + 6x_2 -> \max$$

$$\begin{aligned} -4x_1 + 3x_2 &\le 6 \\ -x_1 + 3x_2 &\le 15 \\ x_1 - 4x_2 &\le 4 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Solution:

Standard form:

$$z = 18x_1 + 6x_2 -> \max$$

$$\begin{aligned} -4x_1 + 3x_2 + x_3 \qquad\qquad &= 6 \\ -x_1 + 3x_2 \qquad + x_4 \qquad &= 15 \\ x_1 - 4x_2 \qquad\qquad + x_5 &= 4 \\ x_i \ge 0,\ i = 1,2,...,5. \end{aligned}$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_0
x_3	-4	3	1	0	0	6
x_4	-1	3	0	1	0	15
x_5	1	-4	0	0	1	4
z	-18	-6	0	0	0	0
x_3	0	**-13**	1	0	4	22
x_4	0	**-1**	0	1	1	19
x_1	1	**-4**	0	0	1	4
z	0	-78	0	0	18	72

The objective function is not bounded on the set of feasible solutions. Therefore, there is no optimal solution.

R. 5. 3. (***Two-Phase Method***)
Consider the standard linear optimisation problem:

$$f = c^T x \to \max!$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

and we assume that $b \geq 0$. We construct an „*artificial*" linear optimisation problem

$$\tilde{f} = -\sum_{i=1}^{m} y_i \to \max!$$
$$\text{s.t.} \quad Ax + y = b$$
$$x, y \geq 0$$

where $y = (y_1, ..., y_m)^T$ denotes the vector of „*artificial*" variables. Obviously, if x is a feasible solution to the original problem, $y = 0$ is the optimal solution to the „*artificial*" linear optimisation problem and $\tilde{f} = 0$, otherwise $\tilde{f} < 0$, indicating that there is no feasible solution to the original problem.
Based on this observation we construct the following two-phase algorithm:

Phase 1:
Solve the "*artificial*" linear optimisation problem by the simplex method to find x and y. If $\tilde{f} = 0$, then x is a feasible solution to the original problem. And then go to Phase 2. Otherwise, there is no feasible solution to the optimisation problem. Stop.

Phase 2:
Starting from the feasible basic solution found in Phase 1 solve the original linear optimisation problem by the simplex method. This can start from the final tableau in Phase 1 by dropping the artificial variable y.

Ex. 5. 5.

$$z = 12x_1 + 18x_2 -> \max$$
$$x_1 + 3x_2 \leq 12$$
$$x_1 + 2x_2 \leq 10$$
$$2x_1 + 5x_2 \geq 30$$
$$x_1,\ x_2 \geq 0$$

Solution:

<u>Standard form</u>:

$$
\begin{aligned}
& z = 12x_1 + 18x_2 -> \max \\
& x_1 + 3x_2 + x_3 \qquad\qquad = 12 \\
& x_1 + 2x_2 \quad + x_4 \qquad = 10 \\
& 2x_1 + 5x_2 \qquad\qquad - x_5 = 30 \\
& \quad x_i \geq 0,\ i = 1,2,...,5.
\end{aligned}
$$

$$
\begin{aligned}
& \tilde{z} = -x_6 > \max \\
& x_1 + 3x_2 + x_3 \qquad\qquad\qquad = 12 \\
& x_1 + 2x_2 \quad + x_4 \qquad\qquad = 10 \\
& 2x_1 + 5x_2 \qquad\qquad - x_5 + x_6 = 30 \\
& \quad x_i \geq 0,\ i = 1,2,...,6.
\end{aligned}
$$

$$x_6 = 30 - 2x_1 - 5x_2 + x_5$$

$$\tilde{z} = -30 + 2x_1 + 5x_2 - x_5 \rightarrow \max!$$

Simplex Tableau (Phase 1)

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_0
x_3	1	3	1	0	0	0	12
x_4	1	2	0	1	0	0	10
x_6	2	5	0	0	-1	1	30
z	-12	-18	0	0	0	0	(0)
$\tilde{z}$	**-2**	-5	0	0	1	0	-30
x_2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	0	4
x_4	$\frac{1}{3}$	0	$-\frac{2}{3}$	1	0	0	2
x_6	$\frac{1}{3}$	0	$-\frac{5}{3}$	0	-1	1	10
z	-6	0	6	0	0	0	(72)
$\tilde{z}$	$-\frac{1}{3}$	0	$\frac{5}{3}$	0	1	0	-10
x_2	0	1	1	-1	0	0	2
x_1	1	0	-2	3	0	0	6
x_6	0	0	-1	-1	-1	1	8
z	0	0	-6	18	0	0	(108)
$\tilde{z}$	0	0	1	1	1	0	-8

Because of $\max \tilde{z} \neq 0$ there is no (basic) feasible solution. Therefore, there is also no optimal solution

Ex. 5. 6.

$$z = 20x_1 + 40x_2 -> \min$$
$$6x_1 + x_2 \geq 18$$
$$x_1 + 4x_2 \geq 12$$
$$2x_1 + x_2 \geq 10$$
$$x_1,\ x_2 \geq 0$$

Solution:

Standard form:

$$-20x_1 - 40x_2 -> \max$$
$$6x_1 + x_2 - x_3 = 18$$
$$x_1 + 4x_2 - x_4 = 12$$
$$2x_1 + x_2 - x_5 = 10$$
$$x_i \geq 0,\ i = 1,2,...,5.$$

$$\tilde{z} = -(x_6 + x_7 + x_8) -> \max$$
$$6x_1 + x_2 - x_3 + x_6 = 18$$
$$x_1 + 4x_2 - x_4 + x_7 = 12$$
$$2x_1 + x_2 - x_5 + x_8 = 10$$
$$x_i \geq 0,\ i = 1,2,...,8.$$

$$x_6 = 18 - 6x_1 - x_2 + x_3$$
$$x_7 = 12 - x_1 - 4x_2 + x_4$$
$$x_8 = 10 - 2x_1 - x_2 + x_5$$
$$x_6 + x_7 + x_8 = 40 - 9x_1 - 6x_2 + x_3 + x_4 + x_5$$

$$\tilde{z} = -40 + 9x_1 + 6x_2 - x_3 - x_4 - x_5 -> \max$$

Simplex Tableau (Phase 1)

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_0
x_6	**6**	1	-1	0	0	1	0	0	18
x_7	1	4	0	-1	0	0	1	0	12
x_8	2	1	0	0	-1	0	0	1	10
z	20	40	0	0	0	0	0	0	(0)
$\tilde{z}$	**-9**	-6	1	1	1	0	0	0	-40
x_1	1	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$	0	0	3
x_7	0	$\frac{23}{6}$	$\frac{1}{6}$	-1	0	$-\frac{1}{6}$	1	0	9
x_8	0	$\frac{2}{3}$	$\frac{1}{3}$	0	-1	$-\frac{1}{3}$	0	1	4
z	0	$\frac{110}{3}$	$\frac{10}{3}$	0	0	$-\frac{10}{3}$	0	0	(-60)
$\tilde{z}$	0	$-\frac{9}{2}$	$-\frac{1}{2}$	1	1	$\frac{3}{2}$	0	0	-13
x_1	1	0	$-\frac{4}{23}$	$\frac{1}{23}$	0	$\frac{4}{23}$	$-\frac{1}{23}$	0	$\frac{60}{23}$
x_2	0	1	$\frac{1}{23}$	$-\frac{6}{23}$	0	$-\frac{1}{23}$	$\frac{6}{23}$	0	$\frac{54}{23}$
x_8	0	0	$\frac{7}{23}$	$\frac{4}{23}$	-1	$-\frac{7}{23}$	$-\frac{4}{23}$	1	$\frac{56}{23}$
z	0	0	$\frac{40}{23}$	$\frac{220}{23}$	0	$-\frac{40}{23}$	$-\frac{220}{23}$	0	$\left(-\frac{3360}{23}\right)$
$\tilde{z}$	0	0	$-\frac{7}{23}$	$-\frac{4}{23}$	1	$\frac{30}{23}$	$\frac{27}{23}$	0	$-\frac{56}{23}$
x_1	1	0	0	$\frac{1}{7}$	$-\frac{4}{7}$	0	$-\frac{1}{7}$	$\frac{4}{7}$	4
x_2	0	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	0	$\frac{2}{7}$	$-\frac{1}{7}$	2
x_3	0	0	1	$\frac{4}{7}$	$-\frac{23}{7}$	-1	$-\frac{4}{7}$	$\frac{23}{7}$	8
z	0	0	0	$\frac{60}{7}$	$\frac{40}{7}$	*	*	*	-160
$\tilde{z}$	0	0	0		0	1	1	1	0

$$x^* = \begin{pmatrix} 4 & 2 & 8 & 0 & 0 \end{pmatrix}^T, \qquad z^* = 160$$

Chapter 5

Linear Optimization (Simplex Method)

Exercises

5. 1.
A store sells men's and women's tennis shoes. It makes a profit of $1 per pair of men's shoes and $1.20 per pair of women's shoes. It takes two minutes of a salesperson's time and two minutes of a cashier's time to sell a pair of men's shoes. It takes three minutes of a salesperson's time and one minute of a cashier's time per pair of women's shoes. The store is open eight hours per day, during which time there are two salespersons and one cashier on duty.
How many pairs of shoes of each type should the store sell in order to maximize profit each day?

1. Formulate the problem as a linear optimisation model
2. Solve the model by the simplex method.

5. 2.
A company owns a small paint factory that produces both interior and exterior house paints for wholesale distribution. Two basic raw materials, A and B, are used to manufacture the paints. The maximum availability of A is 6 tons a day; that of B is 8 tons a day. The daily requirements of the raw materials per ton of interior and exterior paints are summarised in the following table:

	Tons of Raw Material Per Ton of Paint		Maximum Availability (tons)
	Exterior	Interior	
A	1	2	6
B	2	1	8

A market survey has established that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. The survey also shows that the maximum demand for interior paint is limited to 2 tons daily. The wholesale price per ton is 3000 € for exterior point and 2000 € for interior point.
The company would like to maximise its daily gross income.

1. Formulate the problem as a linear optimisation model.
2. Solve the model by the simplex method.

5. 3.
A furniture maker has 6 units of wood and 28 hours of free time, in which he will make decorative screens. Two models have sold well in the past, so he will restrict himself to those two. He estimates that model I requires 2 units of wood and 7 h of time, while model II

requires 1 unit of wood and 8 h of time. The prices of the models are 120 € and 80 €, respectively.
The furniture maker wishes to maximise his sales revenue.

1. Formulate the problem as a linear optimisation model.
2. Represent the feasible solutions graphically (*Hint:* the variables are integer!)
3. Solve the model by the simplex method.

5. 4.

A firm makes dog food out of chicken and grain. Chicken has 10 grams of protein and 5 grams of fat per ounce, and grain has 2 grams of protein und 2 grams of fat per ounce. A bag of dog food must contain at least 200 grams of protein and at least 150 grams of fat.
Chicken costs 10 cents per ounce and grain costs 1 cent per ounce.
The company wants to know how many ounces of chicken and grain to use in each bag of dog food in order to minimise cost.

1. Formulate the problem as a linear optimisation model
2. Solve the model by the simplex method.

Chapter 5

Linear Optimization (Simplex Method)

Solutions

5. 1.

1.

Let

x_1 : number of pairs of men's shoes

x_2 : number of pairs of women's shoes

The model

$$z = x_1 + 1.20x_2 \to \max!$$

$$2x_1 + 3x_2 \le 960$$

$$2x_1 + x_2 \le 480$$

$$x_1, x_2 \ge 0: \text{ integer}$$

2.

The standard form

$$z = x_1 + 1.20x_2 \to \max!$$

$$2x_1 + 3x_2 + x_3 = 960$$

$$2x_1 + x_2 + x_4 = 480$$

$$x_i \ge 0,\ i = 1,2,3,4$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_0
x_3	2	3	1	0	960
x_4	2	1	0	1	480
z	-1	-1.2	0	0	0
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	320
x_4	$\frac{4}{3}$	0	$-\frac{1}{3}$	1	160
z	$-\frac{1}{5}$	0	$\frac{2}{5}$	0	384
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	240
x_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	120
z	0	0	$\frac{7}{20}$	$\frac{3}{20}$	408

$x^* = \begin{pmatrix} 240 & 120 & 0 & 0 \end{pmatrix}^*, \qquad z^* = 408$

5. 2.

1.

Denote by

x_1: sales amount of exterior paint

x_2: sales amount of interior paint.

<u>*The model*</u>:

$$z(x_1, x_2) = 3000x_1 + 2000x_2 \to \max!$$

$$x_1 + 2x_2 \le 6$$

$$2x_1 + x_2 \le 8$$

$$-x_1 + x_2 \le 1$$

$$x_2 \le 2$$

$$x_1, x_2 \ge 0.$$

2.

<u>*Standard form*</u>*:*

$$z(x_1, x_2) = 3000x_1 + 2000x_2 \to \max!$$

$$\begin{aligned} x_1 + 2x_2 + x_3 \quad\quad\quad &= 6 \\ 2x_1 \quad + x_2 \quad\ + x_4 \quad\quad &= 8 \\ -x_1 \quad x_2 \quad\quad\quad + x_5 \quad &= 1 \\ x_2 \quad\quad\quad\quad + x_6 &= 2 \\ x_i \geq 0,\ i = 1,2,\ldots,6. \end{aligned}$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_0
x_3	1	2	1	0	0	0	6
x_4	**2**	1	0	1	0	0	8
x_5	-1	1	0	0	1	0	1
x_6	0	1	0	0	0	1	2
z	-3000	-2000	0	0	0	0	0
x_3	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	2
x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	4
x_5	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	0	5
x_6	0	1	0	0	0	1	2
z	0	-500	0	1500	0	**0**	12000
x_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	$\frac{4}{3}$
x_1	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{10}{3}$
x_5	0	0	-1	1	1	0	3
x_6	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	$\frac{2}{3}$
z	0	0	$\frac{1000}{3}$	$\frac{4000}{3}$	0	0	$\frac{38000}{3}$

$$x^* = \begin{pmatrix} \frac{10}{3} & \frac{4}{3} & 0 & 0 & 3 & \frac{2}{3} \end{pmatrix}^T, \qquad z^* = \frac{38000}{3}.$$

5. 3.

1.

Denote by

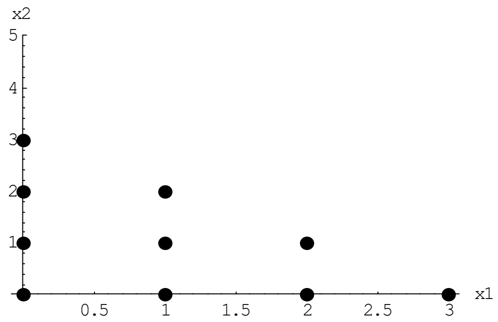
x_1 : number of model I screens to be produced,

x_2 : number of model II screens to be produced.

<u>*The model*</u>:

$$P(x_1, x_2) = 120x_1 + 80x_2 \to \max!$$
$$2x_1 + x_2 \le 6$$
$$7x_1 + 8x_2 \le 28$$
$$x_1, x_2 \ge 0: \text{ integer.}$$

2.



3.

<u>*Standard form*</u>:

$$P(x_1, x_2) = 120x_1 + 80x_2 \to \max!$$
$$2x_1 + x_2 + x_3 = 6$$
$$7x_1 + 8x_2 + x_4 = 28$$
$$x_1, x_2, x_3, x_4 \ge 0$$
$$x_1, x_2: \text{ integer.}$$

Simplex Tableau

BV	x_1	x_2	x_3	x_4	x_0
x_3	**2**	1	1	0	6
x_4	7	8	0	1	28
z	**-120**	-80	0	0	0
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	3
x_4	0	$\frac{9}{2}$	$-\frac{7}{2}$	1	7
z	0	**-20**	60	0	360
x_1	1	0	$\frac{8}{9}$	$-\frac{1}{9}$	$\frac{20}{9}$
x_2	0	1	$-\frac{7}{9}$	$\frac{2}{9}$	$\frac{14}{9}$
z	0	0	$\frac{400}{9}$	$\frac{40}{9}$	$\frac{3520}{9}$

$$x^* = \begin{pmatrix} \frac{20}{9} & \frac{14}{9} & 0 & 0 \end{pmatrix}^*, \qquad z^* = \frac{3520}{9}$$

The simplex method *does not* yield an optimal *integer* solution. Based upon the graphical solution of the problem, we have the following feasible points:

(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), 0, 3)

The corresponding values of the objective function are:

0, 120, 240, 360, 80, 200, 320, 160, 280, 240.

Therefore, we have:

$$x^* = \begin{pmatrix} 3 & 0 & 0 & 0 \end{pmatrix}^*, \qquad z^* = 360.$$

5. 4.

Denote by

x_1 : amount of chicken

x_2 : amount of grain.

<u>*The model*</u>:

$$z = 10x_1 + x_2 \to \min!$$
$$10x_1 + 2x_2 \ge 200$$
$$5x_1 + 2x_2 \ge 150$$
$$x_1, x_2 \ge 0.$$

<u>*Standard form*</u>:

$$-10x_1 - x_2 \to \max!$$
$$10x_1 + 2x_2 - x_3 \qquad = 200$$
$$5x_1 + 2x_2 \qquad - x_4 = 150$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

A basic feasible solution is not immediately available.

$$-10x_1 - x_2 \to \max!$$
$$10x_1 + 2x_2 - x_3 \qquad + x_5 \qquad = 200$$
$$5x_1 + 2x_2 \qquad - x_4 \qquad + x_6 = 150$$
$$x_i \ge 0,\ i = 1,2,...,6.$$

$$x_5 = 200 - 10x_1 - 2x_2 + x_3$$
$$x_6 = 150 - 5x_1 - 2x_2 + x_4$$
$$x_5 + x_6 = 350 - 15x_1 - 4x_2 + x_3 + x_4$$

$$\tilde{z} = -\left(x_5 + x_6\right) = -350 + 15x_1 + 4x_2 - x_3 - x_4 \to \max!$$

s.t.

$$10x_1 + 2x_2 - x_3 \qquad + x_5 \qquad = 200$$
$$5x_1 + 2x_2 \qquad - x_4 \qquad + x_6 = 150$$
$$x_i \ge 0,\ i = 1,2,...,6.$$

Simplextableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_0
x_5	10	2	-1	0	1	0	200
x_6	5	2	0	-1	0	1	150
z	10	1	0	0	0	0	(0)
$\tilde{z}$	-15	-4	1	1	0	0	-350
x_1	1	$\frac{1}{5}$	$-\frac{1}{10}$	0	$\frac{1}{10}$	0	20
x_6	0	1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	50
z	0	-1	1	0	-1	0	(-200)
$\tilde{z}$	0	-1	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	-50
x_1	1	0	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	10
x_2	0	1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	50
z	0	0	$\frac{3}{2}$	-1	$-\frac{3}{2}$	1	-150
$\tilde{z}$	0	0	0	0	1	1	0
x_4	5	0	-1	1			50
x_2	5	1	$-\frac{1}{2}$	0			100
z	5	0	$\frac{1}{2}$	0			-100

$$x^* = \begin{pmatrix} 0 & 100 & 0 & 50 \end{pmatrix}^T, \qquad z^* = 100 \text{ cents}$$

Chapter 6

Linear Optimization

Duality

D. 6. 1. (***Symmetric Duality***)
The following problems are said to be *symmetrically dual* to one another:

(SP) $\max\left\{z = c^T x \mid Ax \le b,\ x \ge 0\right\}$

(SD) $\min\left\{Z = b^T \lambda \mid A^T \lambda \ge c,\ \lambda \ge 0\right\}$

D. 6. 2. (**As*ymmetric Duality***)
The following problems are said to be *asymmetrically dual* to one another:

(ASD) $\min\left\{z = c^T x \mid Ax = b,\ x \ge 0\right\}$

(AP) $\max\left\{Z = b^T \lambda \mid A^T \lambda \ge c\right\}$

R. 6. 1. (***Primal and Dual***)
Usually the given problem is called the *primal* and the one to be constructed is called the *dual*.

Th. 6. 1.
Dual of dual is primal.

Th. 6. 2. (***Weak Duality***)[1]

If $\bar{x}$ is feasible for (SP) and $\bar{\lambda}$ is feasible for (DP), then

$$c^T \bar{x} \le \bar{\lambda}^T b\,.$$

Proof:

$$c^T \bar{x} \le (\bar{\lambda}^T A)\bar{x} = \bar{\lambda}^T (A\bar{x}) \le \bar{\lambda}^T b$$

C. 6. 1.

If $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively, and if $c^T \bar{x} = \bar{\lambda}^T b$, then $\bar{x}$ and $\bar{\lambda}$ are optimal for (SP) and (SD), respectively.

---

[1] The following statements will be proved only for the symmetric duality. They also hold for the asymmetric duality.

Proof:

Suppose $x^{\bullet}$ is any feasible solution for (SP). Then $c^T x^{\bullet} \le \bar{\lambda}^T b = c^T \bar{x}$. Similarly, if $\lambda^{\bullet}$ is any solution for (SD), then $\lambda^{\bullet T} b \ge \bar{\lambda}^T b$.

C. 6. 2.

If (SP) has unbounded objective function value, then (SD) is infeasible. If (SD) has unbounded objective value, then (SP) is infeasible

Proof:

Suppose (SD) is feasible. Let $\bar{\lambda}$ be a particular feasible solution. Then for all $\bar{x}$ feasible for (SP) we have $c^T \bar{x} \le \bar{\lambda}^T b$. So (SP) has bounded objection value if it is feasible, and therefore cannot be unbounded. The second statement is proved similarly.

Th. 6. 3.

Assume (SP) is feasible. Then (SP) is unbounded if and only if the following system is feasible:

$$\text{(USP)} \qquad \begin{cases} Aw \le 0 \\ c^T w > 0 \\ w \ge 0 \end{cases}$$

Proof:

Suppose $\bar{x}$ is feasible for (SP).

First assume that $\bar{w}$ is feasible for (USP) and $t \ge 0$ is a real number. Then

$$A(\bar{x} + t\bar{w}) = A\bar{x} + tA\bar{w} \le b + 0 = b$$

$$\bar{x} + t\bar{w} \ge 0 + t0 = 0$$

$$c^T(\bar{x} + t\bar{w}) = c^T \bar{x} + tc^T \bar{w}$$

Hence $\bar{x} + t\bar{w}$ is feasible for (SP), and by choosing t approximately large, we can make $c^T(\bar{x} + t\bar{w})$ as large as desired since $c^T \bar{w}$ is a positive number.

Conversely, suppose that (SP) has unbounded objective function value, then by C. 6. 2., (SD) is infeasible. That is, the following system has no solution:

$$\begin{cases} \lambda^T A \ge c^T \\ \lambda \ge 0 \end{cases}$$

or

$$\begin{cases} A^T \lambda \ge c^T \\ \lambda \ge 0 \end{cases}$$

By the "Theorem of Alternatives"[2], the following system is feasible:

$$\begin{cases} w^T A^T \leq 0^T \\ w^T c > 0 \\ w \geq 0 \end{cases}$$

or

$$\begin{cases} Aw \leq 0 \\ c^T w > 0 \\ w \geq 0 \end{cases}$$

Hence (USP) is feasible.

Th. 6. 4.
Assume (SD) is feasible. Then (SD) is unbounded if and only if the following system is feasible:

(USD) $$\begin{cases} v^T A \geq 0^T \\ v^T b < 0 \\ v \geq 0 \end{cases}$$

C. 6. 3.
(SP) is feasible if and only if (USD) is infeasible. (SD) is feasible if and only if (USP) is infeasible

C. 6. 4.
If (SP) is infeasible, then either (SD) is infeasible or (SD) is unbounded. If (SD) is infeasible, then either (SP) is infeasible or (SP) is unbounded.

Th. 6. 5.
Suppose (SP) and (SD) are both feasible. Then (SP) and (SD) each have finite objective function values, and moreover these two values are equal

Proof:

We know by Weak Duality Theorem that if $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively, then $c^T \bar{x} \leq \bar{\lambda}^T b$. In particular, neither (SP) nor (SD) is unbounded. So it suffices to show that the following system is feasible:

---

[2] Theorem of Alternatives: Either the system $Ax \geq b$, $x \geq 0$ has a solution, or the system

$$\begin{cases} \lambda^T A \leq 0^T \\ \lambda^T b > 0 \\ \lambda \geq 0 \end{cases}$$

has a solution, but not both.

$$
(I) \qquad \begin{cases} Ax \le b \\ x \ge 0 \\ \lambda^T A \ge c^T \\ \lambda \ge 0 \\ c^T x \ge \lambda^T b \end{cases}
$$

For if $\bar{x}$ and $\bar{\lambda}$ are feasible for this system, then by Weak Duality Theorem in fact it would have to be the case that $c^T \bar{x} \ge \bar{\lambda}^T b$

We now write the above system in matrix form:

$$
\begin{pmatrix} A & 0 \\ 0 & -A^T \\ -c^T & b^T \end{pmatrix} \cdot \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ -c \\ 0 \end{pmatrix}, \qquad x, \lambda \ge 0
$$

We assume that this system is infeasible and derive a contradiction. If it is not feasible, then by the Theorem of Alternatives the following system has a solution $\bar{v}, \bar{w}, \bar{t}$,

$$
(II) \qquad \begin{cases} \begin{pmatrix} v^T & w^T & t \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -A^T \\ -c^T & b^T \end{pmatrix} \ge \begin{pmatrix} 0^T & 0^T \end{pmatrix} \\ \begin{pmatrix} v^T & w^T & t \end{pmatrix} \begin{pmatrix} b \\ -c \\ 0 \end{pmatrix} < 0 \\ v, w, t \ge 0 \end{cases}
$$

So we have

$$
\begin{aligned}
\bar{v}^T A - \bar{t} c^T &\ge 0^T \\
-\bar{w}^T A^T + \bar{t} b^T &\ge 0^T \\
\bar{v}^T b - \bar{w}^T c &< 0^T \\
v, w, t &\ge 0
\end{aligned}
$$

Case 1: Suppose $\bar{t} = 0$. Then

$$
\begin{aligned}
&\bar{v}^T A \ge 0^T \\
&A\bar{w} \le 0 \\
&\bar{v}^T b < c^T \bar{w} \\
&\bar{v}, \bar{w} \ge 0
\end{aligned}
$$

Now we cannot have both $c^T \bar{w} \leq 0$ and $\bar{v}^T b \geq 0$; otherweise $0 \leq \bar{v}^T b < c^T \bar{w} \leq 0$, which is a contradiction.

Case 1a: Suppose $c^T \bar{w} > 0$. Then $\bar{w}$ is a solution to (USP), so (SD) is infeasible by C. 6.3., a contardiction.

Case 1b: Suppose $\bar{v}^T b < 0$. Then $\bar{v}$ is a solution to (USD), so (SP) is infeasible by C. 6.3., a contardiction.

Case 2: Suppose $\bar{t} > 0$. Set $\bar{x} = \bar{w}/\bar{t}$ and $\bar{\lambda} = \bar{v}/\bar{t}$. Then

$$A\bar{x} < b$$

$$\bar{x} \geq 0$$

$$\bar{\lambda}^T A \geq c^T$$

$$\bar{\lambda} \geq 0$$

$$c^T \bar{x} \geq \bar{\lambda}^T b$$

Hence we have a pair of feasible solutions to (SP) and (SD), repectively, that violates Weak Duality, a contradiction.
We have now shown that (*II*) has no solution, Therefore, (*I*) has a solution.

C. 6. 5.
Suppose (SP) has a finite optimal objective value. Then so does (SD), and these two values are equal.
Suppose (SD) has a finite optimal objective function value. Then so does (SP), and these two values are equal.

Proof:
We will prove the first statement only.. If (SP) has a finite optimal objective function value, then it is feasible, but not unbounded. So (USP) has no solution by Theorem Th. 6. 3. Therefore (SD) is feasible by C. 6. 3. Now apply Theorem Th. 6. 5.

R. 6. 2.
The results gained above can now be summarised in the following central theorem.

Th. 6. 6. (*Strong Duality*)
Exactly one of the following holds for the pair (SP) and (SD):

1. They are both infeasible.
2. One is infeasible and the other unbounded.
3. They are both fesible and have equal finite optimal objective function values.

C. 6. 8.

If $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively, then $\bar{x}$ and $\bar{\lambda}$ are optimal for (SP) and (SD), respectively, if and only if $c^T \bar{x} = \bar{\lambda}^T b$.

C. 6. 9.

Suppose $\bar{x}$ is feasible for (SP). Then $\bar{x}$ is optimal for (SP) if and only if there exists $\bar{\lambda}$ feasible for (SD) such that $c^T \bar{x} = \bar{\lambda}^T b$.

Similarly suppose $\bar{\lambda}$ is feasible for (DS). Then $\bar{\lambda}$ is optimal for (SD) if and only if there exists $\bar{x}$ feasible for (SP) such that $c^T \bar{x} = \bar{\lambda}^T b$.

D. 6. 3. (***Complementary Slackness***)

Suppose $\bar{x} \in R^n$ and $\bar{\lambda} \in R^m$. Then $\bar{x}$ and $\bar{\lambda}$ satisfy *complementary slackness* if

1. For all j, either $\bar{x}_j = 0$ or $\sum_{i=1}^{m} a_{ij} \bar{\lambda}_i = c_j$ or both; and
2. For all i, either $\bar{\lambda}_i = 0$ or $\sum_{i=1}^{n} a_{ij} \bar{x}_j = b_i$ or both.

Th. 6. 7.

Suppose $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively. Then $c^T \bar{x} = \bar{\lambda}^T b$ if and only if $\bar{x}$ and $\bar{\lambda}$ satisfy complementary slackness.

C. 6. 10.

Suppose $\bar{x}$ and $\bar{\lambda}$ are feasible for (SP) and (SD), respectively, then $\bar{x}$ and $\bar{\lambda}$ are optimal for (SP) and (SD), respectively if and only if they $\bar{\lambda}$ satisfy complementary slackness.

C. 6. 11.

Suppose $\bar{x}$ is feasible for (SP). Then $\bar{x}$ is optimal for (SP) if and only if there exists $\bar{\lambda}$ feasible for (SD) such that $\bar{x}$ and $\bar{\lambda}$ satisfy complementary slackness.

Similarly suppose $\bar{\lambda}$ is feasible for (SD). Then $\bar{\lambda}$ is optimal for (SD) if and only if there exists $\bar{x}$ feasible for (SP) such that $\bar{x}$ and $\bar{\lambda}$ satisfy complementary slackness.

<u>Ex. 6. 1.</u> (*See Ex. 5. 6.*)
1. Solve the folloowing problem by dualisation:

$$
\begin{aligned}
z = 20x_1 + 40x_2 -> \min \\
6x_1 \quad + x_2 \geq 18 \\
x_1 + 4x_2 \geq 12 \\
2\, x_1 + \quad x_2 \geq 10 \\
x_1,\, x_2 \geq 0.
\end{aligned}
$$

2. Check the complementary slackness for this problem.

Solution:
1.

$$
\begin{aligned}
Z = 18\lambda_1 + 12\lambda_2 + 10\lambda_3 \to \max \\
6\lambda_1 \quad + \lambda_2 + 2\lambda_3 \leq 20 \\
\lambda_1 + 4\lambda_2 \quad + \lambda_3 \leq 40 \\
\lambda_1, \lambda_2, \lambda_3 \geq 0
\end{aligned}
$$

<u>Standard form</u>:

$$
\begin{aligned}
Z = 18\lambda_1 + 12\lambda_2 + 10\lambda_3 \to \max \\
6\lambda_1 \quad + \lambda_2 + 2\lambda_3 + \lambda_4 \qquad = 20 \\
\lambda_1 + 4\lambda_2 \quad + \lambda_3 \qquad + \lambda_5 = 40 \\
\lambda_i \geq 0,\; i = 1, 2, \ldots, 5.
\end{aligned}
$$

Simplex Tableau

BV	λ_1	λ_2	λ_3	λ_4	λ_5	λ_0
λ_4	6	1	2	1	0	20
λ_5	1	4	1	0	1	40
Z	-18	-12	-10	0	0	0
λ_1	1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{10}{3}$
λ_5	0	$\frac{23}{6}$	$\frac{2}{3}$	$-\frac{1}{6}$	1	$\frac{110}{3}$
Z	0	-9	-4	3	0	60
λ_1	1	0	$\frac{7}{23}$	$\frac{4}{23}$	$-\frac{1}{23}$	$\frac{40}{23}$
λ_2	0	1	$\frac{4}{23}$	$-\frac{1}{23}$	$\frac{6}{23}$	$\frac{220}{23}$
Z	0	0	$-\frac{56}{23}$	$\frac{60}{23}$	$\frac{54}{23}$	$\frac{3360}{23}$
λ_3	$\frac{23}{7}$	0	1	$\frac{4}{7}$	$-\frac{1}{7}$	$\frac{40}{7}$
λ_2	$-\frac{4}{7}$	1	0	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{60}{7}$
Z	8	0	0	4	2	160

$$x^* = \begin{pmatrix}4 & 2 & 8 & 0 & 0\end{pmatrix}^T, \quad \lambda^* = \begin{pmatrix}0 & 60/7 & 40/7 & 0 & 0\end{pmatrix}^T, \; z^* = Z^* = 160$$

2.

x_j^*	λ_j^*	Constraint j
4	0	> 0
2	1380/161	$= 0$
8	40/7	$= 0$
0	0	> 0
0	0	> 0

Chapter 6

Linear Optimization (Duality)

Exercises

6. 1.

An oil company has two refineries. Each day, refinery R_1 produces 200 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $12000 to operate. Each day, refinery R_2 produces 100 barrels of high-grade oil, 100 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $10000 to operate. The company must produce at least 800 barrels of high-grade oil, 900 barrels of medium-grade oil, and 1,000 barrels of low-grade oil.
The problem is to find the number of days that each refinery should be operated to lead to minimum cost for the oil company

1. Formulate the problem as a model of linear optimisation.
2. Solve the problem by the simplex method.
3. Give and interpret the solutions of both the primal and the dual problems.

6. 2.

A nutritionist is planning a menu consisting of two main foods F_1 and F_2. Each ounce of F_1 contains 2 units of fat, 1 unit of carbohydrates, and 4 units of protein. Each unit of F_2 contains 3 units of fat, 3 units of carbohydrates, and 3 units of protein. The nutritionist wants the meal to provide at least 18 units of fat, at least 12 units of carbohydrates, and at least 24 units of protein. An ounce of F_1 costs 20 cents and an ounce of F_2 costs 25 cents.

The question is how many ounces of each food should be bought to minimise the cost of the meal and yet satisfy the nutritionist's requirements.

1. Formulate the problem as a model of linear optimisation.
2. Solve the problem by the simplex method.
3. Give and interpret the solutions of both the primal and the dual problems.

Chapter 6

Linear Optimization (Duality)

Solutions

6. 1.
1.
Denote by

$x_i,\ i=1,2:$ number of days operated in R_i

$$
\begin{aligned}
&z = 12000x_1 + 10000x_2 \to \min!\\
&200x_1 + 100x_2 \ge 800\\
&300x_1 + 100x_2 \ge 900\\
&200x_1 + 200x_2 \ge 1000\\
&x_1, x_2 \ge 0 : \text{integer}
\end{aligned}
$$

2.
The dual:

$$
\begin{aligned}
&Z = 800\lambda_1 + 900\lambda_2 + 1000\lambda_3 \to \max!\\
&200\lambda_1 + 300\lambda_2 + 200\lambda_3 \le 12000\\
&100\lambda_1 + 100\lambda_2 + 200\lambda_3 \le 10000\\
&\lambda_1, \lambda_2, \lambda_3 \ge 0
\end{aligned}
$$

<u>Standard form:</u>

$$
\begin{aligned}
&Z = 800\lambda_1 + 900\lambda_2 + 1000\lambda_3 \to \max!\\
&200\lambda_1 + 300\lambda_2 + 200\lambda_3 + \lambda_4 \qquad = 12000\\
&100\lambda_1 + 100\lambda_2 + 200\lambda_3 \qquad + \lambda_5 = 10000\\
&\lambda_i \ge 0,\ i = 1,2,\ldots,5.
\end{aligned}
$$

Simplex Tableau

BV	λ_1	λ_2	λ_3	λ_4	λ_5	λ_0
λ_4	200	300	200	1	0	12000
λ_5	100	100	200	0	1	10000
Z	-800	-900	-1000	0	0	0
λ_1	1	$\frac{5}{2}$	1	$\frac{1}{100}$	0	60
λ_5	0	-50	100	$-\frac{1}{2}$	1	40000
Z	0	300	-200	4	0	48000
λ_1	1	2	0	$\frac{1}{100}$	$-\frac{1}{100}$	20
λ_3	0	$-\frac{1}{2}$	1	$-\frac{1}{100}$	$\frac{1}{100}$	40
Z	0	200	0	3	2	56000

Solution of the primal:

$$x^* = \begin{pmatrix}3 & 2 & 0 & 200 & 0\end{pmatrix}^T, \qquad z^* = 56000$$

Solution of the dual:

$$\lambda^* = \begin{pmatrix}20 & 0 & 40 & 0 & 0\end{pmatrix}^T, \qquad Z^* = 56000\,.$$

Increasing the right-hand side of only the first constraint in the primal by one unit will lead to an increase of the value of the objective function by 20 units.

Increasing the right-hand side of only the second constraint in the primal by one unit will have no influence on the value of the objective function.

Increasing the right-hand side of only the third constraints in the primal by one unit will lead to an increase of the value of the objective function by 40 units.

6. 2.

1.

Denote by

$x_i,\ i = 1,2:$ amount of F_i

$$
\begin{aligned}
z = 20x_1 + 25x_2 &\to \min \\
2x_1 + 3x_2 &\geq 18 \\
x_1 + 3x_2 &\geq 12 \\
4x_1 + 3x_2 &\geq 24 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

2.

The dual:

$$
\begin{aligned}
Z = 18\lambda_1 + 12\lambda_2 + 24\lambda_2 &\to \max \\
2\lambda_1 + \lambda_2 + 4\lambda_3 &\leq 20 \\
3\lambda_1 + 3\lambda_2 + 3\lambda_3 &\leq 25 \\
\lambda_1, \lambda_2, \lambda_3 &\geq 0
\end{aligned}
$$

Standard form:

$$
\begin{aligned}
Z = 18\lambda_1 + 12\lambda_2 + 24\lambda_2 &\to \max \\
2\lambda_1 + \lambda_2 + 4\lambda_3 + \lambda_4 &= 20 \\
3\lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_5 &= 25 \\
\lambda_i \geq 0,\ i = 1,2,\ldots,5.
\end{aligned}
$$

Simplex Tableau

BV	λ_1	λ_2	λ_3	λ_4	λ_5	λ_0
λ_4	2	1	4	1	0	20
λ_5	3	3	3	0	1	25
Z	-18	-12	-24	0	0	0
λ_3	$\frac{1}{2}$	$\frac{1}{4}$	1	$\frac{1}{4}$	0	5
λ_5	$\frac{3}{2}$	$\frac{9}{4}$	0	$-\frac{3}{4}$	1	10
Z	-6	-6	0	6	0	120
λ_3	0	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{5}{3}$
λ_1	1	$\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{2}{3}$	$\frac{20}{3}$
Z	0	3	0	3	4	160

Solution of the primal:

$$x^* = \begin{pmatrix} 0 & 3 & 4 & 0 & 3 \end{pmatrix}^T, \qquad z^* = 160$$

Solution of the dual:

$$\lambda^* = \begin{pmatrix} \frac{20}{3} & 0 & \frac{5}{3} & 0 & 0 \end{pmatrix}^T, \qquad Z^* = 160\,.$$

Increasing the right-hand side of the only the first constraint in the primal by one unit will lead to an increase of the value of the objective function by $20/3$ units.

Increasing the right-hand side of only the second constraint in the primal by one unit will have no influence on the value of the objective function.

Increasing the right-hand side of only the third constraint in the primal by one unit will lead to an increase of the value of the objective function by $5/3$ units.

Chapter 7

Linear Optimization

Transportation Problem

D. 7. 1. (***Hitchcock-Koopmans's Transportation Problem***)
The *(classical) Hitchcock-Koopmans's transportation problem* is defined as follows:

$$Minimise\ z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to

$$\sum_{i=1}^{m} x_{ij} = a_i, \quad i = 1,2,...,m$$
$$\sum_{j=1}^{n} x_{ij} = b_j, \quad j = 1,2,...,n$$

$$x_{ij} \geq 0, \quad i = 1,2,...,m; j = 1,2,...,n.$$

Here are:

x_{ij} : the amount of goods moved from origin i to destination j
c_{ij} : the cost of moving a unit amount of goods from origin i to destination j
a_i : the supply available at origin i
b_j : the demand for goods at destination j
m : total number of origins (sources)
n : total number of destinations (sinks).

The classical transportation is called *closed* (*or balanced*)if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j .$$

It is called *open* (or *unbalanced*)if

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j .$$

Ex. 7. 1.
A certain commodity is to be transported from the sources S_1, S_2 and S_3 to the destinations D_1, D_2, D_3 and D_4. The following table shows the availabilities of the commodity, the requirements of the destinations and the costs per unit for the transportation of the commodity from the three sources to the four destinations:

Destination $\rightarrow$ Sources $\downarrow$	D_1	D_2	D_3	D_4	Availabilities
S_1	5	2	4	3	30
S_2	6	4	9	5	40
S_3	2	3	8	1	55
Requirements $\rightarrow$	15	20	40	50	

Determine a transportation plan that minimises the total transportation cost.

Th. 7. 1.
The classical transportation problem has always a feasible solution.

Proof:
Let

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j := a.$$

We show that

$$x_{ij} = \frac{a_i b_j}{a}, \quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,n,$$

is a feasible solution of the classical transportation problem:

$$x_{ij} = \frac{a_i b_j}{a} \geq 0, \quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,n.$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{a} = \frac{a_i \sum_{j=1}^{n} b_j}{a} = a_i, \quad i = 1,2,\ldots,m.$$

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{a} = \frac{b_j \sum_{j=1}^{n} a_i}{a} = b_j, \quad j = 1,2,\ldots,n.$$

Th. 7. 2
The set of solution of the classical transportation problem is unbounded.

Proof:

$$\begin{cases} \sum_{j=1}^{n} x_{ij} = a_i < \infty \\ \sum_{i=1}^{m} x_{ij} = b_j < \infty \quad \Rightarrow \quad 0 \le x_{ij} < \infty \\ x_{ij} \ge 0, \ \forall \text{i,j} \end{cases}$$

C. 7. 1.
The classical transportation problem has always (at least) an optimal solution.

Th. 7. 3.
In the system of constraints of the classical transportation problem at least one equation is dependent on the remaining constraints.

Proof (Only for $m = 2, \ n = 3$*)*

x_{11}	x_{12}	x_{13}	a_1
x_{21}	x_{22}	x_{23}	a_2
b_1	b_2	b_3	

$$\begin{array}{lllllll} \text{(i)} & x_{11} + x_{12} + x_{13} & & & & & = a_1 \\ \text{(ii)} & & & x_{21} + x_{22} + x_{23} & & & = a_2 \\ \text{(iii)} & x_{11} & & + x_{21} & & & = b_1 \\ \text{(iv)} & & x_{12} & & + x_{22} & & = b_2 \\ \text{(v)} & & & x_{13} & & + x_{23} & = b_3 \end{array}$$

Subtracting equation (ii) from the sum of the equation3 (iii), (iv) and (v), we obtain equation (i).

C. 7. 2.

$$r(A) \le m + n - 1,$$

$r(A)$: Rank of the coefficient matrix A.

R. 7. 1. (***Illustration of C. 2. 7.***)

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$r(A) \leq 2 + 3 - 1 = 4.$$

C. 7. 3.
A feasible basic solution of the classical transportation has at most $m + n - 1$ positive components.

D. 7. 3. (***Degenerate Solution***)
A feasible basic solution (of the classical transportation problem) with less than $m + n - 1$ positive components is called a *degenerate solution*.

Th. 7. 4.
Assuming $a_i \geq 0,\ i = 1,2,...,m;\ b_j \geq 0,\ j = 1,2,...,n,$ to be integer, then every feasible basic solution of the classical transportation problem will also be integer.

R. 7. 2.
Because of the above results the transportation problem is considered as a *special case* of linear optimization:

R. 7. 3.
The transportation algorithm comprises two phases:
In *Phase 1* a basic feasible solution will be found. In *Phase 2,* the basic feasible solution will be successively improved until an optimal solution has been found.

Alg. 7. 1. (***Transportation Algorithm***)

Phase 1:

Use one of the following three methods:

- Northwest corner method
- The minimum cell cost method
- Vogel's approximation method (VAM)

A. Northwest Corner

S1: Assign largest possible allocation to the cell in the upper-left-hand corner of the tableau.
S2: Repeat Step 1 until all allocations have been assigned.
S3: Stop. A basic feasible solution has been reached.

B. *Minimum Cell Cost*

S1: Find a cell that has the least cost
S2: Allocate as much as possible to this cell.
S3: Block those cells that cannot be allocated to
S4: Repeat above stels until all allocations have been assigned.

C. *Vogel's Approximation Method (VAM)*

S1: For each column and row, determine its penalty cost by subtraction their two of least cost
S2: Select row/column that has the highest penalty cost in Step 1.
S3: Allocate as much as possible to the selected row/column that has the least cost.
S4: Block those cells that cannot be further allocated to.
S5: Repeat above steps until all allocations have been assigned.

<u>Phase 2</u>:

S1: Set up the equation

$$u_i + v_j = c_{ij}$$

for basic variables (occupied cells).

S2: Solve the equation by assigning a value (preferably 0) to one of the variables u_i and v_j

S3: Calculate

$$c_{ij}^{'} = u_i + v_j$$

for nonbasic variables (empty cells).

S4: Compute the opportunity cost for each empty cell:

$$d_{ij} = c_{ij} - c_{ij}^{'} .$$

S5: If

$d_{ij} \geq 0$ for all empty sets, the given solution is optimal; otherwise continue.

S6: Select a cell with the smallest negative opportunity cost as the cell to be included in the next solution.

S7: Draw a *closed loop* for the unoccupied cell selected in *S6*. Note that the right angle turn in the loop is permitted only at occupied cells and at the original unoccupied cell.
(A closed loop is a sequence of cells in the transportation problem such that

1. each pair of consecutive cells lies in either the same row or the same column
2. no three consecutive cells lie in the same row or column
3. the first and last cells of a sequence lie in the same row or column
4. no cell appears more than once in the sequence
5. The first cell is unoccupied and all the others are occupied.

S8: Assign alternate plus and minus signs at the unoccupied cells on the corner points of the loop aith the plus sign at the cell being evaluated.

S9: Determine the maximum number of units that can be shipped to this unocupied cell. The smallest valte with a negative position can be shipped to the entering cell. Now, add this quantity to the cells on the corner points of the loop marked with plus signs and subtract it from those cells marked with minus sign.
In this way an unoccupeis cell becomes a occupied cell.

S10: Repeat the whole procedure until an optimal solution is obtained.

Ex. 7. 2.
Consider the transportation problem *in Ex. 7. 1.:*

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities
S_1	5	2	4	3	30
S_2	6	4	9	5	40
S_3	2	3	8	1	55
Requirements →	15	20	40	50	

Find an initial solution by applying the following methods:

1. Northwest corner
2. Minimum cell cost
3. VAM

Solution:

1.

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities
S_1	5 **15**	2 **15**	4	3	30
S_2	6	4 **5**	9 **35**	5	40
S_3	2	3	8 **5**	1 **50**	55
Requirements →	15	20	40	50	125

$z_0 = 530$

2.

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities
S_1	5	2 **20**	4 **10**	3	30
S_2	6 **10**	4	9 **30**	5	40
S_3	2 **5**	3	8	1 **50**	55
Requirements →	15	20	40	50	125

$z_0 = 470$

3.

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availabilities	d_i^r
S_1	5	2	4 **30**	3	30	1
S_2	6	4 **20**	9 **10**	5 **10**	40	1
S_3	2 **15**	3	8	1 **40**	55	1, 2
Requirements →	15	20	40	50	125	
d_j^c	3, 4	1	4, 1	2, 4		

$z_0 = 410$

Ex. 7. 3.
A transportation company ships truckloads of grain from 4 silos $S_i,\ i=1,...,4$, to 5 mills $M_j,\ j=1,...,5$. The supply and the demand together with the transportation costs per truckload on the different routes are summarised in the following table:

	M_1	M_2	M_3	M_4	M_5	Supply
S_1	6	2	8	7	5	40
S_2	4	3	7	5	9	70
S_3	2	1	3	6	4	60
S_4	5	6	4	8	3	30
Demand	30	60	50	40	20	200

The company seeks the minimum-cost shipping schedule between the silos and the mills.

Solution:
We use VAM to obtain an initial solution:

0. Iteration

	M_1	M_2	M_3	M_4	M_5	Supply	d_i^r
S_1	6	2 **40**	8	7	5	40	3
S_2	4	3 **20**	7	5 **40**	9 **10**	70	1, 2
S_3	2 **30**	1	3 **20**	6	4 **10**	60	1, 2, 1
S_4	5	6	4 **30**	8	3	30	1
Demand	30	60	50	40	20	200	
d_j^c	2	1, 2	1	1	1		

$z_0 = 700$

	M_1	M_2	M_3	M_4	M_5	Supply	u_i
S_1	6 6	− 2 **40**	8 7	7 4	+ 5 8	40	-1
S_2	4 7	+ 3 **20**	7 8	5 **40**	− 9 **10**	70	0
S_3	2 **30**	1 -2	3 **20**	6 0	4 **10**	60	-5
S_4	5 3	6 -1	4 **30**	8 1	3 5	30	-4
Demand	30	60	50	40	20	200	
v_j	7	3	8	5	9		

	M_1	M_2	M_3	M_4	M_5	Supply	u_i
S_1	6 3	2 **30**	8 4	7 4	5 **10**	40	1
S_2	4 4	3 **30**	7 5	5 **40**	9 6	70	2
S_3	2 **30**	1 1	+ 3 **20**	6 3	− 4 **10**	60	0
S_4	5 3	6 2	− 4 **30**	8 4	+ 3 5	30	1
Demand	30	60	50	40	20	200	
v_j	1	3	3	4	9		

$z_1 = 680$

	M_1	M_2	M_3	M_4	M_5	Supply	u_i
S_1	6 5	+ 2 **30**	8 6	7 4	− 5 **10**	40	0
S_2	+ 4 6	− 3 **30**	7 7	5 **40**	9 6	70	1
S_3	+ 2 **30**	1 -1	+ 3 **30**	6 1	4 2	60	-3
S_4	5 3	6 0	− 4 **20**	8 2	+ 3 **10**	30	-2
Demand	30	60	50	40	20	200	
v_j	5	2	6	4	5		

$z_2 = 660$

	M_1	M_2	M_3	M_4	M_5	Supply	u_i
S_1	6 5	2 **40**	8 4	7 4	5 3	40	-1
S_2	4 **10**	3 **20**	7 5	5 **40**	9 4	70	0
S_3	2 **20**	1 3	3 **40**	6 3	4 2	60	-2
S_4	5 3	6 4	4 **10**	8 4	3 **20**	30	-1
Demand	30	60	50	40	20	200	
v_j	4	3	5	5	4		

$z_2 = 640$

We have now reached the optimal solution:

$x_{12}^* = 40,\ x_{21}^* = 10,\ x_{22}^* = 20,\ x_{24}^* = 40,\ x_{31}^* = 20,\ x_{32}^* = 40,\ x_{43}^* = 10, x_{45}^* = 20$

All other components are equal to zero. $z^* = 640$.

<u>Ex. 7. 4.</u> (*A Degenerate Problem)*

Three factories F_1, F_2 and F_3 supply four dealers D_1, D_2, D_3 and D_4 with a certain product. The available informations are given in the following table:

Factory	Dealer				Supply
	D_1	D_2	D_3	D_4	
F_1	2	2	2	4	1000
F_2	4	6	4	3	700
F_3	3	2	1	0	900
Demand	900	800	500	400	2600

1. Use the minimum cell cost method as initial solution.
2. Find an optimal transportation plan.

Solution:

1.

Dealer → Factory ↓	D_1	D_2	D_3	D_4	Supply
F_1	[2] **900**	[2] **100**	[2]	[4]	1000
F_2	[4]	[6] **700**	[4]	[3]	700
F_3	[3]	[2]	[1] **500**	[0] **400**	900
Demand	900	800	500	400	2600

$z_0 = 6700$

2.

The number of positive basic variables is $5 < m + n - 1 = 3 + 4 - 1$. Therefore, the initial basic feasible solution is degenerate.

To resolve degeneracy, we make use of an arbitrarily small quantity $\varepsilon > 0$ (practically $\varepsilon = 0$).

Dealer → Factory ↓	D_1	D_2	D_3	D_4	Supply	u_i
F_1	[2] - **900**	[2] + **100**	[2] 1	[4] 0	1000	0
F_2	[4] + 6	[6] - **700**	[4] 5	[3] 4	700	4
F_3	[3] 2	[2] **0**	[1] **500**	[0] **400**	900	0
Demand	900	800	500	400	2600	
v_j	2	2	1	0		

Dealer → Factory ↓	D_1	D_2	D_3	D_4	Supply	u_i
F_1	[2] **200**	[2] **800**	[2] 1	[4] 0	1000	0
F_2	[4] **700**	[6] 4	[4] 3	[3] 2	700	2
F_3	[3] 2	[2] **0**	[1] **500**	[0] **400**	900	0
Demand	900	800	500	400	2600	
v_j	2	2	1	0		

$z_1 = z^* = 5300.$

Ex. 7. 5. (***Multiple Solutions***)

A concrete company transports concrete from three plants P_1, P_2 and P_3 to four construction sites S_1, S_2, S_3 and S_4. The following table shows the necessary data:

Plant	Construction Sites				Supply
	S_1	S_2	S_3	S_4	
P_1	5	3	6	2	19
P_2	4	7	9	1	37
P_3	3	4	7	5	34
Demand	16	18	31	25	90

Find an optimal transportation programme.

Solution:

We use Vogel's approximation method to find an initial solution:

Site → Plant ↓	S_1	S_2	S_3	S_4	Supply	d_i^r
P_1	5	3 **18**	6 **1**	2	19	1, 2, 1
P_2	4 **12**	7	9	1 **25**	37	3, 5
P_3	3 **4**	4	7 **30**	5	34	1, 4
Demand	16	18	31	25	90	
d_c^c	1	1	1	1		

Site → Plant ↓	S_1	S_2	S_3	S_4	Supply	u_i
P_1	5 2	- 3 **18**	+ 6 **1**	2 -1	19	0
P_2	4 **12**	7 5	9 8	1 **25**	37	2
P_3	3 **4**	4 +4	7 - **30**	5 0	34	1
Demand	16	18	31	25	90	
v_j	2	3	6	-1		

$z_0 = 355 = z^*$.

Because of $c_{32} = c'_{32}$ we have a multiple optimal solution

Site→ Plant ↓	S_1	S_2	S_3	S_4	Supply	u_i
P_1	5 2	3 3	6 **19**	2 -2	19	-1
P_2	4 **12**	7 3	9 8	1 **25**	37	1
P_3	3 **4**	4 **18**	7 **12**	5 -1	34	0
Demand	16	18	31	25	90	
v_j	3	4	7	-1		

$z_1 = 355 = z^*$

$$X^* = \alpha \begin{pmatrix} 0 & 18 & 1 & 0 \\ 12 & 0 & 0 & 25 \\ 4 & 0 & 30 & 0 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0 & 0 & 19 & 0 \\ 12 & 0 & 0 & 25 \\ 4 & 18 & 12 & 0 \end{pmatrix}$$

$$0 \le \alpha \le 1$$

R. 7. 3. (***Prohibited Routes***)
Sometimes one or more of the routes in a transportation problem are *prohibited*. That is, units cannot be transported from a particular source to s particular destination. When this situation occurs, we must make sure that no units in the optimal solution are allocated to the cell representing this route. For that purpose a value of $M > 0$, arbitrarily large, is assigned as transportation cost for such a cell.

Ex. 7. 6.
The following table shows the supply of the canneries C_1, C_2, C_3 and C_4 to the factories $F_1, F_2, ..., F_6$ together with the corresponding unit transportation costs. The route from C_2 to F_5 is for the time being due to construction works closed:

	F_1	F_2	F_3	F_4	F_5	F_5	Supply
C_1	7	5	7	7	5	3	60
C_2	9	11	6	11		5	20
C_3	11	10	6	2	2	8	90
C_4	9	10	9	6	9	12	50
Demand	60	20	40	20	40	40	220

Determine an optimal transportation plan.

Solution:
We find an initial solution by applying Vogel's approximation method:

	F_1	F_2	F_3	F_4	F_5	F_6	Supply	d_i^r
C_1	[7] **20**	[5]	[7]	[7]	[5]	[3] **40**	60	2, 4,0
C_2	[9] **10**	[11]	[6] **10**	[11]	[**M**]	[5]	20	1, 3
C_3	[11]	[10]	[6] **30**	[2] **20**	[2] **40**	[8]	90	0, 4, 2, 5
S_4	[9] **30**	[10] **20**	[9]	[6]	[9]	[12]	50	3, 0, 3,0
Demand	60	20	40	20	40	40	220	
d_j^c	2	5	0, 1	4, 5	3, 4	2, 7		

$z_0 = 1180$

	F_1	F_2	F_3	F_4	F_5	F_6	Supply	u_i
C_1	- [7] **20**	+ [5] 8	[7] 4	[7] 0	[5] 0	[3] **40**	60	-2
C_2	[9] **10**	[11] 10	[6] **10**	[11] 2	[**M**] 2	[5] 5	20	0
C_3	[11] 9	[10] 10	[6] **30**	[2] **20**	[2] **40**	[8] 5	90	0
S_4	+ [9] **30**	- [10] **20**	[9] 6	[6] 2	[9] 2	[12] 5	50	0
Demand	60	20	40	20	40	40	220	
v_j	9	10	6	2	2	5		

	F_1	F_2	F_3	F_4	F_5	F_6	Supply	u_i
C_1	[7] **0**	[5] **20**	[7] 4	[7] 0	[5] 0	[3] **40**	60	-2
C_2	[9] **10**	[11] 7	[6] **10**	[11] 2	[**M**] 2	[5] 5	20	0
C_3	[11] 9	[10] 7	[6] **30**	[2] **20**	[2] **40**	[8] 5	90	0
S_4	[9] **50**	[10] 7	[9] 6	[6] 2	[9] 2	[12] 5	50	0
Demand	60	20	40	20	40	40	220	
v_j	9	7	6	2	2	5		

$z_1 = 1120$

Because of $c_{26} = c_{26}^{'} = 5$ there are multiple solutions:

	F_1	F_2	F_3	F_4	F_5	F_6	Supply	u_i
C_1	7 + **0**	5 **20**	7 4	7 0	5 0	3 - **40**	60	-2
C_2	9 - **10**	11 7	6 **10**	11 2	**M** 2	5 + 5	20	0
C_3	11 9	10 7	6 **30**	2 **20**	2 **40**	8 5	90	0
S_4	9 **50**	10 7	9 6	6 2	9 2	12 5	50	0
Demand	60	20	40	20	40	40	220	
v_j	9	7	6	2	2	5		

	F_1	F_2	F_3	F_4	F_5	F_6	Supply
C_1	7 **10**	5 **20**	7	7	5	3 **30**	60
C_2	9	11	6 **10**	11	**M**	5 **10**	20
C_3	11	10	6 **30**	2 **20**	2 **40**	8	90
S_4	9 **50**	10	9	6	9	12	50
Demand	60	20	40	20	40	40	220

$$X^* = \alpha \begin{pmatrix} 0 & 20 & 0 & 0 & 0 & 40 \\ 10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 30 & 20 & 40 & 0 \\ 50 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 10 & 20 & 0 & 0 & 0 & 30 \\ 0 & 0 & 10 & 0 & 0 & 10 \\ 0 & 0 & 30 & 20 & 40 & 0 \\ 50 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$z^* = 1120$$

$$0 \le \alpha \le 1$$

<u>Ex. 7. 7.</u> (*Unbalanced Transportation Problem*)
A company has three factories F_1 and F_2 which supply warehouses W_1, W_2 and W_3 with a certain product. Monthly factory capacities, monthly warehouse demands and unit transportation cost are given in the following table:

Factory	W_1	W_2	W_3	Supply
F_1	28	17	26	500
F_2	19	12	16	300
Demand	250	250	500	

Find an optimal solution to the problem.

Solution:
Since the demand is higher than the supply, a dummy factory F_d will be introduced:

Factory	W_1	W_2	W_3	Supply
F_1	28	17	26	500
F_2	19	12	16	300
F_d	0	0	0	200
Demand	250	250	500	1000

We use VAM to obtain an initial solution:

0. Iteration

	W_1	W_2	W_3	Supply	d_i^r
F_1	28 **50**	17 **250**	26 **200**	500	9, 2
F_2	19	12	16 **300**	300	4, 3
F_d	0 **200**	0	0	200	0
Demand	250	250	500		
d_j^c	19, 9	12, 5	16, 8, 10		

$z_0 = 15650$

	W_1	W_2	W_3	Supply	u_i
F_1	28 **50**	17 **250**	26 **200**	500	0
F_2	19 18	12 **7**	16 **300**	300	-10
F_d	0 **200**	0 -9	0 -2	200	-28
Demand	250	250	500		
v_j	28	17	26		

$z_0 = 15650 = z^*$.

Chapter 7

Linear Optimization (Transportation Problem)

Exercises

7. 1.

There are three warehouses W_1, W_2 and W_3. They have 250, 130 and 235 tons of paper accordingly. There are four publishers P_1, P_2, P_3 and P_4.. They ordered 75, 230, 240 and 70 tons of paper to publish new books.
There are the following costs in dollars of transportation of one ton of paper:

From \ T0	P_1	P_2	P_3	P_4
W_1	15	20	16	21
W_2	25	13	5	11
W_3	15	15	7	17

7. 2.

The following table shows the supply and demand of a certain good from the producers P_1, P_2 and P_3 to the consumers C_1, C_2 and C_3 as well as the unit transportation costs for each route:

From \ To	C_1	C_2	C_3	Supply
P_1	32	60	200	20
P_2	40	68	80	30
P_3	120	104	60	45
Demand	30	35	30	95

1. Find a transportation plan with minimum transportation costs.
2. The route $P_3 \to C_3$ is going to be closed due to some necessary construction works. How should the previous optimal solution be adapted to the new situation?

Chapter 7

Linear Optimization (Transportation Problem)

Solution

7. 1.
We use the northwest corner method to find an initial solution:

To → From ↓	P_1	P_2	P_3	P_4	Supply
W_1	15 **75**	20 **175**	16	21	250
W_2	25	13 **55**	5 **75**	11	130
W_3	15	15	7 **165**	17 **70**	235
Demand	75	230	240	70	615

$z_0 = 6940$

To → From ↓	P_1	P_2	P_3	P_4	Supply	u_i
W_1	15 **75**	20 **175**	16 12	21 22	250	0
W_2	25 8	13 **55**	- 5 **75**	+ 11 15	130	-7
W_3	15 10	15 15	+ 7 **165**	- 17 **70**	235	-5
Demand	75	230	240	70	615	
v_j	15	20	12	22		

To → From ↓	P_1	P_2	P_3	P_4	Supply	u_i
W_1	15 **75**	20 **175**	16 12	21 18	250	7
W_2	25 8	13 **55**	5 **5**	11 **70**	130	0
W_3	15 10	15 15	7 **235**	17 13	235	2
Demand	75	230	240	70	615	
v_j	8	13	5	11		

To → From ↓	P_1	P_2	P_3	P_4	Supply	u_i
W_1	15 **75**	20 **175**	16 12	21 18	250	7
W_2	25 8	13 − **55**	5 + **5**	11 **70**	130	0
W_3	15 10	15 15 +	7 − **235**	17 13	235	2
Demand	75	230	240	70	615	
v_j	8	13	5	11		

To → From ↓	P_1	P_2	P_3	P_4	Supply
W_1	15 **75**	20 **175**	16	21	250
W_2	25	13	5 **60**	11 **70**	130
W_3	15	15 **55**	7 **180**	17	235
Demand	75	230	240	70	615

$z^* = 7780$

$$X^* = \alpha \begin{pmatrix} 75 & 175 & 0 & 0 \\ 0 & 55 & 75 & 0 \\ 0 & 0 & 165 & 70 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 75 & 175 & 0 & 0 \\ 0 & 0 & 60 & 70 \\ 0 & 55 & 180 & 0 \end{pmatrix}, \qquad 0 \le \alpha \le 1$$

7. 2.

1.

We use Vogel's method to find an initial solution:

To→ From ↓	C_1	C_2	C_3	Supply	d_i^r
P_1	32 **20**	60	200	20	28
P_2	40 **10**	68 **20**	80	30	28
P_3	120	104 **15**	60 **30**	45	97
Demand	30	35	30		
d_j^c	8	8	73		

$z_0 = 4170$

2.

To → From ↓	C_1	C_2	C_3	Supply	u_i
P_1	32 **20**	60 60	200 M-44	20	-8
P_2	40 **10**	68 - **20**	80 + M-36	30	0
P_3	120 76	104 + **15**	M - **30**	45	36
Demand	30	35	30		
v_j	40	68	M-36		

To→ From ↓	C_1	C_2	C_3	Supply	u_i
P_1	32 **20**	60 180-M	200 72	20	-8
P_2	40 - **10**	68 184-M	80 + **20**	30	0
P_3	120 + M-40	104 **35**	M - **10**	45	M-80
Demand	30	35	30		
v_j	40	184-M	80		

To→ From ↓	C_1	C_2	C_3	Supply	u_i
P_1	32 **20**	60 16	200 72	20	-88
P_2	40 **0**	68 24	80 **30**	30	-80
P_3	120 **10**	104 **35**	M 160	45	0
Demand	30	35	30		
v_j	120	104	160		

$$X^* = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 0 & 30 \\ 10 & 35 & 0 \end{pmatrix}, \qquad z^* = 7880$$

Chapter 8

Project Management

CPM/PERT

R. 8. 1.
The CPM and PERT are widely used techniques for the purpose of planning, scheduling and controlling the process and completion of large and complex projects.

R. 8. 2 (***Critical Path Method - CPM***)
The *Critical Path Method* (*CPM*) is one of several related techniques for doing project planning. It is for projects that are made up of individual "*activities*". If some of the activities require other activities to finish before they can start, then the project becomes a complex web of activities.
CPM was developed by Du Pont and the emphasis was on the trade-off between the cost of the project and its overall completion time.
CPM and PERT (see later) were independently developed in the late 1950s. They have been among the most widely used OR techniques. These methods have been used for a variety of projects, including the following types:

1. Construction of a new plant
2. Research and development of a new product
3. NASA space exploration projects
4. Movie productions
5. Building a ship
6. Government-sponsored projects for developing a new weapon system
7. Relocation of a major facility
8. Maintenance of a nuclear reactor
9. Installation of a management of information system
10. Conducting an advertising campaign.

CPM is mainly used for the jobs of repetitive nature where the activity time estimates can be predicted with considerable certainty due to the existence of past experience.

CPM can help us answer following questions:

- How long a project will take to complete?
- Are the activities on schedule?
- Is the project within budget?
- Which activities are *critical*, meaning that they have to be done on time or else the whole project will take longer?
- How can the project be finished early at the least cost?

<u>ALG. 8. 1</u> (*CPM*)

Step 1 (List the activities):

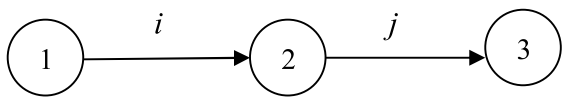
Set up a table giving following informations:

- The name of activities and their description
- The required predecessors
- The duration of each activity
-

Step 2 (Draw the Diagram):

There are two types of formats:

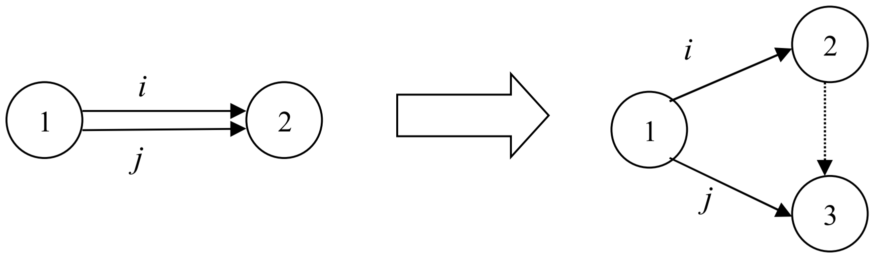
1. Activity-on-Arrow (AoA)
   In an AoA format, activities are displayed by means of arrows in the network. The nodes are events denoting the start and/or finish of a set of activities of the project. The technological links between activity i or activity j :



Since activities can be labeled with their corresponding start and end node event, it is said that activity (2, 3) is a *successor* of (1, 2) and activity (1, 2) is a *predecessor* of (2, 3).
The following rules have been suggested to construct AoA networks:

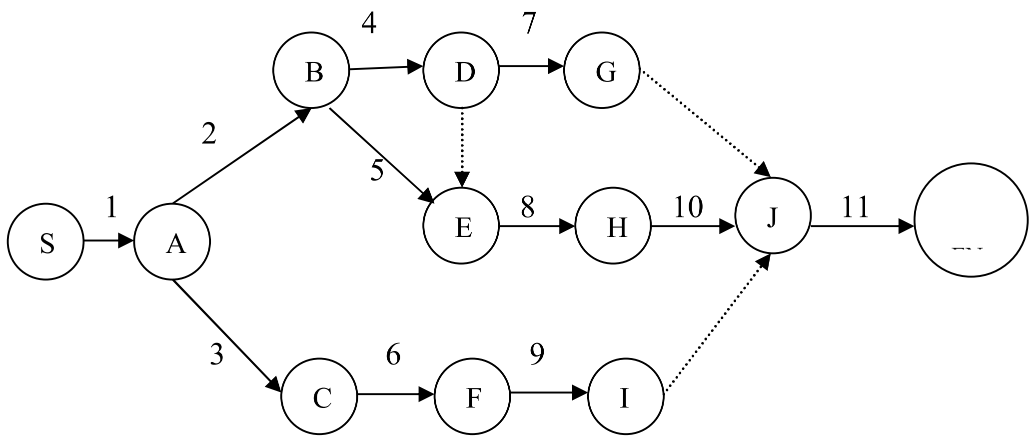
1. Before any activity may begin, all activities preceding it must be completed.
2. Arrows imply logical precedence only. Neither the length nor their “compass” direction has any difference.
3. Event numbers must not be duplicated in any network.
4. Any two events may be directly connected by no more than one activity.
5. Networks may have only one initial event (with no predecessor) and only one terminal event (with no successor)
6. The introduction of dummy activities is often necessary to model all precedence relations.

Dummies are introduced for the unique identification of activities and/or for displaying certain precedence relations. These activities are represented by dashed arrows in the network and do not consume time or resources.
The following figure displays an example project with a dummy arrow to identify all activities in a unique way. The network contains two activities that can be performed in parallel (i.e. there is no technological precedence relation between the two activities. Rule 4states that two events cannot be connected by more than one activity to ensure the unique identification of each activity (both i and j can be labeled as activity (1, 2)). Therefore, an extra dummy activity needs to be embedded in the project work, represented by the dashed arrows. In doing so, the network starts and ends with a single event node (rule 5) and each activity has been defined by a unique start/end event combination (rule 4):



Example:

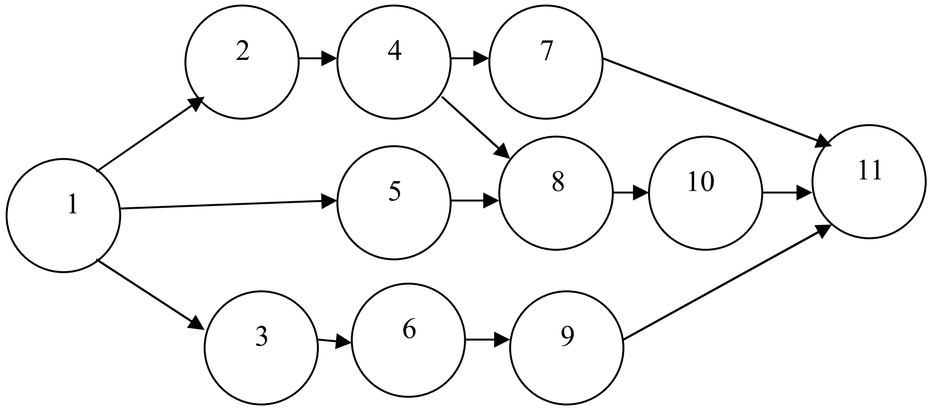
Activity	Predecessors
1	-
2	1
3	1
4	2
5	2
6	3
7	4
8	4, 5
9	6
10	8
11	7, 9, 10



2. Activity-on-Node (AoN)
   An AoN network displays the activities by nodes and precedence relations by arrows. Dummy activities are not necessary, apart from a single initial start and a single end activity, which makes an AoN network always unique.
   Following steps should be performed in order to construct an AoN network:

   1. Draw a node for each network activity.
   2. Draw an arrow for each immediate precedence relation between two activities.
   3. Possibly add a dummy start and dummy end node to force that the network begins with a single start activity and finish with a single end activity.

   Below, we have the AoN network of the above example:



Step 3 (Calculate the Critical Path):

The *critical path* determines the shortest time to complete the project. It is the longest duration path through a network of activities. *Critical activities* are activities on the critical path.

Denote by

$K := \{P_0, ..., P_n\}$

T_i^e : earliest possible point in time on which the activity $i \rightarrow j$ can start

T_j^e : earliest possible point in time on which the activity $i \rightarrow j$ can finish

T_i^l : latest possible point in time on which the activity $i \rightarrow j$ can start

T_j^l : latest possible point in time on which the activity $i \rightarrow j$ can finish

t_{ij} : duration of the activity $i \rightarrow j$.

Then we have:

$$T_0^e := 0$$

$$T_j^e = \max_i \left\{T_i^e + t_{ij}\right\}, \quad i < j;\ j = 1, 2, ..., n;\ (P_i, P_j) \in K$$

$$T_n^l := T_n^e$$

$$T_i^l = \min_j \left\{T_j^l - t_{ij}\right\}, \quad i < j;\ i = n-1, n-2, ..., 0,\ (P_i, P_j) \in K.$$

A path is *critical* if and only if all activities lying on it fulfil the following equation:

$$T_j^e - T_i^e - t_{ij} = 0.$$

Step 4 (Calculate the Float Times):

Slack time is the difference between the latest time and the earliest time of an event:

$$T_i^l - T_i^e$$

Positive slack is the amount of time an event can be delayed without delaying the project completion.

The most common types of floats are:

1. *Total float*
   It is the amount of time a single activity can be delayed without delaying project completion:

   $$\Delta^T t_{ij} = T_j^l - T_i^e - t_{ij}$$

2. *Free float*
   It is the amount of time a single activity can be delayed without delaying the early start of any successor activity:

   $$\Delta^F t_{ij} = T_j^e - T_i^e - t_{ij}$$

3. *Independent float*
   It is the amount of time that can be used without affecting either the predecessor or successor events. The independent float represents the amount of float time available for an activity when its preceding activities are completed at their latest and its succeeding activities are to begin at their earliest time:

   $$\Delta^I t_{ij} = \max\{0,\ T_j^e - T_i^l - t_{ij}\}$$

4. *Conditional slack*
   It is defined as:

   $$\begin{aligned}\Delta^C t_{ij} &:= \Delta^T t_{ij} - \Delta^F t_{ij} \\ &= T_j^l - T_j^e\end{aligned}$$

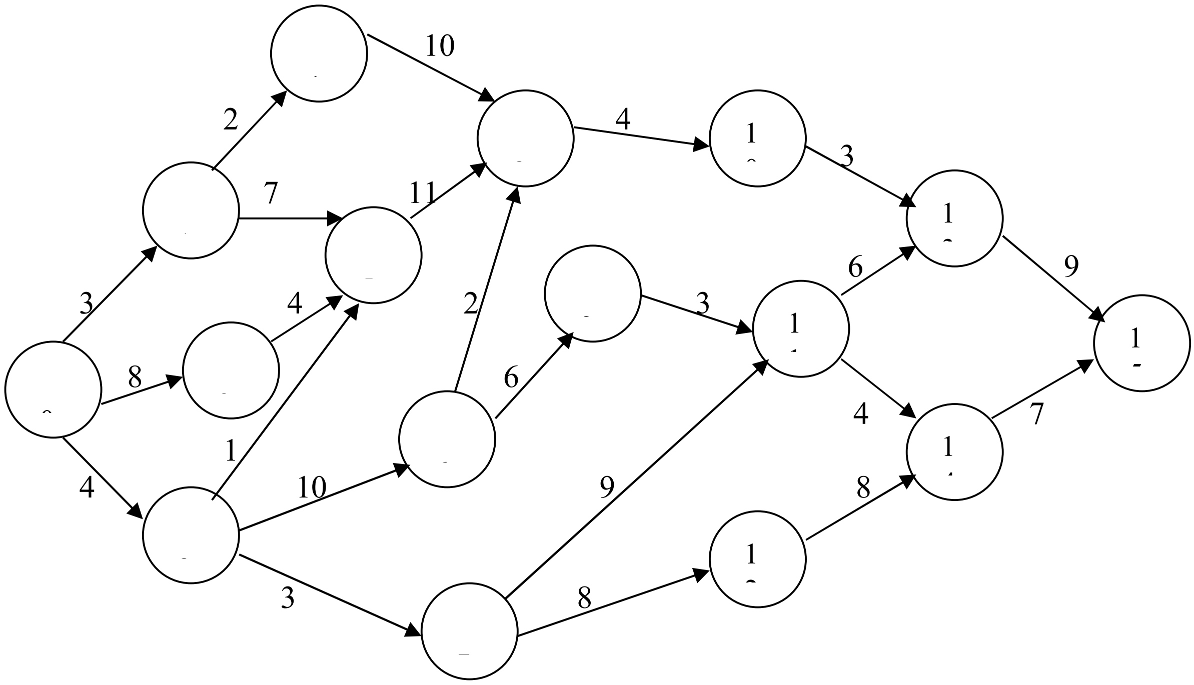
Slack time is the difference between the latest time and the earliest time of an event:

$$T_i^l - T_i^e$$

Positive slack is the amount of time an event can be delayed without delaying the project completion.

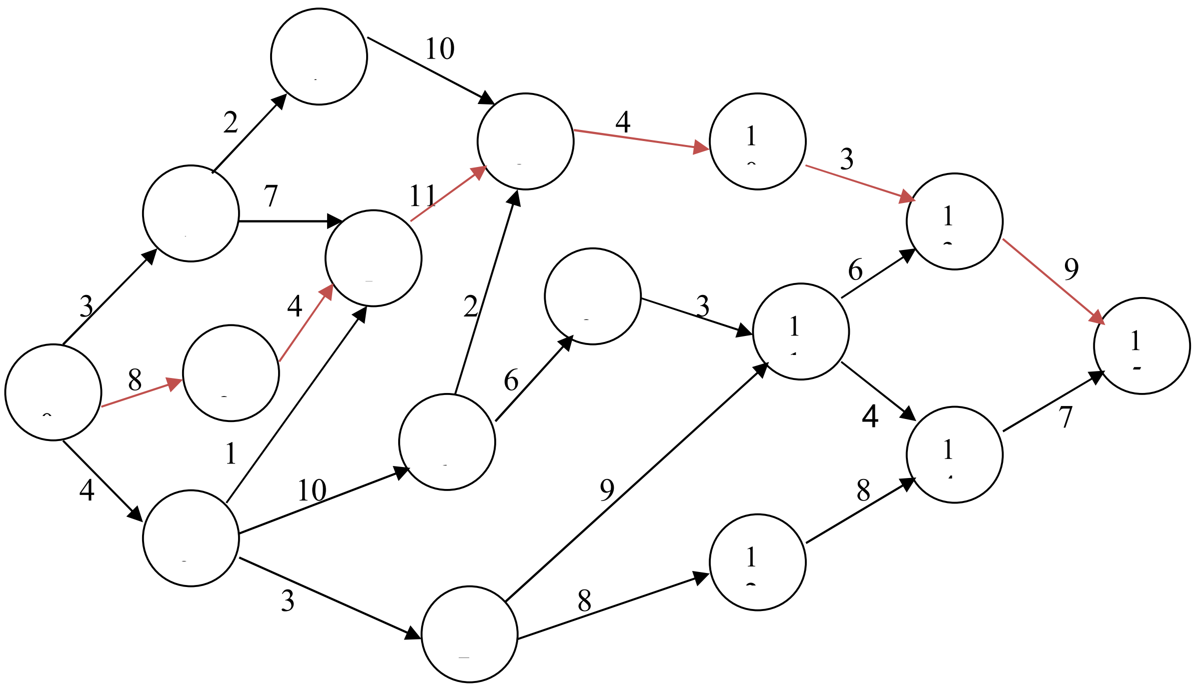
Ex. 8. 1.

Step 2:



Step 3:

T^e	Events	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0		3	8	4												
3	1					2	7										
8	2						4										
4	3						1	10	3								
5	4									10							
12	5									11							
14	6									2	6						
7	7												9	8			
23	8											4					
20	9												3				
27	10														3		
23	11														6	4	
15	12															8	
30	13																9
27	14																7
39	15																
T^l		0	5	8	5	13	12	15	15	23	21	27	24	24	30	32	39


10
2
4
1
3
7
11
1
3
4
2
3
6
9
8
6
1
4
1
7
1
4
10
9
8
1
3
8
1

Step 4:

i	j	t_{ij}	T_i^e	T_j^l	$T_i^e + t_{ij}$	$T_j^l - t_{ij}$	$\Delta^T t_{ij}$	$\Delta^F t_{ij}$	$\Delta^I t_{ij}$	$\Delta^C t_{ij}$
0	1	3	0	5	3	3	2	0	0	2
0	2	8	0	8	8	8	0	0	0	0
0	3	4	0	5	4	4	1	0	0	1
1	4	2	3	13	5	5	8	0	0	8
1	5	7	3	12	10	10	2	2	0	0
2	5	4	8	12	12	12	0	0	0	0
3	5	1	4	12	5	5	7	7	6	0
3	6	10	4	15	14	14	1	0	0	1
3	7	3	4	15	7	7	8	0	0	8
4	8	10	5	23	15	15	8	8	0	0
5	8	11	12	23	23	23	0	0	0	0
6	8	2	14	23	16	16	7	7	6	0
6	9	6	14	21	20	20	1	0	0	1
7	11	9	7	24	16	16	8	7	0	1
7	12	8	7	24	15	15	9	0	0	9
8	10	4	23	27	27	27	0	0	0	0
9	11	3	20	24	23	23	1	0	0	1
10	13	3	27	30	30	30	0	0	0	0
11	13	6	23	30	29	29	1	1	0	0
11	14	4	23	32	27	27	5	0	0	5
12	14	8	15	32	23	23	9	4	0	5
13	15	9	30	29	39	39	0	0	0	0
14	15	7	27	39	34	34	5	5	0	0

R. 8. 2 (***Program Evaluation Review Technique -PERT***)
In PERT activities are shown as a network of precedence relationships using AoA format

PERT was developed by the US Navy for the planning and control of the Polaris missile programme and the emphasis on completing the programme in the shortest possible time.

PERT is used mainly for non-repetitive jobs, where the time and cost estimates tend to be quite uncertain. This technique uses probabilistic time estimates.

<u>ALG. 8. 2</u> (*PERT*)

Step 1 (List the activities):

Set up a table giving following informations:

- The name of activities and their description
- The required predecessors
- Three duration estimates:

  i) *Optimistic*
  This is the shortest possible time in which the activity can be completed, denoted by $a_{ij},\ i, j = 1, 2, ..., n$.

  ii) *Most probable*
  This is the most likely time in which the activity can be completed under normal circumstances, denoted by $m_{ij},\ i, j = 1, 2, ..., n$.

  iii) *Pessimistic*
  This is the longest time the activity might need, denoted by $b_{ij},\ i, j = 1, 2, ..., n.$.

Step 2 (Draw the Diagram):

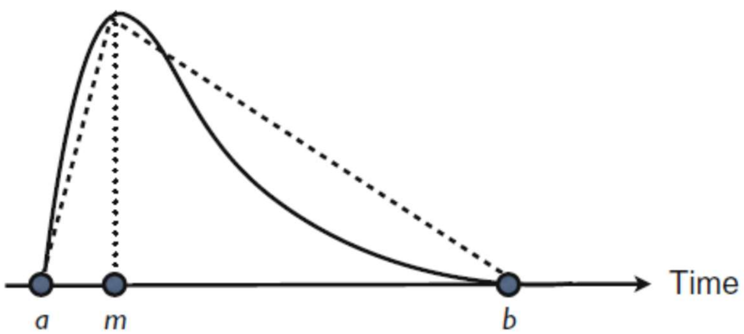
See ALG 8. 1.

Step 3 (Calculate the Expected Time $\bar{t}_{ij}$ and the Standard Deviation $\sigma_{t_{ij}}$):

$$\bar{t}_{ij} = \frac{a_{ij} + 4m_{ij} + b_{ij}}{6}, \quad i, j = 1, 2, ..., n;\ (P_i, P_j) \in K$$

$$\sigma_{t_{ij}} = \frac{b_{ij} - a_{ij}}{6}, \quad i, j = 1, 2, ..., n;\ (P_i, P_j) \in K$$

PERT assumes that each activity duration is a random variable between two extreme value (i.e. a_{ij} and b_{ij}) a follows a beta probability distribution. A typical beta distribution function and its triangular approximation looks as follows:



Step 4 (***Calculate the Critical Path***)

Denote by

$K := \{P_0, \ldots, P_n\}$

$\bar{T}_i^e$: earliest possible point in time on which the activity $i \to j$ can start

$\bar{T}_j^e$: earliest possible point in time on which the activity $i \to j$ can finish

$\sigma^2_{T_i^e}$: variance for the earliest possible point in time on which the activity $i \to j$ can start

$\sigma^2_{T_j^e}$: variance for the earliest possible point in time on which the activity $i \to j$ can finish

$\bar{T}_i^l$: latest possible point in time on which the activity $i \to j$ can start

$\bar{T}_j^l$: latest possible point in time on which the activity $i \to j$ can finish

$\sigma^2_{T_i^l}$: variance for the latest possible point on which the activity $i \to j$ can start

$\sigma^2_{T_j^l}$: variance for the latest possible point in time on which the activity $i \to j$ can finish

t_{ij} : duration of the activity $i \to j$

Then we have:

$$\bar{T}_0^e := 0,$$

$$\bar{T}_j^e = \max_i \{\bar{T}_i^e + \bar{t}_{ij}\},\ i < j;\ j = 1, 2, \ldots, n;\ (P_i, P_j) \in K,$$

$$\sigma^2_{T_0^e} := 0,$$

$$\sigma^2_{T_j^e} = \max_i \{\sigma^2_{T_i^e} + \sigma^2_{t_{ij}}\},\ i < j;\ j = 1, 2, \ldots, n;\ (P_i, P_j) \in K.$$

$$\bar{T}_n^l := \bar{T}_n^e,$$

$$\bar{T}_i^l = \min_j \{\bar{T}_j^l - \bar{t}_{ij}\},\ i < j;\ i = n-1,\ n-2, \ldots 0;\ (P_i, P_j) \in K.$$

$$\sigma^2_{T^l_n} := 0,$$

$$\sigma^2_{T^l_i} = \max_j \{\sigma^2_{T^l_j} + \sigma^2_{t_{ij}}\},\ i < j;\ i = n-1,\ n-2, \ldots 0;\ (P_i, P_j) \in K.$$

Step 5 (***Calculation of probabilities***)

$$P\left[(T^l_i - T^e_i) \le x\right] = P\left[x \le -\frac{T^l_i - T^e_i}{\sqrt{\sigma^2_{T^l_i} + \sigma^2_{T^e_i}}}\right]$$

$$x = \frac{(T^l_i - T^e_i) - (\bar{T}^l_i - \bar{T}^e_i)}{\sqrt{\sigma^2_{T^l_i} + \sigma^2_{T^e_i}}}$$

$$P\left[\left(T^l_i - T^e_i\right) \le x\right] = P\left[x \le -\frac{T^l_i - T^e_i}{\sqrt{\sigma^2_{T^l_i} + \sigma^2_{T^e_i}}}\right]$$

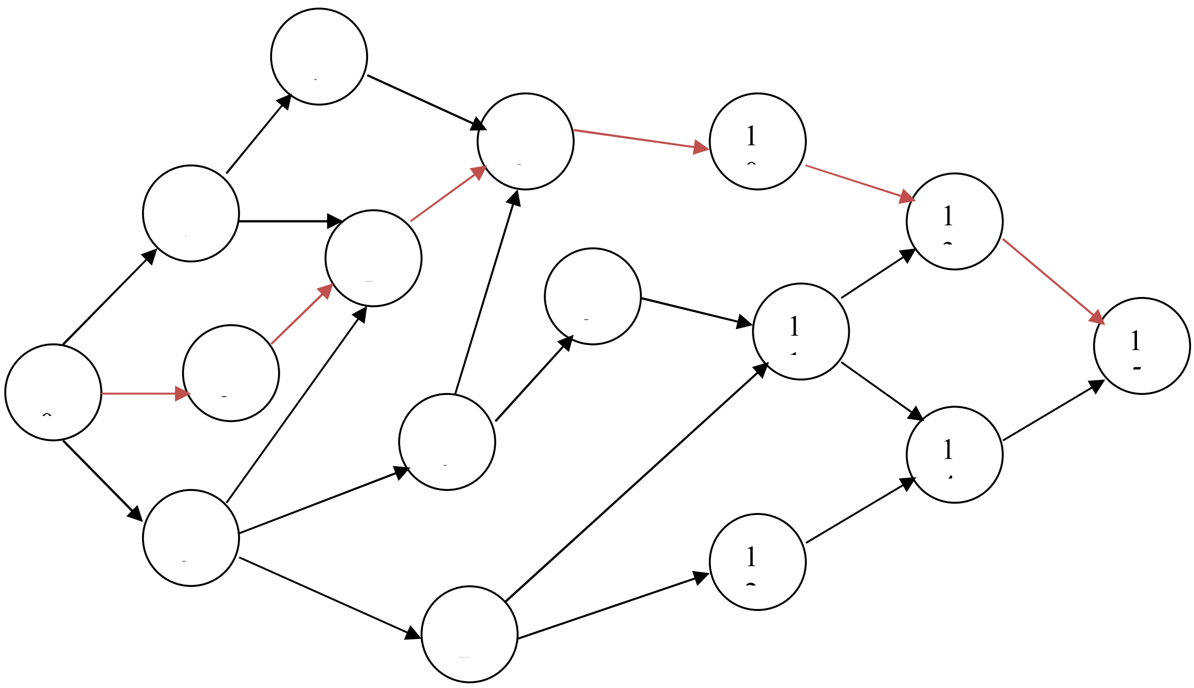
<u>Ex. 8. 1.</u> (*Continued*)

Steps 3:

i	j	a_{ij}	m_{ij}	b_{ij}	$\bar{t}_{ij}$	$\sigma_{t_{ij}}$	$\sigma^2_{t_{ij}}$
0	1	1	3	6	3.17	0.83	0.69
0	2	4	8	10	7.67	1.00	1.00
0	3	3	4	5	4.00	0.33	0.11
1	4	1	2	4	2.17	0.50	0.25
1	5	4	7	12	7.33	1.33	1.78
2	5	3	4	6	4.17	0.50	0.25
3	5	1	1	2	1.17	0.17	0.03
3	6	8	10	13	10.17	0.83	0.69
3	7	1	3	4	2.83	0.50	0.25
4	8	7	10	15	10.33	1.33	1.78
5	8	11	11	11	11.00	0.00	0.00
6	8	2	2	4	2.33	0.33	0.11
6	9	3	6	8	5.83	0.83	0.69
7	11	6	9	12	9.00	1.00	1.00
7	12	5	8	10	7.83	0.83	0.69
8	10	2	4	7	4.17	0.83	0.69
9	11	1	3	4	2.83	0.50	0.25
10	13	2	3	4	3.00	0.33	0.11
11	13	3	6	8	5.83	0.83	0.69
11	14	1	4	7	4.00	1.00	1.00
12	14	7	8	9	8.00	0.33	0.11
13	15	6	9	10	8.67	0.67	0.44
14	15	5	7	10	7.17	0.83	0.69

Steps 4:

$\sigma^2_{T_i^e}$	$\bar{T}_i^e$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.00	0.00	0		3.17 0.69	7.67 1.00	4.00 0.11												
0.69	3.17	1					2.17 0.25	7.33 1.78										
1.00	7.67	2						4.17 0.25										
0.11	4.00	3						1.17 0.03	10.17 0.69	2.83 0.25								
0.94	5.34	4									10.33 1.78							
2.47	11.84	5									11.00 0.00							
0.80	14.17	6									2.33 0.11	5.88 0.69						
0.36	6.83	7												9.00 1.00	7.83 0.69			
2.72	22.84	8											4.17 0.69					
1.49	20.00	9												2.83 0.25				
3.41	27.01	10														3.00 0.11		
1.74	22.83	11														5.83 0.69	4.00 1.00	
1.05	14.66	12															8.00 0.11	
3.52	30.01	13																8.67 0.44
2.74	26.83	14																7.17 0.69
3.96	38.68	15																
		$\bar{T}_j^l$	0.00	4.51	7.67	5.35	12.51	11.84	15.52	15.18	22.84	21.35	27.01	24.18	23.51	30.01	31.51	38.68
		$\sigma^2_{T_j^l}$	3.96	3.27	1.49	3.32	3.02	1.24	2.63	2.69	1.24	1.94	0.55	1.69	0.80	0.44	0.69	0.00



Step 5

i	$\bar{T}_i^e$	$\sigma^2_{T_i^e}$	$\bar{T}_i^l$	$\sigma^2_{T_i^l}$	$\bar{T}_i^l - \bar{T}_i^e$	$P\left[(T_i^l - T_i^e)\right] \leq 0$
0	0.00	0.00	0.00	3.96	0.00	0.50
1	3.17	0.69	4.51	3.27	1.34	0.25
2	7.67	1.00	7.67	1.49	0.00	0.50
3	4.00	0.11	5.35	3.32	1.35	0.24
4	5.34	0.94	12.51	3.02	7.17	0.00
5	11.84	2.47	11.84	1.24	0.00	0.50
6	14.17	0.80	15.52	2.63	1.35	0.24
7	6.83	0.36	15.18	2.09	8.35	0.00
8	22.84	2.72	22.84	1.24	0.00	0.50
9	20.00	1.49	21.35	1.94	1.35	0.24
10	27.01	3.41	27.01	0.55	0.00	0.50
11	22.83	1.74	24.18	1.69	1.35	0.24
12	14.66	1.05	23.51	0.80	8.85	0.00
13	30.01	3.52	30.01	0.44	0.00	0.50
14	26.83	2.74	31.51	0.69	4.68	0.01
15	38.68	3.96	38.68	0.00	0.00	0.50

Chapter 8

Project Management

Exercises

8. 1.
The owner of a shopping centre is considering modernising and expanding the current 32-business shopping complex. He hopes to add 8 to 10 new business or tenants to the shopping complex. The specific activities that make up the expansion project, together with information on immediate predecessor and completion time, are listed in the following table:

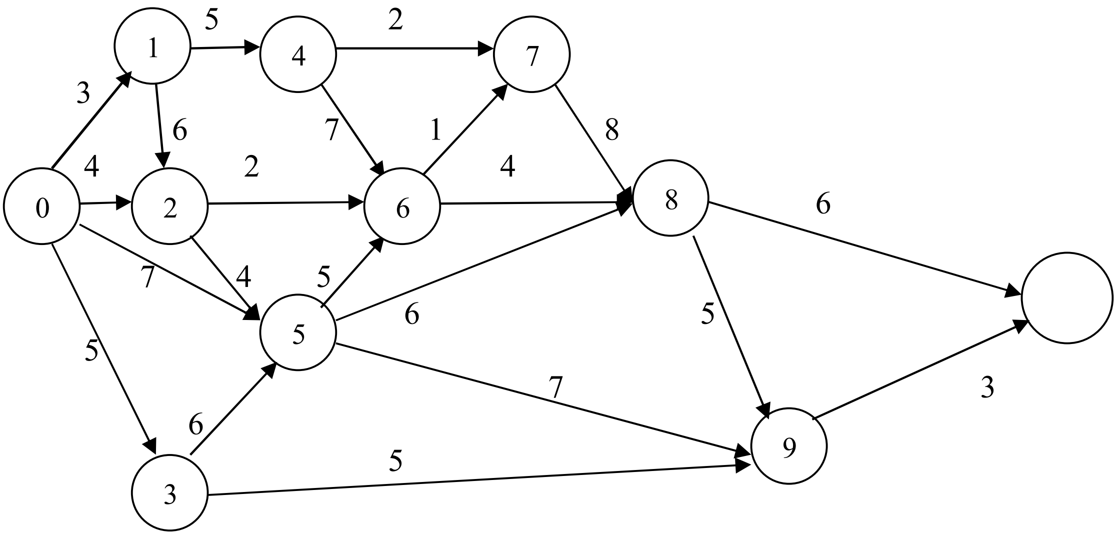
Activity	Activity Description	Immediate Predecessor	Completion Time (Weeks)
A	Prepare architectural drawings	-	5
B	Identify potential new tenants	-	6
C	Develop prospectus for tenants	A	4
D	Select contractor	A	3
E	Prepare building permits	A	1
F	Obtain approval for building permits	E	4
G	Perform construction	D, F	14
H	Finalise contracts with tenants	B, C	12
I	Tenants move in	G, H	2

Answer the following questions:

1. What is the total completion time?
2. What are the scheduled start and completion time for each activity?
3. Which activities are critical and must be completed exactly as scheduled in order to keep the project on schedule?
4. How long can the non-critical activities be delayed before they cause a delay in the completion time for the project?

8. 2

The following diagram represents a certain project:

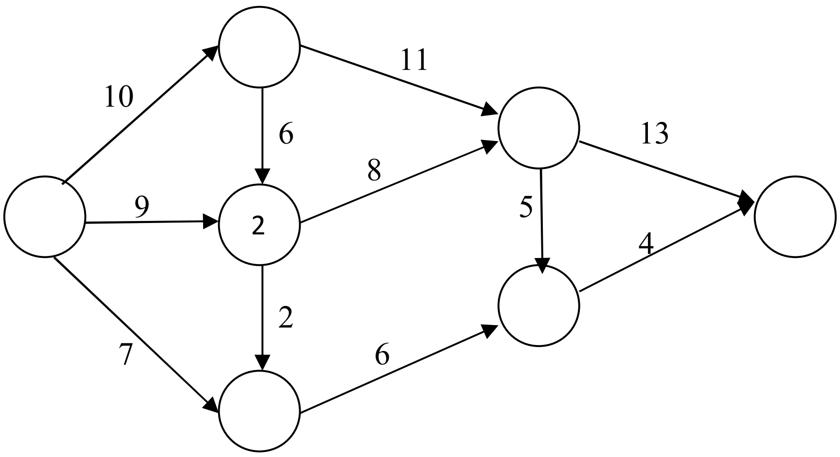


1. Determine the critical path.
2. Calculate the following floats.

   a. Total
   b. Free
   c. Conditional
   d. Independent

8. 3.

Consider the following nerwork diagram:



1. Find the critical path and the completion time.
2. Calculate all floats.

8. 4.

1. Construct a network diagram for a project consisting of the following activities:

Activity	Immediate Predecessor(s)
A	-
B	-
C	A, B
D	B
E	D
F	C, E
G	D

2. Suppose the project has the following activity times:

Activity	Time (days)
A	3
B	4
C	5
D	6
E	7
F	8
G	9

a) Find the critical path.
b) What is the project completion time?

3. Find and interpret the total floats (slacks) and free floats for the activities.

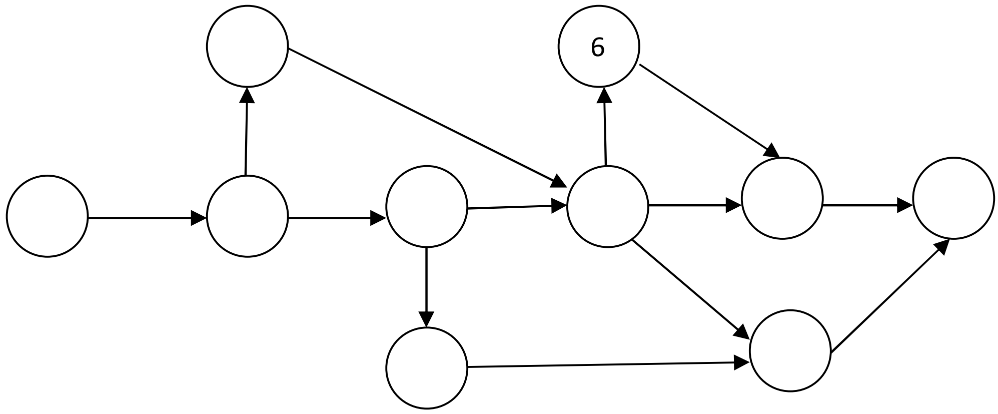
4. Suppose now the folowing estimates of activity times (days) are provided:

Activity	Optimistic	Most likely	Pessimistc
A	1	3	5
B	3	4	5
C	4	5	6
D	3	5	7
E	5	6	13
F	4	7	10
G	6	8	10

a) Determing the expected completion time and variance for the project.
b) What is the probability that the project will be completed within

i. 20 days?
ii. 25 days?

8. 5.
Consider a project with the following network diagram:



The following table lists the corresponding activities and precedence relationships:

i	j	a_{ij}	m_{ij}	b_{ij}
0	1	3	5	8
1	2	12	13	16
1	3	8	11	15
2	5	13	15	21
3	4	6	8	10
3	5	7	8	10
4	8	6	7	9
5	6	13	14	16
5	7	3	4	5
5	8	10	14	19
6	7	7	9	12
7	9	4	5	6
8	9	15	16	18

1. Determine the "critical path".
2. What is the probability that the project will be completed in less than 65 days?
3. Calculate the probability that for each event the latest possible time to start before the earliest possible time to start lies.

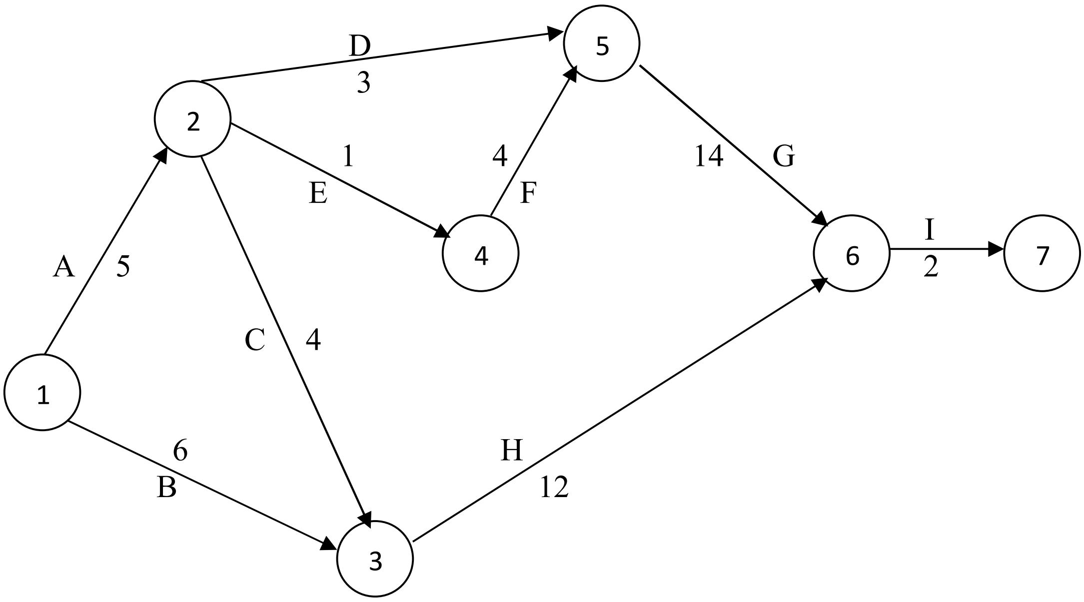
Chapter 8

Project Management

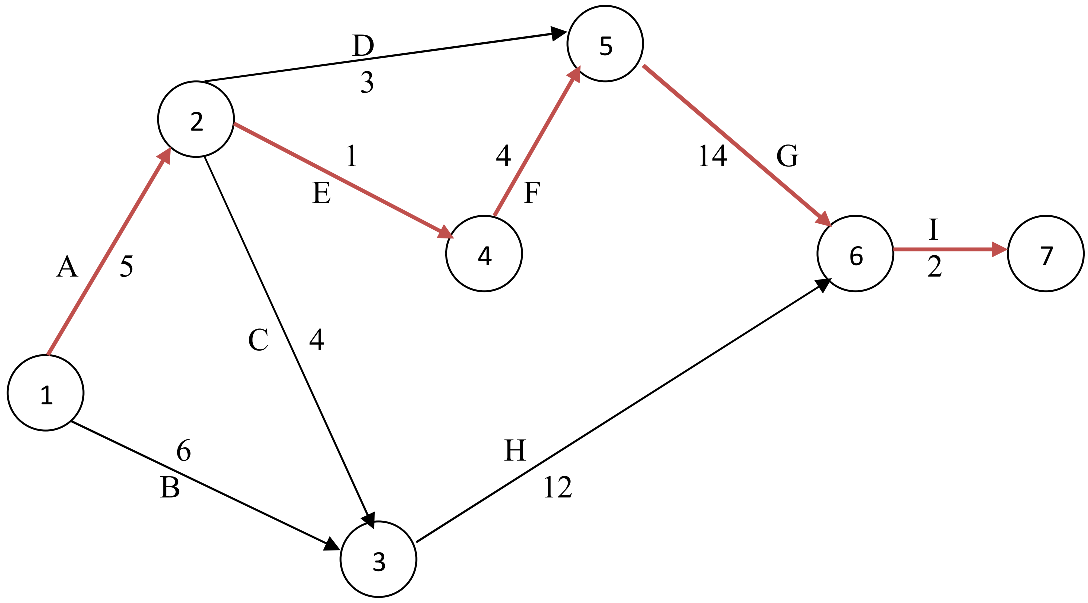
Solutions

8. 1.

1.



T^e	Events	1	2	3	4	5	6	7
<u>0</u>	1		5	6				
<u>5</u>	2			4	1	3		
9	3						12	
<u>6</u>	4					4		
<u>10</u>	5						14	
<u>24</u>	6							2
26	7							
T^l		<u>0</u>	<u>5</u>	12	<u>6</u>	<u>10</u>	<u>24</u>	26



The project can be completed in 26 weeks if the individual activities are completed on schedule.

2.
See the following table:

i	j	Activity	t_{ij}	$\Delta^T t_{ij} = T_j^l - T_i^e - t_{ij}$	Critical Path?
1	2	A	5	0	Yes
1	3	B	6	6	No
2	3	C	4	3	No
2	4	E	1	0	Yes
2	5	D	3	2	No
3	6	H	12	3	No
4	5	F	4	0	Yes
5	6	G	14	0	Yes
6	7	I	2	0	Yes

3.

A, E, F, G, I

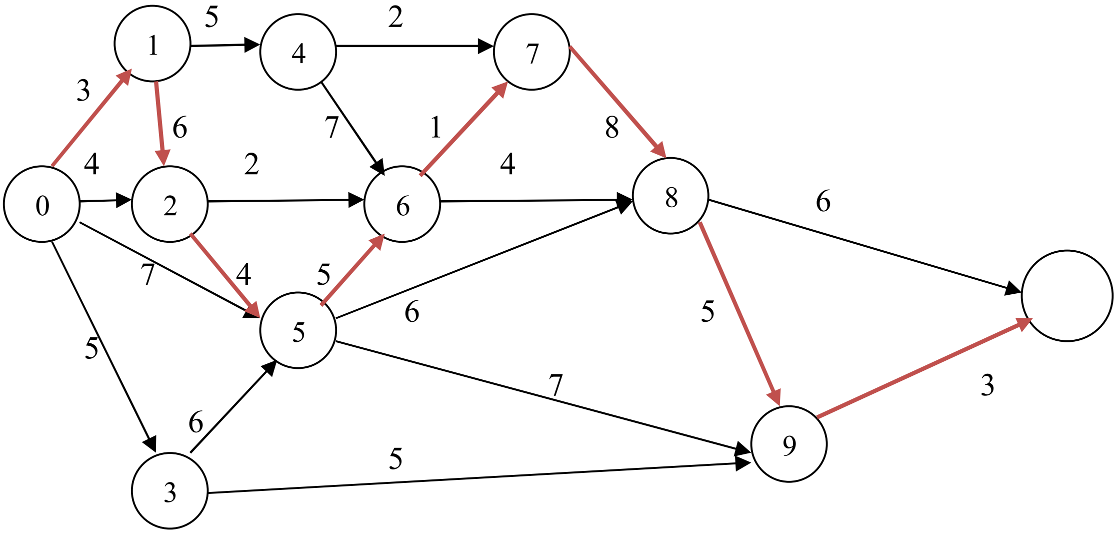
4.
It is the critical path that determines the project completion time. Changing time of the non-critical activities within the permissible range will not affect the project completion time. But changing time of the critical activities may cause the project completion time to change.

8. 2.

1.

0	t_{ij}	T^e	T^l
0	$t_{01}=3$ $t_{02}=4$ $t_{03}=5$ $t_{05}=7$	0	$\min\{13-7,\ 7-5,\ 9-4,\ 3-3\}=0$
1	$t_{12}=6$ $t_{14}=5$	3	$\min\{11-5,\ 9-6\}=3$
2	$t_{25}=4$ $t_{26}=2$	$\max\{0+4,\ 3+6\}=9$	$\min\{18-2,\ 13-4\}=9$
3	$t_{35}=6$ $t_{39}=5$	5	$\min\{32-5,\ 13-6\}=7$
4	$t_{46}=7$ $t_{47}=2$	8	$\min\{19-2,\ 18-7\}=11$
5	$t_{56}=5$ $t_{58}=6$ $t_{59}=7$	$\max\{7,\ 9+4,\ 5+6\}=13$	$\min\{32-7,\ 27-6,\ 18-5\}=13$
6	$t_{67}=1$ $t_{68}=4$	$\max\{9+2,\ 8+7,\ 13+5\}=18$	$\min\{27-4,\ 19-1\}=18$
7	$t_{78}=8$	$\max\{8+2,\ 18+1\}=19$	$\min\{27-8\}=19$
8	$t_{89}=5$ $t_{8,10}=6$	$\max\{13+6,\ 18+4,\ 19+8\}=27$	$\min\{35-6,\ 32-5\}=27$
9	$t_{9,10}=3$	$\max\{5+5,\ 13+7,\ 27+5\}=32$	$\min\{35-3\}=32$
10		$\max\{27+6,\ 32+3\}=35$	35

T^e	Events	0	1	2	3	4	5	6	7	8	9	10
0	0		3	4	5		7					
3	1			6		5						
9	2						4	2				
5	3						6				5	
8	4							7	2			
13	5							5		6	7	
18	6								1	4		
19	7									8		
27	8										5	6
32	9											3
35	10											
T^l		0	3	9	7	11	13	18	19	27	32	35

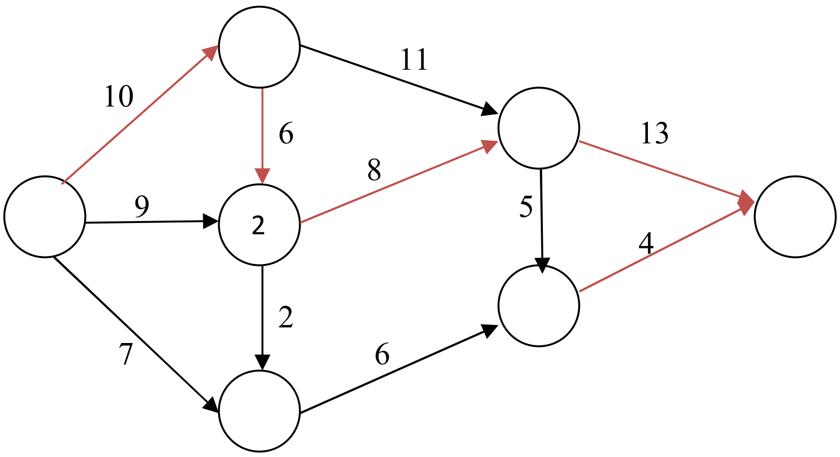


2.

i	j	t_{ij}	T_i^e	T_j^l	$T_i^e + t_{ij}$	$\Delta^T t_{ij}$	$\Delta^F t_{ij}$	$\Delta^C t_{ij}$	$\Delta^I t_{ij}$
1	2	3	4	5	6	7	8	9	10
0	1	3	0	3	3	0	0	0	0
0	2	4	0	9	4	5	5	0	5
0	3	5	0	7	5	2	0	2	0
0	5	7	0	13	7	6	6	0	6
1	2	6	3	19	9	0	0	0	0
1	4	5	3	11	8	3	0	3	0
2	5	4	9	13	13	0	0	0	0
2	6	2	9	18	11	7	7	0	7
3	5	6	5	13	11	2	2	0	0
3	9	5	5	32	10	22	22	0	20
4	6	7	8	18	15	3	3	0	0
4	7	2	8	19	10	9	9	0	6
5	6	5	13	18	18	0	0	0	0
5	8	6	13	27	19	8	8	0	8
5	9	7	13	32	20	12	12	0	12
6	7	1	18	19	19	0	0	0	0
6	8	4	18	27	22	5	5	0	5
7	8	8	19	27	27	0	0	0	0
8	9	5	27	32	32	0	0	0	0
8	10	6	27	35	33	2	2	0	2
9	10	3	32	35	35	0	0	0	0

8. 3.

T^e	$i \backslash j$	0	1	2	3	4	5	6
$\underline{0}$	0		10	9	7			
$\underline{10}$	1			6		11		
$\underline{16}$	2				2	8		
18	3						6	
$\underline{24}$	4						5	13
29	5							4
$\underline{37}$	6							
T^l		$\underline{0}$	$\underline{10}$	$\underline{16}$	27	$\underline{24}$	33	$\underline{37}$



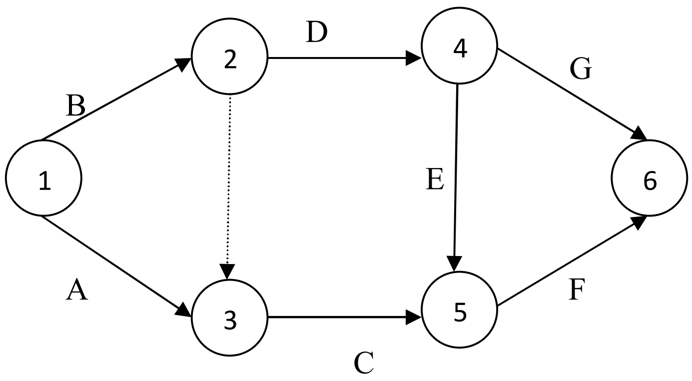
The project will be completed in 37 weeks.

2.

i	j	t_{ij}	T_i^e	T_j^l	$T_i^e + t_{ij}$	$\Delta^T t_{ij}$	$\Delta^F t_{ij}$	$\Delta^C t_{ij}$	$\Delta^I t_{ij}$
0	1	10	0	10	10	0	0	0	0
0	2	9	0	16	9	7	7	0	7
0	3	7	0	27	7	20	11	9	11
1	2	6	10	16	16	0	0	0	0
1	4	11	10	24	21	3	3	0	3
2	3	2	16	27	18	9	0	9	0
2	4	8	16	24	24	0	0	0	0
3	5	6	18	33	24	9	5	4	0
4	5	5	24	33	29	4	0	4	0
4	6	13	24	37	37	0	0	0	0
5	6	4	29	37	33	4	4	0	0

8. 4

1.

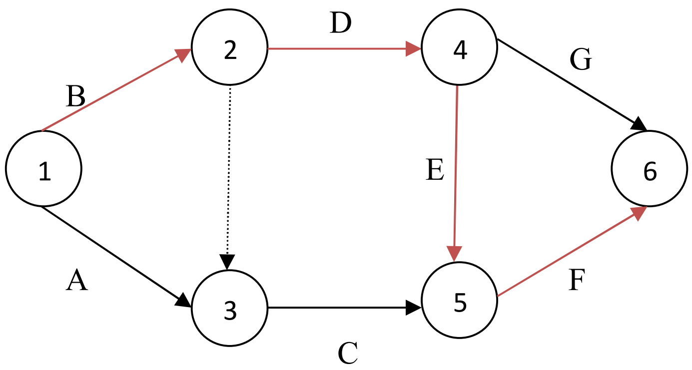


2.

T^e	Events	1	2	3	4	5	6
0	1		4	3			
4	2			0	6		
3	3					5	
10	4					7	9
17	5						8
25	6						
T^l		0	4	12	10	17	25

a)

B→D→E→F



b)

4 + 6 + 7 + 8 = 25 days.

3.

i	j	t_{ij}	$\Delta^T t_{ij} = T_j^l - T_i^e - t_{ij}$	Critical	$\Delta^F t_{ij} = T_j^e - T_i^e - t_{ij}$
1	2	4	0	Yes	0
1	3	3	9	No	0
2	4	6	0	Yes	0
3	5	5	9	No	9
4	5	7	0	Yes	0
4	6	9	0	Yes	6
5	6	8	0	Yes	0

A slack signifies the time which an activity can be delayed without delaying the project. A zero slack for an activity indicates that it cannot be delayed without delaying the project and hence it is called a critical activity. On the other hand, a positive slack for an activity means that it can be delayed by the length of the slack without delaying the project and hence it is called a non-critical activity.
It is the amount of time a single activity can be delayed without delaying the early start of any successor activity.

4.
a)

i	j	a_{ij}	m_{ij}	b_{ij}	$\bar{t}_{ij}$	$\sigma^2_{t_{ij}}$
1	2	3	4	5	4	0.1111
1	3	1	3	5	3	0.4444
2	4	3	5	7	5	0.4444
3	5	4	5	6	5	0.1111
4	5	5	6	13	7	1.7778
4	6	6	8	10	8	0.4444
5	6	4	7	10	7	1.0000

$\bar{T}^e$	$\sigma^2_{T^e}$		1	2	3	4	5	6
0	0.0000	1		4 0.1111	3 0.4444			
4	0.1111	2			0 0	5 0.4444		
4	0.4444	3					5 0.1111	
9	0.5555	4					7 1.7778	8 0.4444
16	2.3333	5						7 1.0000
23	3.3333	6						
	$\bar{T}^l$	$\bar{T}^l$	0	4	11	9	16	23
		$\sigma^2_{T^l}$	3.3333	3.2222	1.1111	2.7778	1.0000	0

Critical path: $\mathbf{B} \to \mathbf{D} \to \mathbf{E} \to \mathbf{F}$

Expected completion time $= \bar{t}_{12} + \bar{t}_{24} + \bar{t}_{45} + \bar{t}_{56} = 4 + 5 + 7 + 7 = 23$

Project variance $= \sigma^2_{T^e_{12}} + \sigma^2_{T^e_{24}} + \sigma^2_{T^e_{45}} + \sigma^2_{T^e_{56}}$

$= 0.1111 + 0.4444 + 1.7778 + 1.0000 = 3.3333$

b)

Let T be the completion time:

i) $P(T \le 20) \approx P(T < 20) = \Theta\left(\dfrac{20-23}{1.83}\right) = \Theta(-1.64) = 1 - \Theta(1.64)$

$= 1 - 0.948449 = 0.051551$

ii) $P(T \le 25) \approx P(T < 25) = \Theta\left(\dfrac{25-23}{1.83}\right) = \Theta(1.09) = 0.862143.$

8. 5.

1.

i	j	a_{ij}	m_{ij}	b_{ij}	$\bar{t}_{ij}$	$\sigma_{t_{ij}}$	$\sigma^2_{t_{ij}}$
0	1	3	5	8	5.17	0.83	0.69
1	2	12	13	16	13.33	0.67	0.44
1	3	8	11	15	11.17	1.17	1.37
2	5	13	15	21	15.67	1.33	1.78
3	4	6	8	10	8.00	0.67	0.44
3	5	7	8	10	8.17	0.50	0.25
4	8	6	7	9	7.17	0.50	0.25
5	6	13	14	16	14.17	0.50	0.25
5	7	3	4	5	4.00	0.33	0.11
5	8	10	14	19	14.17	1.50	2.25
6	7	7	9	12	9.17	0.83	0.69
7	9	4	5	6	5.00	0.33	0.11
8	9	15	16	18	16.17	0.50	0.25

$\sigma^2_{T_i^e}$	$\bar{T}_i^e$		0	1	2	3	4	5	6	7	8	9
0.00	0.00	0		5.17 0.69								
0.69	5.17	1			13.33 0.44	11.17 1.37						
1.13	18.50	2						15.67 1.78				
2.06	16.34	3					8.00 0.44	8.17 0.25				
2.50	24.34	4									7.17 0.25	
2.91	34.17	5							14.17 0.25	4.00 0.11	14.17 2.25	
3.16	48.34	6								9.17 0.69		
3.85	57.51	7										5.00 0.11
5.16	48.34	8										16.17 0.25
5.41	64.51	9										
		$\bar{T}_j^l$	0.00	5.17	18.50	26.00	41.17	34.17	50.34	59.51	48.34	64.51
		$\sigma^2_{T_j^l}$	5,41	4.72	4.28	2.75	0.50	2.50	0.80	0.11	0.25	0.00

Critical path: $\mathbf{0 \to 1 \to 2 \to 5 \to 8 \to 9}$

2

Expected completion time: $= \bar{t}_{01} + \bar{t}_{12} + \bar{t}_{25} + \bar{t}_{58} + \bar{t}_{89}$

$$= 5.17 + 13.33 + 15.67 + 14.17 + 16.17 = 64.51$$

Project variance $= \sigma^2_{T^e_{01}} + \sigma^2_{T^e_{12}} + \sigma^2_{T^e_{25}} + \sigma^2_{T^e_{58}} + \sigma^2_{T^e_{89}}$

$$= 0.69 + 0.44 + 1.78 + 2.25 + 0.25 = 5.41$$

$$P(T < 65) = \Phi\left(\frac{65 - 64.51}{\sqrt{5.41}}\right) = \Phi(0.21) = 0.5832$$

3.

i	$\bar{T}^e_i$	$\bar{T}^l_i$	$\sigma^2_{T^e_i}$	$\sigma^2_{T^l_i}$	$\sqrt{\sigma^2_{T^e_i} + \sigma^2_{T^l_i}}$	$\bar{T}^l_i - \bar{T}^e_i$	$\dfrac{\bar{T}^l_i - \bar{T}^e_i}{\sqrt{\sigma^2_{T^e_i} + \sigma^2_{T^l_i}}}$	$P\left[(\bar{T}^l_i - \bar{T}^e_i) \le 0\right]$
0	0.00	0.00	0.00	5.41	2.33	0.00	0.00	50%
1	5.17	5.17	0.69	4.72	2.33	0.00	0.00	50%
2	18.50	18.50	1.13	4.28	2.33	0.00	0.00	50%
3	16.34	26.00	2.06	2.75	2.19	9.66	4.41	0.0%
4	24.34	41.17	2.50	0.50	1.73	16.83	9.72	0.0%
5	34.17	34.17	2.91	2.50	2.33	0.00	0.00	50%
6	48.34	50.34	3.16	0.80	1.99	2.00	1.01	15.6%
7	57.51	59.51	3.85	0.11	1.99	2.00	1.01	15.6%
8	48.34	48.34	5.16	0.25	2.33	0.00	0.00	50%
9	64.51	64.51	5.41	0.00	2.33	0.00	0.00	50%

Introduction
to
Input-Output Analysis

R. 9. 1. (***Input-Output Tableau***)
The *"classical Leontief model"* is based on the so-called *"input-output tableau"*:

$i \setminus j$	Production Sectors					Sectors of Final Demand					Total Output
	1	…	j	...	n	1	...	k	…	m	
1	x_{11}	…	x_{1j}	…	x_{1n}	y_{11}	…	y_{1k}	…	y_{1m}	x_1
.	.	…	.	…		.	…	.	…	.	.
.	.	…	.	…		.	…	.	…	.	.
.	.	…	.	…		.	…	.	…	.	.
i	x_{i1}		x_{ij}	…	x_{in}	y_{i1}	…	y_{ik}	…	y_{im}	x_i .
.	.	…	.	…		.	…	.	…	.	.
.	.	…	.	…		.	…	.	…	.	.
.	.	…	.	…		.	…	.	…	.	.
n	x_{n1}	…	x_{nj}	…	x_{nn}	y_{n1}	…	y_{nk}	…	y_{nm}	x_n
1	z_{11}	…	z_{1j}	…	z_{1n}						
.	.	…	.	…							
.	.	…	.	…							
.	.	…	.	…							
l	z_{l1}	…	z_{lj}	…	z_{ln}						
.	.	…	.	…							
.	.	…	.	…							
.	.	…	.	…							
p	z_{p1}	…	z_{pj}	…	z_{pn}						
Total Input	x_1	…	x_j	…	x_n						

Notations:

$n\ (i, j = 1, 2, ..., n)$: Number of production sectors
$m\ (k = 1, 2, ..., m)$: Number of sectors of final demand
$p\ (l = 1, 2, ..., p)$: Number of primary inputs
x_{ij}: Sales by sector i to sector j
y_{ik}: Final demand of type k for sector i
z_{lj}: Primary input of type l used in sector i

Ex. 9. 1.

	Agriculture	Manufacturing	Services	Other	Final Demand	Total Output
Agriculture	10	65	10	5	10	100
Manufacturing	40	25	35	75	25	200
Services	15	5	5	5	90	120
Other	15	10	50	50	100	225
Value Added	20	95	20	90		
Total Input	100	200	120	225		

Consider the following input-output table

Input-Output Table, $Million

	Sent to	Intermediate Sectors			Final Demand		
	Purchases from	Agriculture	Manufacturing	Services	Households	Exports	Total Output
Intermediate Sectors	Agriculture	0	400	0	500	100	
	Manufacturing	350	0	150	800	700	
	Services	100	200	0	300	0	
Primary Input	Imports	250	600	50			
	Wages	200	500	300			
	Other value added	100	300	100			
	Total Input						

Compute the following

1. The total output (= input) of each sector.
2. The final demand of each sector
3. The matrix of direct secondary input-output coefficients.
4. The unknown elements of the complex secondary input-output coefficients:

$$\begin{pmatrix} a & 0.2222 & 0.0556 \\ 0.4167 & 1.1111 & b \\ 0.1500 & 0.1333 & 1.0333 \end{pmatrix}$$

   Interpret the first column of this matrix.

5. The final demand vector for the total output vector $x = (1200 \quad 1900 \quad 550)^T$.
6. The total output vector for the final demand vector $y = (650 \quad 1200 \quad 350)^T$
7. The matrix of direct primary input-output coefficients.
8. The matrix of complex primary input-output coefficients. Interpret the second column.
9. The amount of primary inputs needd for $y = \begin{pmatrix} 4 & 8 & 4 \end{pmatrix}^T$.

9. 3.

1.

$$x = \begin{pmatrix} 1000 & 2000 & 600 \end{pmatrix}^T$$

2.

$$y = \begin{pmatrix} 600 & 1300 & 300 \end{pmatrix}^T$$

3.

$$A = \begin{pmatrix} 0.00 & 0.20 & 0.00 \\ 0.35 & 0.00 & 0.25 \\ 0.10 & 0.10 & 0.00 \end{pmatrix}.$$

4.

$$I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.00 & 0.20 & 0.00 \\ 0.35 & 0.00 & 0.25 \\ 0.10 & 0.10 & 0.00 \end{pmatrix} = \begin{pmatrix} 1.00 & -0.20 & 0.00 \\ -0.35 & 1.00 & -0.25 \\ -0.10 & -0.10 & 1.00 \end{pmatrix}$$

$$\begin{pmatrix} 1.00 & -0.20 & 0.00 \\ -0.35 & 1.00 & -0.25 \\ -0.10 & -0.10 & 1.00 \end{pmatrix} . \begin{pmatrix} a & 0.2222 & 0.0556 \\ 0.4167 & 1.1111 & b \\ 0.1500 & 0.1333 & 1.0333 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a - 0.20 \cdot 0.4167 = 1 \qquad \Rightarrow \; a \approx 1.0833$$

$$-0.10 \cdot 0.0556 - 0.10b + 1.0333 = 1 \quad \Rightarrow \quad b \approx 0.2774$$

Incraesing only the final demand of the first sector will require an additional production of 1.0833 units by the first sector, 0.4167 units of the second and 0.15 units of the third sector.

5.

$$y = (I - A)x = \begin{pmatrix} 1.00 & -0.20 & 0.00 \\ -0.35 & 1.00 & -0.25 \\ -0.10 & -0.10 & 1.00 \end{pmatrix} \begin{pmatrix} 1200 \\ 1900 \\ 550 \end{pmatrix} = \begin{pmatrix} 820.0 \\ 1342.5 \\ 240.0 \end{pmatrix}$$

6.

$$y = (I - A)^{-1} y = \begin{pmatrix} 1.0833 & 0.2222 & 0.0556 \\ 0.4167 & 1.1111 & 0.2774 \\ 0.1500 & 0.1333 & 1.0333 \end{pmatrix} \begin{pmatrix} 650 \\ 1200 \\ 350 \end{pmatrix} = \begin{pmatrix} 990.245 \\ 1702.265 \\ 619.115 \end{pmatrix}.$$

R. 9. 2. (***Some Fundamental Relations***)

$$x_i = \sum_{i=1}^{n} x_{ij} + y_i, \quad i = 1, 2, ..., n$$

$$y_i := \sum_{k=1}^{m} y_k : \qquad \text{final demand of sector } i$$

$$x_j = \sum_{j=1}^{n} x_{ij} + z_j, \quad j = 1, 2, ..., n .$$

$$z_j := \sum_{l=1}^{p} z_{lj}$$

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{n} x_j$$

$$\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} z_j .$$

D. 9. 1. (***Direct Secondary Input Coefficients***)
We call

$$A := \left(a_{ij}\right), \; i, j = 1, 2, ..., n$$

$$a_{ij} := \frac{x_{ij}}{x_j}, \quad i, j = 1, 2, ..., n$$

the *matrix of direct (secondary) input-output coefficients.*

Ex. 9. 1. (***cont.***)

D. 1. 2. (***Classical Leontief Model***)
Based on the above relations the classical Leontief model is defined as follows:

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + y_i, \quad i = 1, 2, ...n ,$$

or

$$x = Ax + y .$$

Here are:

$$x := (x_j) \in R_+^n,$$

$$A := (a_{ij}),\ i, j = 1,2,...,n,$$

$$y := (y_i) \in R_+^n$$

R. 9. 3.

1. Given the production vector x, we get

(9. 9.) $\qquad y = (I - A)x .$

2. Given the final demand y, we obtain

(9. 10.) $\qquad x = (I - A)^{-1} y .$

R. 9. 4.

The *classical Leontief model* is based on the following assumptions:

1. Each sector produces at most one homogeneous good.
2. The final demand is given as a homogeneous factor.
3. A given output can only be produced by a unique combination of factors.
4. The input x_{ij} is proportional to the output x_j.

D. 9. 3. (*Leontief Matrix, Indirect and Complex Input-Output Coefficients*)

1.

The matrix $(I - A)$ is called the *Leontief matrix* (or *technology matrix*).

2.

The matrix $(I - A)^{-1} =: B = (b_{ij}),\ i, j = 1,2,...,n,$ is called the *Leontief inverse*. The elements $b_{ij},\ i, j = 1,2,...,n,$ are the so-called *complex input-output coefficients*.

3.

The matrix $(I - A)^{-1} - (I + A)$ is called the *matrix of indirect input-output coefficients.*

D. 9. 3. (*Direct Primary Input-Output Coefficients*)

The matrix

$$\tilde{A} := (\tilde{a}_{ij}),\ i, j = 1,2,...,n$$

$$\tilde{a}_{ij} := \frac{z_{lj}}{x_j},\quad l = 1,2,,...p;\ j = 1,2,...,n$$

is called the *matrix of direct primary input-output coefficients*.

<u>D. 9. 4.</u> (*Complex Primary Input-Output Coefficients*)
The matrix

$$\tilde{B} = \tilde{A} \cdot (I - A)^{-1}$$

is called the *matrix of complex primary input-output coefficients*.

(The $l-th$ column of $\tilde{B}$ gives the additional primary input needed if only the final demand of the $l-th$ sector is increased by one unit.)

<u>R. 9. 5</u>
Let

$$z := (z_l) \in R_+^p$$

be the vector of primary inputs. Then

$$z = \tilde{B} \cdot y\,.$$

Chapter 9

Input-Output Analysis

Exercises

9. 1.
Consider the following Leontief tableau:

	Manufacture	Non-Manufacture	Consumption	Government	Export	Total
Manufacture	18	18	40	14	30	120
Non-Manufacture	20	37	43	7	18	125
Households	56	52	16	9	27	160
Government	6	7	20	22	0	55
Import	20	11	41	3	5	80
Total	120	125	160	55	80	540

Calculate

1. the total final demand vector,
2. the matrix of direct secondary input-output coefficients. Interpret the coefficient a_{12},
3. the Leontief matrix,
4. the matrix of complex input-output coefficients. Interpret the second column of this matrix,
5. the matrix of indirect input-output coefficients,
6. the total demand vector for the following total output vector:

$$x = \begin{pmatrix}130 & 120\end{pmatrix}^T,$$

7. the total output for the following final demand vector:

$$y = \begin{pmatrix}90 & 70\end{pmatrix}^T,$$

8. the total output of sector "manufacture" and the final demand of sector non-manufacture for $x_2 = 130$ and $y_1 = 80$,

9. the matrix of direct primary input-output coefficients. Interpret the coefficient $\tilde{a}_{31}$,

10. the matrix of complex primary input-output coefficients. Interpret its first column.

11. the amount of primary inputs needed for $y = \begin{pmatrix}90 & 70\end{pmatrix}^T$.

9. 2.
Consider the following input-output table:

	Sector 1	Sector 2	Sector 3	Consumption	Investment	Output
Sector 1	8	5	4	1	2	
Sector 2	0	1	0	9	0	
Sector 3	2	0	2	0	6	
Amortisation	2	2	2			
Wages and salaries	3	1	1			
Profit	5	1	1			
Input						

Compute the following

1. The total output (= input) of each sector.
2. The final demand of each sector
3. The matrix of direct secondary input-output coefficients.
4. The unknown elements of the complex secondary input-output coefficients:

$$\begin{pmatrix} a & 1.01 & 0.91 \\ 0.00 & 1.11 & 0.00 \\ 0.23 & 0.13 & b \end{pmatrix}$$

   Interpret the first column of this matrix.
5. The final demand vector for the total output vector $x = (24 \quad 8 \quad 9)^T$.
6. The total output vector for the final demand vector $y = (4 \quad 8 \quad 8)^T$.
7. The matrix of direct primary input-output coefficients.
8. The matrix of complex primary input-output coefficients. Interpret the second column.
9. The amount of primary inputs needed for $y = \left(4 \quad 8 \quad 4\right)^T$.

Chapter 9

Input-Output Analysis

Solutions

9. 1.

	Manufacture	Non-Manufacture	Consumption	Government	Export	Total
Manufacture	18	18	40	14	30	120
Non-Manufacture	20	37	43	7	18	125
Households	56	52	16	9	27	160
Government	6	7	20	22	0	55
Import	20	11	41	3	5	80
Total	120	125	160	55	80	540

1.

$$y = (84 \quad 68)^T.$$

2.

$$A = \begin{pmatrix} \frac{18}{120} & \frac{18}{125} \\ \frac{20}{120} & \frac{37}{125} \end{pmatrix} = \begin{pmatrix} 0.150 & 0.144 \\ 0.167 & 0.296 \end{pmatrix}.$$

$a_{12} = 0.144$ is the proportion of the sector "manufacture" in one unit of the product of the sector "non-manufacture".

3.

$$I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.150 & 0.144 \\ 0.167 & 0.296 \end{pmatrix} = \begin{pmatrix} 0.850 & -0.144 \\ -0.167 & 0.704 \end{pmatrix}.$$

4.

$$(I - A)^{-1} = \begin{pmatrix} 0.850 & -0.144 \\ -0.167 & 0.704 \end{pmatrix}^{-1} = \begin{pmatrix} 1.2257 & 0.2507 \\ 0.2908 & 1.4799 \end{pmatrix}.$$

If only sector "non-manufacture" increases its final demand by 1 unit, the two sectors should produce additionally 0.2507 and 1.4799.

5.

$$(I-A)^{-1}-(I+A)=\begin{pmatrix}1.2257 & 0.2507\\ 0.2908 & 1.4799\end{pmatrix}-\begin{pmatrix}1.150 & 0.144\\ 0.167 & 1.296\end{pmatrix}=\begin{pmatrix}0.0757 & 0.1067\\ 0.1238 & 0.1839\end{pmatrix}.$$

6.

$$y=(I-A)x=\begin{pmatrix}0.850 & -0.144\\ -0.167 & 0.704\end{pmatrix}\begin{pmatrix}130\\ 120\end{pmatrix}=\begin{pmatrix}93.22\\ 62.77\end{pmatrix}$$

7.

$$x=(I-A)^{-1}y=\begin{pmatrix}1.2257 & 0.2507\\ 0.2908 & 1.4799\end{pmatrix}\begin{pmatrix}90\\ 70\end{pmatrix}=\begin{pmatrix}127.862\\ 129.765\end{pmatrix},$$

8.

$$\begin{pmatrix}0.850 & -0.144\\ -0.167 & 0.704\end{pmatrix}\begin{pmatrix}x_1\\ 130\end{pmatrix}=\begin{pmatrix}80\\ y_2\end{pmatrix}\quad\Rightarrow\quad x_1=116.141,\quad y_2=72.124,$$

9.

$$\tilde{A}=\begin{pmatrix}\frac{56}{120} & \frac{52}{125}\\ \frac{6}{120} & \frac{7}{125}\\ \frac{20}{120} & \frac{11}{125}\end{pmatrix}=\begin{pmatrix}0.467 & 0.416\\ 0.050 & 0.056\\ 0.167 & 0.088\end{pmatrix}.$$

For each unit of the production of sector „manufacture“ 0.167 units should be imported.

10.

$$\tilde{B}=\tilde{A}\cdot(I-A)^{-1}=\begin{pmatrix}0.467 & 0.416\\ 0.050 & 0.056\\ 0.167 & 0.088\end{pmatrix}\cdot\begin{pmatrix}1.2257 & 0.2507\\ 0.2908 & 1.4799\end{pmatrix}=\begin{pmatrix}0.693 & 0.733\\ 0.078 & 0.095\\ 0.230 & 0.172\end{pmatrix}$$

The elements in the first column of this vector give the additional amount of primary inputs if only the final demand of sector "manufacture" increases by one unit.

11.

$$z=\tilde{B}\cdot y=\begin{pmatrix}0.693 & 0.733\\ 0.078 & 0.095\\ 0.230 & 0.172\end{pmatrix}\begin{pmatrix}90\\ 70\end{pmatrix}=\begin{pmatrix}113.68\\ 13.67\\ 32.74\end{pmatrix}.$$

9. 2

1.

$$x = \begin{pmatrix} 20 & 10 & 10 \end{pmatrix}^T$$

2.

$$y = \begin{pmatrix} 3 & 9 & 6 \end{pmatrix}^T$$

3.

$$A = \begin{pmatrix} \frac{8}{20} & \frac{5}{10} & \frac{4}{10} \\ \frac{0}{20} & \frac{1}{10} & \frac{0}{10} \\ \frac{2}{20} & \frac{0}{10} & \frac{2}{10} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.5 & 0.4 \\ 0.0 & 0.1 & 0.0 \\ 0.1 & 0.0 & 0.2 \end{pmatrix}.$$

4.

$$I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.5 & 0.4 \\ 0.0 & 0.1 & 0.0 \\ 0.1 & 0.0 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.5 & -0.4 \\ 0.0 & 0.9 & 0.0 \\ -0.1 & 0.0 & 0.8 \end{pmatrix}$$

$$\begin{pmatrix} 0.6 & -0.5 & -0.4 \\ 0.0 & 0.9 & 0.0 \\ -0.1 & 0.0 & 0.8 \end{pmatrix} \begin{pmatrix} a & 1.01 & 0.91 \\ 0.00 & 1.11 & 0.00 \\ 0.23 & 0.13 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$0.6a - 0.4 \cdot 0.23 = 1 \quad \Rightarrow \quad a = 1.82$$

$$-0.1 \cdot 0.91 + 0.8b = 1 \quad \Rightarrow \quad b = 1.36$$

Incraesing only the final demand of the first sector will require an additional production of 1.82 units by the first sector, 0 units of the second and 0.23 units of the third sector.

5.

$$y = (I - A)x = \begin{pmatrix} 0.6 & -0.5 & -0.4 \\ 0.0 & 0.9 & 0.0 \\ -0.1 & 0.0 & 0.8 \end{pmatrix} \begin{pmatrix} 24 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 6.8 \\ 7.2 \\ 4.8 \end{pmatrix}$$

6.

$$x = (I - A)^{-1} y = \begin{pmatrix} 1.82 & 1.01 & 0.91 \\ 0.00 & 1.11 & 0.00 \\ 0.23 & 0.13 & 1.36 \end{pmatrix} \begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 20.82 \\ 8.88 \\ 10.12 \end{pmatrix}.$$

7.

$$\tilde{A} = \begin{pmatrix} \frac{2}{20} & \frac{2}{10} & \frac{2}{10} \\ \frac{3}{20} & \frac{1}{10} & \frac{1}{10} \\ \frac{5}{20} & \frac{1}{10} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} 0.10 & 0.20 & 0.20 \\ 0.15 & 0.10 & 0.10 \\ 0.25 & 0.10 & 0.10 \end{pmatrix}.$$

The elements of this matrix give the amount of primary inputs per production unit.

8.

$$\tilde{B} = \tilde{A} \cdot (I - A)^{-1} = \begin{pmatrix} 0.10 & 0.20 & 0.20 \\ 0.15 & 0.10 & 0.10 \\ 0.25 & 0.10 & 0.10 \end{pmatrix} \cdot \begin{pmatrix} 1.82 & 1.01 & 0.91 \\ 0.00 & 1.11 & 0.00 \\ 0.23 & 0.13 & 1.36 \end{pmatrix} = \begin{pmatrix} 0.228 & 0.349 & 0.363 \\ 0.296 & 0.276 & 0.273 \\ 0.478 & 0.377 & 0.364 \end{pmatrix}$$

Incraesing only the final demand of the second sector will require an additional production of 0.149 units by the first sector, 0.176 units of the second and 0.277 units of the third sector.

9.

$$z = \tilde{B} \cdot y = \begin{pmatrix} 0.228 & 0.349 & 0.363 \\ 0.296 & 0.276 & 0.273 \\ 0.478 & 0.377 & 0.364 \end{pmatrix} \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 5.156 \\ 4.484 \\ 6.384 \end{pmatrix}.$$

Chapter 10

Queuing Systems

D. 10. 1. (***Queuing Theory***)
Queuing theory is the branch of operations research concerned with waiting lines.

D. 10. 2. (***Queuing System***)
A *queuing system* consists of

1. a user source
2. a queue
3. a service facility with one or more identical parallel servers.

D. 10. 3. (***Queuing Network***)
A *queuing network* is a set of interconnected queuing systems.

R. 10. 1. (***Queuing Characteristics***)

1. Arrival process
2. Service time distribution
3. Number of servers
4. System capacity
5. Population size
6. Service discipline

1. Arrival process

Suppose jobs arrive at times $t_1, t_2, ..., t_j$

- Random variables $\tau_j = t_j - t_{j-1}$ are *inter-interval times*
- There are many possible assumptions for the distribution of the τ_j ,among others:
    - i. Independent
    - ii. Identically distributed
    - iii. Bulk arrivals
    - iv. Bulking
    - v. Correlated

- For *Poisson* arrival, the interval times are
    - i. independent and identically distributed (IDD)
    - ii. exponentially distributed (i.e., CDF $F(x) = 1 - e^{-x/a}$)

  (Notation: M : memoryless)

- Other common arrival time distributions include: Erlang (E), hyper-exponential (H), deterministic (D), general (G results valid for any distribution)

2. *Service time*
   - Interval spent actually receiving service (exclusive of waiting time)
   - Most common assumptions:

     i. IID random variables
     ii. Exponential service time distribution

3. *Number of servers*

   - Servers may or may not be identical
   - Service discipline determines allocation of customers to servers

4. *System capacity*

   - Maximum number of customers in the system (including those in service) may be

     i. finite
     ii. infinite

5. *Population size*

   - Maximum number of potential customers may be

     i. finite
     ii. infinite

6. *Service discipline*

   - First-come-first serve (FCFS)
   - Last-come-first-serve (LCFS)
   - Last-come-first-serve preempt resume (LCFS-PR)
   - Round robin (RR) with quantum size
   - Processor sharing (PS) with infinitesimal quantum size (PS-RR)
   - Infinite server (IS)

R. 10. 2. (***Some Applications***)

1. Airpot check-in
2. Aircraft takeoff/landing sequence
3. Automated teller machines (ATMs)
4. Traffic analysis
5. Phone switchboard
6. Toll booths
7. Police or other spatially distributed services

R.10. 3. (***Classification of Models: Kendall Notation***)
In queuing theory, *Kendall notation* is the standard system used to describe and classify queuing models. First suggested by D. G. Kedall as a three-factor system for characterising queues, it has since been extended to six items:

$$A / S / m / B / K, SD$$

A : Arrival process
S : Service time
m : Number of servers
B : Number of buffers (system capacity)
K : Population size
SD : Service discipline

Ex. 10. 1.

$$M / M / 3 / 20 / 1500 / FCFS$$

- Time between successive arrivals is exponentially distributed.
- Service times are exponentially distributed.
- Three servers.
- 20 buffers (3 in service + 20 waiting). After 20, all arriving jobs are lost.
- Total of 1500 jobs that can be serviced.
- Service discipline is first-come-first-served.

R. 10. 4. (***Single-Channeled System with Random Arrival and S*ervice *Times***)

Specifications:

λ : Average arrival rate

μ : Average service time

$\bar{n}$: Average number in the system (including the element being serviced)

W : Average time spent in the system

W_q : Average wait before service begins

$\bar{n}_q$: Average number waiting for service to begin

L_q : Average number of customers waiting in the queue

$P(0)$: Percentage of idle time

$P(x)$: Percentage of time in which exactly x elements are in the system ($x > 0$)

Formulas:

$$\bar{n} = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}}$$

$$W = \frac{1}{\mu - \lambda}$$

$$W_q = \frac{\lambda}{\mu} \cdot W$$

$$\bar{n}_q = \frac{\lambda}{\mu} \cdot \bar{n}$$

$$L_q = \lambda \cdot W_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$P(0) = 1 - \frac{\lambda}{\mu}$$

$$P(x) = \left(\frac{\lambda}{\mu}\right)^x \left(1 - \frac{\lambda}{\mu}\right)$$

$\frac{\lambda}{\mu}$ is extremely important in the queuing theory. It is sometimes referred to as the *utilisation factor*. Sometimes a separate symbol for it is used ($\rho = \frac{\lambda}{\mu}$). It is expressed in units of "Erlangs" in honour of Danish queuing theory pioneer, A. K. Erlang.

$\rho = \frac{\lambda}{\mu}$ indicates the extreme sensitivity of the system to small changes in λ or μ when λ is close to μ.

Ex. 10. 2.

An airport, under instrument conditions, can land 12 aircraft per hour on average. Aircraft arrive into the landing pattern at the average rate of 9 per hour.
Find

1. the average number of aircraft in the system,
2. the average time spent in the system,
3. the average wait for the service to begin,
4. the average number of aircraft waiting for service to begin,
5. the percentage of the idle time,
6. the percentage of time in which exactly 1 aircraft will be served,
7. the percentage of time in which exactly 2 aircraft will be served.

Solution:

$$\lambda = 9/hr, \quad \mu = 12/hr, \quad \frac{\lambda}{\mu} = \frac{9}{12} = 0.75.$$

1.

$$\bar{n} = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{0.75}{1 - 0.75} = 3 \text{ aircraft.}$$

2.

$$W = \frac{1}{\mu - \lambda} = \frac{1}{12-9} = \frac{1}{3} hr = 20 \text{ min.}$$

3.

$$W_q = \frac{\lambda}{\mu} \cdot W = 0.75 \cdot 20 = 15 \text{ min.}$$

4.

$$\bar{n}_q = \frac{\lambda}{\mu} \cdot \bar{n} = 0.75 \cdot 3 = 2.25 \text{ aircraft.}$$

5.

$$P(0) = 1 - \frac{\lambda}{\mu} = 1 - 0.75 = 0.25 = 25\%.$$

6.

$$P(1) = \left(\frac{\lambda}{\mu}\right)\left(1 - \frac{\lambda}{\mu}\right) = 0.75 \cdot 0.25 = 0.1875 = 18.75\%.$$

7.

$$P(2) = \left(\frac{\lambda}{\mu}\right)^2 \left(1 - \frac{\lambda}{\mu}\right) = 0.75^2 \cdot 0.25 \approx 0.141.$$

Chapter 10

Queuing Systems

Exercises

10. 1.
At a small grocery store, customers arrive according to *Poisson* process with a mean of 15 customers per hour. The length of time it takes to check out is exponentially distributed with mean equal to 3 minutes.

1. Compute

   a) the probability that the checkout counter is idle,
   b) the probability that the checkout counter is busy,
   c) the probability that at least one customer is waiting to check out,
   d) the average number of customers waiting to check out,
   e) the average cost per customer, supposing that it costs the store 3 € for each minute that a customer spends waiting in the queue.

2. For an additional of 400 € per hour, the store can decrease the average service time to 2 minutes. Is the additional expenditure worthwhile?

10. 2.
A mechanic is able to install new car mufflers at an average rate of 3 per hour, according to a negative exponential distribution. Customers seeking this service arrive at the shop on the average of 2 per hours, following a Poisson distribution. They are served on a first-in, first-out basis and come from a very large (almost infinite) population of possible buyers.
Calculate the following:

1. The average number of cars in the system.
2. The average waiting time spent in the system.
3. The average number of cars waiting in the line.
4. The average waiting time per car.
5. The percentage of time the mechanic is busy (utilization factor).
6. The probability that there are no cars in the system.
7. The probability that there is only one car in the system

Chapter 10

Queuing Systems

Solutions

10. 1.

$$\lambda = 15 / hr, \quad \mu = \frac{60}{3} = 20 / hr$$

1.

a)

$$P(0) = 1 - \frac{\lambda}{\mu} = 1 - \frac{15}{20} = 0.25$$

b)

$$1 - P(0) = 1 - 0.25 = 0.75$$

c).

$$P(x > 1) = 1 - P(x \le 0) = 1 - (P(x = 0) + P(x = 1))$$

$$= 1 - \left(\left(\frac{\lambda}{\mu} \right)^0 \left(1 - \frac{\lambda}{\mu} \right) + \left(\frac{\lambda}{\mu} \right)^1 \left(1 - \frac{\lambda}{\mu} \right) \right)$$

$$= 1 - \left(0.75^0 \cdot 0.25 + 0.75 \cdot 0.25 \right) = 0.5625$$

d)

$$\bar{n}_q = \frac{\lambda}{\mu} \cdot \bar{n} = \frac{\lambda}{\mu} \cdot \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = 0.75 \cdot \frac{0.75}{0.25} = 2.25$$

e)

$$W = \frac{1}{\mu - \lambda} = \frac{1}{20 - 15} = 0.2$$

$$W_q = \frac{\lambda}{\mu} \cdot W = 0.75 \cdot 0.20 = 0.15 \ hr / customer$$

$$Cost = 0.15 \cdot 60 \cdot 3 = 27 \ € / customer$$

2.

$$\lambda = 15 / hr, \quad \mu = \frac{60}{2} = 30 / hr$$

Before	After
$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{15^2}{20(20-15)} = 2.25$ $Cost = 2.25 \cdot 60 \cdot 3 = 405$ €/hr	$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{15^2}{30(30-15)} = 0.5$ $Cost = 0.5 \cdot 60 \cdot 3 = 90$ €/hr
$Difference = 405 - 90 = 315$ € / hr	

2.

We have:

$$\mu = 3, \quad \lambda = 2$$

1.

$$\bar{n} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2 \text{ cars}$$

2.

$$W = \frac{1}{3-2} = 1 \text{ hour}$$

3.

$$\bar{n}_q = \frac{2}{3} \cdot 2 = \frac{4}{3} \approx 1.33 \text{ cars}$$

4.

$$W_q = \frac{2}{3} \cdot 1 = \frac{2}{3} \text{ hour (40 minutes)}$$

5.

$$\rho = \frac{2}{3} (i.e. \text{ about} 66.6\% \text{of the time the mechanic is busy})$$

6.

$$P(0) = 1 - \frac{2}{3} = \frac{1}{3} \approx 0.33$$

7.

$$P(1) = \frac{2}{3} \cdot \left(1 - \frac{2}{3}\right) = \frac{2}{9} \approx 0.22$$

Chapter 11

Decision Theory

R. 11. 1.
Being a manager means primarily making decisions. For example:

- Should a company expand its existing facility or build a new facility?
- How many employees should he hire?
- What piece of equipment should be purchased?
- Should his company introduce the new product locally first or proceed to introduce it nationally or even internationally?

R. 11. 2.
Decision Theory is an interdisciplinary area of study concerned with how a decision-maker makes or should make decisions.
The decision theory is classified into

1. Descriptive
2. Prescriptive

theory.

Descriptive decision theory deals with how people actually make decisions.
Prescriptive decision theory formulates how decisions should be made from a finite set of possible alternatives in order to accommodate a set of axioms believed to ensure "optimality".

It must be remembered that in the decision theory there is no "optimum" as such. An "optimal" decision is always to be conceived within a set of rules.

D. 11. 1 (***Decision Table***)
The following table is called a *decision table* (*matrix*):

	s_1	.	.	.	s_j	.	.	.	s_n
a_1	x_{11}	.	.	.	x_{1j}	.	.	.	x_{1n}
.	.				.				.
.	.				.				.
.	.								
a_i	x_{i1}	.	.	.	x_{ij}	.	.	.	x_{in}
.	.				.				.
.	.				.				.
.	.				.				.
a_m	x_{m1}	.	.	.	x_{mj}	.	.	.	x_{mn}

Here are:

a_i: action (or alternative) $i = 1,2,...,m$;

s_j: state (of nature) $j = 1,2,...,n$;

x_{ij}: consequence of choosing the action a_i in the state of s_j.

If the decision problem involves monetary outcomes the x_{ij} may be single numbers. Otherwise we assume that the decision maker can value them numerically, i.e. we shall assume that he can measure the value of x_{ij} to him through some real-valued function v.
By 'measure the value' we mean that

$$v\left(x_{ij}\right) > v\left(x_{kl}\right)$$

if and only if the decision maker would prefer the consequence x_{ij} to the consequence x_{kl}.
Letting $v_{ij} = v(x_{ij})$, the general form of a decision table becomes:

	s_1	.	.	.	s_j	.	.	.	s_n
a_1	v_{11}	.	.	.	v_{1j}	.	.	.	v_{1n}
.	.				.				.
.	.				.				.
.	.				.				.
a_i	v_{i1}	.	.	.	v_{ij}	.	.	.	v_{in}
.	.				.				.
.	.				.				.
.	.				.				.
a_m	v_{m1}	.	.	.	v_{mj}	.	.	.	v_{mn}

The above table is also called the *payoff table*.

D. 11. 2. (***Dominance***)

1. Outcome dominance
2. Event dominance
3. Probabilistic dominance

1. Outcome dominance

In *outcome dominance* one alternative dominates some second alternative, if the worst payoff for this alternative is at least as good as the best payoff for the second alternative.

2. Event dominance

Event dominance occurs if one alternative has a payoff equal to or better than that of a second alternative for each state of nature (regardless of what event occurs, the first alternative is better than the second).

3. Probabilistic dominance

If, for any amount of money, one alternative has a uniformly equal or better chance of obtaining that amount or more, then that alternative dominates another alternative by probabilistic dominance.

Ex. 11. 1.
We illustrate the first two types of dominance by the following example:

Alternatives \ States	s_1	s_2	s_3
a_1	5	-1	2
a_2	2	1	0
a_3	4	2	5
a_4	-1	1	4
$P(s_j)$	0.1	0.4	0.5

The worst payoff outcome for alternative a_3 is 2 (when s_2 occurs), the best payoff for a_2 is also 2 (when s_1 occurs). Hence, alternative a_3 dominates a_2 by *outcome dominance*.

For each state of nature, the payoff for alternative a_3 is greater than that for a_2 and a_4. Hence, a_3 dominates a_2 and a_4 by *event dominance*.

D. 11. 3. (***Types of Decision Problems***)
We distinguish three different decision problems:

(1) Decision under certainty
The true state is known to the decision maker. Therefore, he can predict the consequence of his action with certainty

(2) Decision under uncertainty
The decision maker does not know which of the states will occur and has no way of estimating the probabilities of their occurrence.

(3) Decision under risk
The decision maker does not know the true state of nature by certain, but he can qualify his uncertainty through a probability distribution.

D. 11. 4. (***Decision under Certainty***)
A decision problem under certainty occurs if $n = 1$, i.e. we have the following decision table:

Actions	State s
a_1	v_1
.	.
.	.
.	.
a_i	v_i
.	.
.	.
.	.
a_m	v_m

If the outcomes are known and the values of the outcomes are certain, the task of the decision maker is to compute the optimal alternative or outcome with some optimization criterion in mind.
As an example: if the optimization criterion is cost minimisation and you are considering two different brands of a product, which appear to be equal in value to you, one costing 20% less than the other, then, all other things being equal, you will choose the less expensive brand.
Linear optimisation is an example of a technique for making decisions under the assumption of certainty.

D. 11. 5. (***Decision under Uncertainty***)
Here the decision maker feels that he can say nothing at all about the true state of nature. Not only is he unaware of the true state, but he cannot even quantify his uncertainty in any way. He is only prepared to say that each s_j describes a possible state of the world.

Ex. 11. 2. (***A Newsvendor Problem***)
The demand for a certain newspaper in a small town is 0, 1, 2, or 3 units. A newsvendor purchases the newspaper for 0.10 € and sells each paper for 0.25 €.
Therefore, we shall have the following profit table (matrix):

		s_1 0	s_2 1	s_3 2	s_4 3
a_1	0	0	0	0	0
a_2	1	-10	15	15	15
a_3	2	-20	5	30	30
a_4	3	-30	-5	20	45

How many newspapers should the newsvendor order?

R. 11. 4. (***Decision Rule under Uncertainty***)
How should a decision maker choose in such a situation of strict uncertainty?
We shall discuss the following rules:

(1) The Laplace (equal likelihood) rule,
(2) The maximin rule,
(3) The minimax rule,
(4) The maximax rule,
(5) The minimax regret rule
(6) The pessimism-optimism rule.

R. 11. 5. (***Laplace Rule***)
Choose the action a_k such that

$$\sum_{j=1}^{n} (1/n) v_{kj} = \max_{i=1,2,\ldots,m} \sum_{j=1}^{n} (1/n) v_{ij} .$$

Ex. 11. 3.
We illustrate the Laplace rule by applying it to the newsvendor problem:

		s_1 0	s_2 1	s_3 2	s_4 3	$\sum_{j=1}^{n}(1/n)v_{ij}$
a_1	0	0	0	0	0	0
a_2	1	-10	15	15	15	8.75
a_3	2	-20	5	30	30	**11.25**
a_4	3	-30	-5	20	45	7.50

$$\max(0,\ 8.75,\ 11.25,\ 7.50) = 11.25.$$

Therefore, the newsvendor chooses the action a_3.

R. 11. 6.
The Laplace rule is, in fact, a rule for decisions with risk for the special case that all states of nature occur with the same probability. It is based on the so-called *"principle of insufficient reason*". Laplace argued that "knowing nothing about the true state of nature" is equivalent to "all states having equal probability".

R. 11. 7. (***Maximin Rule/Wald***)
Choose the action a_k such that

$$S_k = \max_{i=1,2,\ldots,m} S_i := \max_i \min_j v_{ij} .$$

We call S_i the *security level* of a_i, i. e. a_i guarantees the decision maker a return of at least S_i.
We note that this rule is a very pessimistic criterion of choice; for its general philosophy is to assume that the worst will happen. It can also mean "pure fear".

Ex. 11. 4.
We illustrate the minmax rule by applying it to the newsvendor problem:

		s_1 0	s_2 1	s_3 2	s_4 3	$\min_j v_{ij}$
a_1	0	0	0	0	0	**0**
a_2	1	-10	15	15	15	-10
a_3	2	-20	5	30	30	-20
a_4	3	-30	-5	20	45	-30

$$\max(0,\ -10,\ -20,\ -30) = 0 .$$

Therefore, the newsvendor chooses the action a_1.

R. 11. 8. (***Minimax Rule***)
Choose the action a_k such that

$$w_{kj} = \min_i \max_j w_{ij} ,$$

where

$$w_{ij} := \max_{i,j} v_{ij} - v_{ij} .$$

Ex. 11. 5.
We illustrate the minimax rule by applying it to the newsvendor problem:

w_{ij}		s_1 0	s_2 1	s_3 2	s_4 3	$\max_j w_{ij}$
a_1	0	45	45	45	45	**45**
a_2	1	55	30	30	30	55
a_3	2	65	40	15	15	65
a_4	3	75	50	25	0	75

$$\min\left(45,\ 55,\ 60,\ 75\right) = 45 .$$

Therefore, the newsvendor chooses the action a_1.

R. 11. 9.
The maximin and the minimax rules will always lead to the same choice of action.

R. 11. 10. (***Maximax Rule***)
Choose the action a_k such that

$$v_{kj} = \max_i \max_j v_{ij} .$$

We note that this rule is a very optimistic criterion of choice; for its general philosophy is to assume that the best case will happen. It can also mean "pure greed" ("Go for the gold!")

Ex. 11. 6.
We illustrate the maximax rule by applying it to the newsvendor problem:

		s_1 0	s_2 1	s_3 2	s_4 3	$\max_j v_{ij}$
a_1	0	0	0	0	0	0
a_2	1	-10	15	15	15	15
a_3	2	-20	5	30	30	30
a_4	3	-30	-5	20	45	**45**

$$\max(0,\ 15,\ 30,\ 45) = 45 .$$

Therefore, the newsvendor chooses the action a_4 .

R. 11. 11. (***Minimax Regret Rule/Savage-Niehans***)
Choose the action a_k such that

$$v_{kj} = \min_i \max_j r_{ij} ,$$

where

$$r_{ij} = \max_i v_{ij} - v_{ij} .$$

The matrix $R := \left(r_{ij}\right),\ i = 1,2,...,m;\ j = 1,2,...,n,$ is the so-called *"regret matrix"*.
This rule might suggest "fear of guilt".

Ex. 11. 7.
We illustrate the *maximax* regret rule by applying it to the newsvendor problem:

r_{ij}		s_1 0	s_2 1	s_3 2	s_4 3	$\max_j r_{ij}$
a_1	0	0	15	30	45	45
a_2	1	10	0	15	30	30
a_3	2	20	10	0	15	**20**
a_4	3	30	20	10	0	30

$$\min(45,\ 30,\ 20,\ 30) = 20.$$

Therefore, the newsvendor chooses the action a_3.

R. 11. 12. ***(Pessimism-Optimism Rule/Hurwicz)***
Choose the action a_k such that

$$v_{kj} = \max_i \left\{ \alpha \cdot \max_j v_{ij} + (1-\alpha) \min_j v_{ij} \right\},$$
$$0 \le \alpha \le 1,$$

where

α: optimism index,
$1-\alpha$: pessimism index.

This rule might mean "combining greed and fear".

This approach attempts to strike a balance between the *maximax* and *maximin* criteria
A cautious decision maker will set $\alpha = 1$ which reduces the Hurwicz criterion to the *maximin* criterion. An adventurous decision maker will set $\alpha = 0$ which reduces the Hurwicz criterion to the *maximax* criterion.

Ex. 11. 8.
We illustrate the pessimism-optimism rule by applying it to the newsvendor problem:
Let $\alpha := 0.7$

		s_1 0	s_2 1	s_3 2	s_4 3	$\alpha \cdot \max_j v_{ij} + (1-\alpha)\min_j v_{ij}$
a_1	0	0	0	0	0	$0.3 \cdot 0 + 0.7 \cdot 0 = 0$
a_2	1	-10	15	15	15	$0.3 \cdot (-10) + 0.7 \cdot 15 = 7.5$
a_3	2	-20	5	30	30	$0.3 \cdot (-20) + 0.7 \cdot 30 = 15$
a_4	3	-30	-5	20	45	$0.3 \cdot (-30) + 0.7 \cdot 45 = \mathbf{22.5}$

$$\max_i(0;\ 7.5;\ 15;\ 22.5\ ) = 22.5\,.$$

Therefore, the newsvendor chooses the action a_4.

R. 11. 13.
The above rules do not take into account the probability associated with the outcomes for each alternative; they merely focus on the value of the outcomes. The criticism of decision criteria under uncertainty is aimed at their failure to include important information about the chance of each state occurring.

D. 11. 6. (***Decision under Risk***)
A decision problem under risk occurs when the decision maker does not know the true state of nature by certain, but he can qualify his uncertainty through a probability distribution.

R. 11. 14.
How should a decision maker choose in a situation of risk? We shall discuss the following rules:

(1) The $\mu-$Rule,
(2) The "Bernoulli"-Rule,
(3) The $\mu\sigma-$Rule.

R. 11. 15. (***The $\mu-$ Rule***)
Choose the action a_k such that

$$E_k = \max_i \sum_{j=1}^{n} v_{ij} \cdot p(s_j)\,,$$

where

$p(s_j)$: the probability for the occurrence of s_j,
E_k : the expected value of a_k.

D. 11. 7. (***Expected Monetary Value***)
We call

$$EMV = \max_i \sum_{j=1}^{n} v_{ij} \cdot p(s_j)$$

the *expected monetary value (EMV)*.

Ex. 11. 9.
We illustrate the $\mu-$rule by applying it to the newsvendor problem:

		s_1	s_2	s_3	s_4	
		$p(s_1)=0.10$	$p(s_2)=0.35$	$p(s_3)=0.40$	$p(s_4)=0.15$	
		0	1	2	3	E_i
a_1	0	0	0	0	0	0.00
a_2	1	-10	15	15	15	12.50
a_3	2	-20	5	30	30	**16.25**
a_4	3	-30	-5	20	45	10.00

$$\max(0.00,\ 12.50,\ 16.25,\ 10.00) = 16.25.$$

Therefore, the newsvendor chooses the action a_3. The expected monetary value for our problem is equal to 18.25.

D. 11. 8. (***Expected Value under Certainty***)
The expected value under certainty is defined to be:

$$EVUC = \sum_{j=1}^{n} \max_i v_{ij} \cdot p(s_j).$$

Ex. 11. 10.
The expected value under certainty for our newsvendor problem is

$$EVUC = 0 \cdot 0.10 + 15 \cdot 0.35 + 30 \cdot 0.40 + 45 \cdot 0.15 = 24.$$

D. 11. 9. (***Expected Value of Perfect Information***)

$$EVPI := EVUC - EMV.$$

Ex. 11. 11.
The expected value of perfect information for our newsvendor problem is

$$EVPI = 24 - 16.25 = 7.75.$$

R. 11. 16. ***(The "Bernoulli"-Rule)***
Choose the action a_k such that

$$E_k = \max_i \sum_{j=1}^{n} u_{ij}\left(v_{ij}\right) \cdot p(s_j),$$

where

$u_{ij}\left(v_{ij}\right),\ i = 1,2,...,m;\ j = 1,2,...,n$: the utility of v_{ij}.

R. 11. 17. ***(Risk attitude)***
The risk attitude is determined as follows:

$$u''(v)\begin{cases} < 0 & \text{risk-averse} \\ = 0 & \text{risk-neutral} \\ > 0 & \text{risk-seeking} \end{cases}$$

Ex. 11. 12
We illustrate the Bernoulli rule by applying it to the newsvendor problem using the utility function

$$u_{ij} = -0.02v_{ij}^2 + 3v_{ij},\quad i, j = 1,2,3,4:$$

u_{ij}		s_1	s_2	s_3	s_4	
		$p(s_1) = 0.10$	$p(s_2) = 0.35$	$p(s_3) = 0.40$	$p(s_4) = 0.15$	
		0	1	2	3	E_i
a_1	0	0.0	0.0	0.0	0.0	0.000
a_2	1	-32.0	40.5	40.5	40.5	33.250
a_3	2	-68.0	14.5	72.0	72.0	37.875
a_4	3	-108.0	-15.5	52.0	94.5	18.750

$$\max(0.000,\ 32.250,\ 37.875,\ 18.750) = 37.875.$$

Therefore, the newsvendor chooses the action a_3.

R. 11. 18. ***(The $\mu - \sigma$-Rule)***
Choose the action a_k such that

$$\Phi_k = \max_i \Phi\left(\mu_i, \sigma_i\right),$$

where

$\Phi\left(\mu, \sigma\right)$: *preference function*.

R. 11. 19. (***Risk attitude***)
The risk attitude is determined as follows:

$$\frac{\partial \Phi}{\partial \sigma}\begin{cases} <0 & \text{risk-averse} \\ =0 & \text{risk-neutral} \\ >0 & \text{risk-seeking} \end{cases}$$

Ex. 11. 13.
We illustrate the Bernoulli rule by applying it to the newsvendor problem using the preference function

$$\Phi(\mu,\sigma) = 5\mu - 0.5\sigma.$$

$\mu_1 = 0.00, \quad \mu_2 = 12.50, \quad \mu_3 = 16.25, \quad \mu_4 = 10.00.$

$\sigma_1 = 0.00, \quad \sigma_2 = 7.50, \quad \sigma_3 = 16.7238602, \quad \sigma_4 = 21.50581317.$

		s_1	s_2	s_3	s_4			
		$p(s_1) = 0.10$	$p(s_2) = 0.35$	$p(s_3) = 0.40$	$p(s_4) = 0.15$			
		0	1	2	3	μ_i	σ_i	Φ_i
a_1	0	0	0	0	0	0.00	0.00	0.00
a_2	1	-10	15	15	15	12.50	7.50	58.75
a_3	2	-20	5	30	30	16.25	16.72	**72.89**
a_4	3	-30	-5	20	45	10.00	21.51	39.25

$$\max(0.00,\ 78.75.25,\ 72.89,\ 39.25) = 72.89.$$

Therefore, the newsvendor chooses the action a_3.

R. 11. 20. (***Decision Tree***)
A *decision tree* is a pictorial representation of a decision situation, normally found in discussions of decision-making under uncertainty or risk. It shows decision alternatives, states of nature, probabilities attached to the state of nature, and conditional benefits and losses. The tree approach is most useful in a sequential decision situation.
The decision tree is read from left to right. The leftmost node in a decision tree is called the *root node* (or the *decision* node) and is represented by a small square.
The branches emanating to the right from a decision node represent the set of decision alternatives that are available. One, and only one, of these alternatives can be selected. The small circles in the tree are called *chance nodes*. The number shown in parentheses on each branch of a chance node is the probability that the outcome shown on that branch will occur at the chance node.

The right end of each path through the tree is called an *endpoint*, and each point represents the final outcome of following a path from the root node of the decision tree to that endpoint.

R. 11. 21. (***Drawing a Decision Tree Step-by-Step***)

***Step 1*: Grow the decision tree.

***Step 2*: Assign probabilities to the event outcomes on the tree.
***Step 3*: Assign the cash flows to the tree.

***Step 4*: Fold back the decision tree and compute the expected values for each decision.

Ex. 11. 14. (***Bighorn Oil Company***)[3]
Bighorn Oil Company has leased the drilling rights on a large parcel of land in Wyoming that may or may not contain an oil reserve. A competitor has offered to lease the land for $200000 cash in return for drilling rights and all rights to any oil that might be found.
The offer will expire in three days. If Bighorn does not take the deal, it will be faced with the decision of whether to drill for oil on its own. Drilling costs are projected to be $400000. The company feels that there are four possible outcomes from drilling:

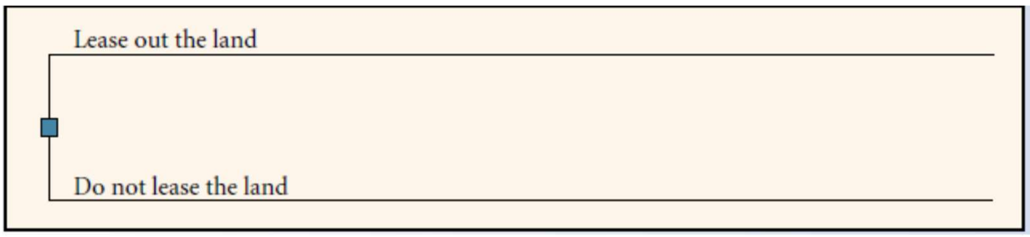
1. dry hole (no oil or natural gas)
2. natural gas
3. natural gas and some oil
4. oil only

If drilling yields a dry hole, the land will be basically worthless, because it is located in the badlands of Wyoming. If natural gas is discovered, Bighorn will recover only its drilling costs. If natural gas and some oil is discovered, revenue is projected to be $800,000. Finally, if only oil is discovered, revenues will be $1,600,000.

Draw a suitable decision tree

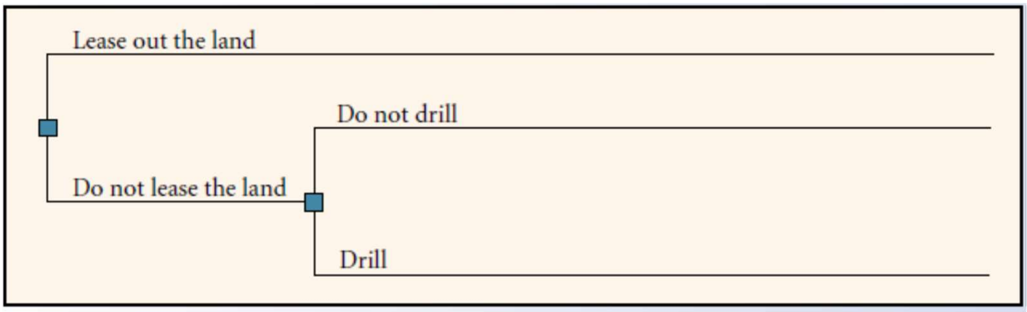
Solution:

***Step 1*: Grow the decision tree.
The initial decision to be made is whether to accept the lease:

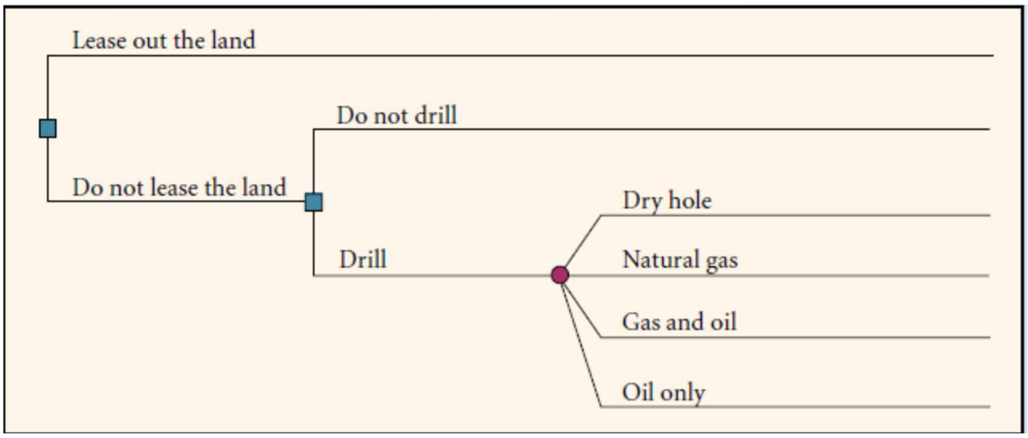


If the land is leased, no further decisions are required. However, if the land is not leased, Bighorn faces the decision of whether to drill on the property. The tree then grows to:

[3] Groebner, D. F.; Shannon, Patrick. W.; Frey, Phillip. C.; Smith, Kent. D.: Business Statistics , 2012



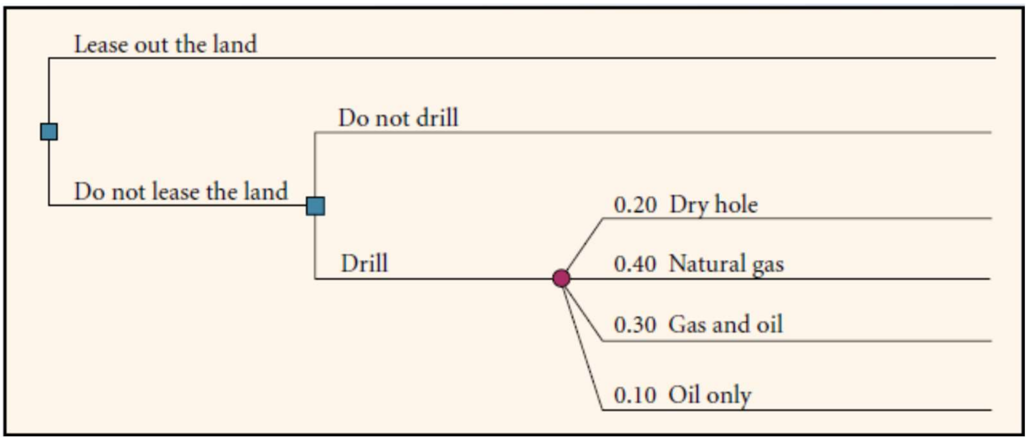
Now, if Bighorn decides to drill, there are four possible events that could occur:



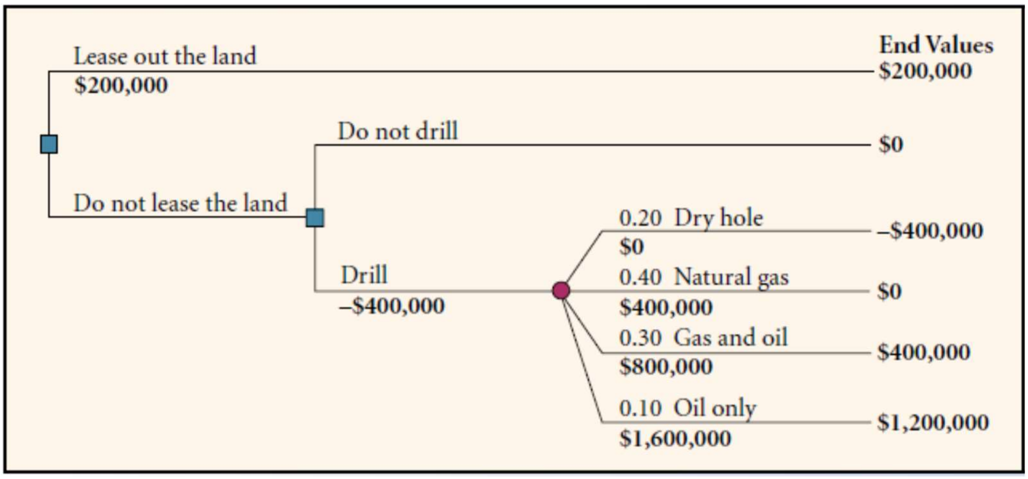
Step 2**:** Assign probabilities to the event outcomes on the tree.
In this example, the only event deals with the production result if Bighorn decides to drill. The company has subjectively assessed the probability of each of the four possible outcomes as follows:

Outcome	Probability
Dry hole	0.20
Natural gas	0.40
Gas and Oil	0.30
Oil only	0.10

The revised decision tree reflects these probabilities and becomes:



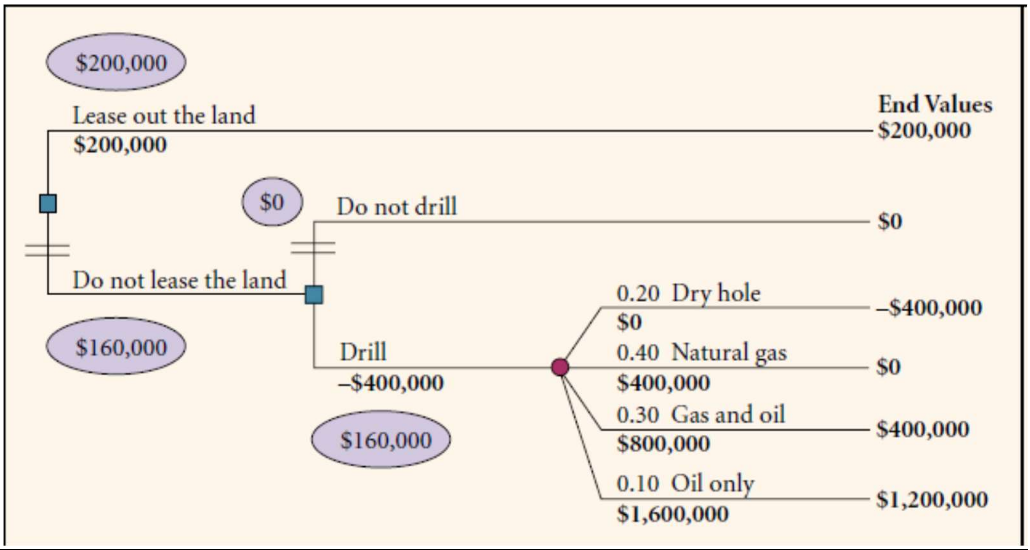
***Step 3*:** Assign the cash flows to the tree.
At each branch of the tree at which a revenue or cost occurs, show the dollar value. These revenues and costs are then totaled across the tree, and the end values for each branch are determined. These cash flows are palace on the tree as follows:



***Step 4*:** Fold back the decision tree and compute the expected values for each decision.
We need to compute the expected value for each decision alternative. This is done from the right side of the tree and working back to the left. We first determine:

$$E(\text{"Drill"}) = -400000 \cdot 0.20 + 0 \cdot 0.40 + 400000 \cdot 0.30 + 1200000 \cdot 0.10 = 160000$$

As we fold back the tree, we block all decision alternatives that do not have the highest expected value. This is shown in the decision tree as follows:



Note that we always select the decision with the highest expected payoff. In this example, the *best decision* is to lease the land and accept $200000 payment because it exceeds the $160000 expected value of the non-lease option.

Chapter 11

Decision Theory
Exercises

11. 1.
A bicycle store would like to order bicycles for the coming month. Orders for the bicycles must be placed in quantities of 20. The cost per bicycle is $70 dollars if they order 20, $67 if they order 40, $65 if they order 60 and $64 if they order 80.
The bicycles will be sold for $100 each. Any bicycles left over at the end of month can be sold at $45 each.
If the bicycle store runs out of bicycles, then it will suffer a loss of "goodwill" among its customers. This goodwill is estimated to be $5 per customer who was unable to buy a bicycle. It is further estimated that the demand for bicycles in coming month will be 10, 30, 50, or 70, with probabilities of 0.2, 0.4, 0.3, and 0.1 respectively.

1. Create a payoff table.
2. Find the best action under each of the following decision criteria:

   i. Maximax
   ii. Maximin
   iii. Minimax
   iv. Minimax regret
   v. Expected value

11. 2.
Mrs. Greedy is thinking of investing in the stock market. Suppose she is considering four alternatives: investing $8000, $4000, $2000, or $1000. These are the four choices that are within her control. The consequences of her investment, in terms of her profit or loses, are dependent on the market and beyond her control.
Mrs. Greedy's a *payoff table* is as follows:

Profit

	$p_1 = 0.1$	$p_2 = 0.5$	$p_3 = 0.4$
Invest:	Strong market	Fair market	Poor market
$8000	$800	$200	-$400
$4000	$400	$100	-$200
$2000	$200	$50	-$100
$1000	$100	$25	-$50

Given the utility function

$$u(v) = 2v - \frac{v^2}{100000},$$

1. Find an "optimal" investment strategy.
2. Determine the decision-maker's risk attitude.

11. 3.

A dress buyer for a large department store must place order with a dress manufacturer 9 months before the dresses are needed. One decision is as to the number of knee-length dresses to stock. The ultimate gain to the department store depends both on this decision and on the fashion prevailing 9 months later.

The buyer's estimate of gains (in thousands of €) are given in the following table:

	Knee lengths are high fashion	Knee lengths are acceptable	Knee lengths are not acceptable
Probability	0.40	0.35	0.25
Order none	-50	0	80
Order a little	-10	30	35
Order moderately	60	45	-30
Order a lot	80	40	-45

Given the following preference function

$$\Phi(a_i) = 2\mu_i - 0.05\sigma_i,$$

recommend a decision.

11. 4.

A company is investigating the possibility of producing and marketing backyard storage sheds. Undertaking this project would require the construction of either a large or a small manufacturing plant. The market for the product produced – storage sheds – could either be favourable or unfavourable. The company has, of course, the option of not developing the new product at all.

Develop a decision tree for this situation.

Chapter 11

Decision Theory

Solutions

11. 1.

1.

Payoff Table

Action	State of Nature			
	s_1 : demand 10 $p_1 = 0.2$	s_1 : demand 30 $p_2 = 0.4$	s_1 : demand 50 $p_3 = 0.3$	s_1 : demand 70 $p_4 = 0.1$
a_1 : buy 20	50	550	450	350
a_2 : buy 40	-330	770	1270	1170
a_3 : buy 60	-650	450	1550	2050
a_4 : buy 80	-970	130	1230	2330

(*Explanation:*

1. Demand is 50, buy 60:
   The store buys 60 at $65 each for $3900. That is -$3900 since that is the money it spends. Now, it sells 50 bicycles at $100 each for $5000. The store has 10 bicycles left over at the end of month, and it sells those at $45 each of $450. That makes

   5000 + 450 -3900 = $1550

2. Demand is 70, buy 40:
   The store buys 40 at $67 each for $2680. That is -$2680 since that is money it spends. Now it sells 40 bicycles at $100 each for $4000. The other customers that want a bicycle und cannot get one cost the store $5 in goodwill for each of them. That is 30 customers at -$5 each or -$150. That makes:

   4000 – 2680 -150 =$1170.)

i. Maximax

v_{ij}		s_1 10	s_2 30	s_3 50	s_4 70	$\max_j v_{ij}$
a_1	20	50	550	450	350	550
a_2	40	-330	770	1270	1170	1270
a_3	60	-650	450	1550	2050	2050
a_4	**80**	-970	130	1230	2330	**2330**

$\therefore$ The store buys 80.

ii. Maximin

v_{ij}		s_1 10	s_2 30	s_3 50	s_4 70	$\min_j v_{ij}$
a_1	**20**	50	550	450	350	**50**
a_2	40	-330	770	1270	1170	-330
a_3	60	-650	450	1550	2050	-650
a_4	80	-970	130	1230	2330	-970

$\therefore$ The store buys 20.

iii. Minimax

w_{ij}		s_1 10	s_2 30	s_3 50	s_4 70	$\max_j w_{ij}$
a_1	**20**	2280	1680	1880	1980	**2280**
a_2	40	2660	1560	1060	1160	2660
a_3	6**0**	2980	1880	780	280	2980
a_4	80	3300	2200	1100	0	3300

$\therefore$ The store buys 20.

iv. Minimax regret

r_{ij}		s_1 10	s_2 30	s_3 50	s_4 70	$\max_j r_{ij}$
a_1	20	0	220	1100	1980	1980
a_2	40	380	0	280	1160	1160
a_3	**60**	700	320	0	280	**700**
a_4	80	1020	640	320	2330	2330

∴ The store buys 60.

v. Expected value

		s_1	s_2	s_3	s_4	
		0.2	0.4	0.3	0.1	
		10	30	50	70	μ_i
a_1	20	50	550	450	350	400
a_2	**40**	-330	770	1270	1170	**740**
a_3	60	-650	450	1550	2050	720
a_4	80	-970	130	1230	2330	460

∴ The store buys 40.

11. 2.

1.

Utility Matrix

	$p_1 = 0.1$	$p_2 = 0.5$	$p_3 = 0.4$	
Invest:	Strong market	Fair market	Poor market	μ_i
\$8000	1593.6	399.6	-801.6	38.52
\$4000	798.4	199.9	-400.4	19.63
\$2000	399.6	99.98	-200.1	9.91
\$1000	199.9	49.99	-100.03	4.98

$$Max\{38.74,\ 19.63,\ 9.91,\ 4.98\}=38.74$$

Therefore, Mrs. Greedy should invest \$8000.

2.

$$u'(v) = 2 - \frac{2v}{100000}, \quad u''(v) = -\frac{1}{50000} < 0.$$

Therefore, Mrs. Greedy is risk averse.

11. 3.

Probabilities	0.40	0.35	0.25			
	s_1	s_2	s_3	$\mu(a_i)$	$\sigma(a_i)$	$\Phi_i(\mu_i,\sigma_i)$
a_1	-50	0	80	0.00	50.99	-2.5495
a_2	-10	30	35	15.25	20.70	29.4650
a_3	60	45	-30	32.25	36.52	62.6740
a_4	80	40	-45	34.75	49.18	**67.0410**

$\therefore$ The best decision would be to order a lot.

11. 4.