Mathematics for Students of Economics

Introduction to Logic

D. 1. (*Proposition*)
A *proposition* is a statement, which is either *true* (T) or *false* (F).

Ex. 1.
Say whether each of the following sentences is a proposition. In case of a proposition, determine its truth value:

1. The capital of Germany is Berlin.
2. How old is your father?
3. $5 \cdot 2 = 20$.
4. Switch the radio on.
5. Every even number greater than 2 is the sum of two primes.
6. $x > 13,\ x \in R$

Solution:
1. A proposition with the truth value T.
2. No proposition.
3. A proposition with the truth value F.
4. No proposition.
5. A proposition whose truth value is not known, the so-called *Goldbach's conjecture*.
6. No proposition. Substituting a real number for x will turn the statement into a proposition having a truth value.
   (Such a proposition is sometimes called a *proposition form*.)

R. 1. (*Two Principles*)

1. A proposition is either true or false.
2. A Proposition cannot be at the same time true and false.

(The two principles will later be formulated as assertions and proved.)

D. 2. (*Logical Quantifiers*)
We have the following *quantifiers*:

1. The *universal quantifier*: $\forall$ means "for all", "for every".
2. The *existence quantifier*: $\exists$ means "there exists at least one".
3. The *extended existence qualifier*: $\exists!$ "there exists exactly one".

R. 2. (*Logical Connectors*)
We have the following *connectors*:

1. Negation
2. Disjunction
3. Conjunction
4. Implication
5. Equivalence

D. 3. (*Negation*)
The *negation* of p is the proposition $\neg p$ which is true if, and only if, p is false:

p	$\neg p$
T	F
F	T

Ex. 2.
p: "Everybody knows Einstein."
$\neg p$: "At least one person does not know Einstein."

D. 4. (*Disjunction*)
The proposition p *or* q is true if, and only if, at least one of the two propositions is true. The *disjunction* will be denoted by $\vee$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex. 3.
Denote by

p: "$20:4=5$"
q: "$2>10$".

The (compound) proposition "$20:4=5$ or $2>10$" is true.

D. 5. (*Conjunction*)
The proposition p *and* q is true if, and only if, both propositions are true. The *conjunction* will be denoted by $\wedge$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex. 4.
Denote by

p: "$20:4=5$"
q: "$2>10$".

The (compound) proposition "$20:4=5$ and $2>10$" is false.

D. 6. (*Implication*)
The proposition p *implies* q is false if, and only if, p is true and q is false. The *implication* will be denoted by $\Rightarrow$.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex. 5.
Denote by

p: "$20:4=5$"
q: "$2>10$".

1. The (compound) proposition "If $20:4=5$ then $2>10$" is false.
2. The (compound) proposition "If $2>10$ then $20:4=5$" is true.

R. 2. (*Sufficient Condition, Necessary Condition*)
In the implication

$$p \Rightarrow q$$

p is the *sufficient condition* for q to be fulfilled; q is the *necessary condition* for p to be fulfilled.

D. 7. (*Equivalence*)
The *equivalence* $p \Leftrightarrow q$ is true whenever p and q have the same logical value.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex. 6.
Denote by

p: "$20:4=6$"
q: "$2>10$".

The compound proposition "$20:4=6$ if and only if $2>10$" is true.

R. 3. (*Sufficient and Necessary Condition*)
In the equivalence

$$p \Leftrightarrow q$$

p is both *sufficient and necessary condition* for q to be fulfilled; q is both *necessary and sufficient condition* for p to be fulfilled.

R. 4. (*Or Exclusion*)
In electronics and some programming languages, there is also an *or exclusion* (the so called *xor connector* defined as follows:
Let p and q be two propositions. The connection $p \ xor \, q$ is true if, and only if, p and q have different logical values.

p	q	$p \ xor \ q$
T	T	F
T	F	T
F	T	T
F	F	F

R. 5.
The following *truth table* summarises the above-mentioned logical connectors for the two propositions p and q:

Truth Table

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$	$p \ xor \ q$
T	T	F	T	F	T	T	F
T	F	F	T	F	F	F	T
F	T	F	T	F	T	F	T
F	F	T	F	T	T	T	F

D. 8. (*Tautology*)
A proposition which is always true is called a *tautology*.

D. 9. (*Contradiction*)
A proposition which is always false is called a *contradiction*.

Th. 1. (*See R.1.*)

1. A proposition is either true or false.
2. A Proposition cannot be at the same time true and false.

Proof:
Let p be a proposition.

1.
We prove the compound proposition:

$$p \vee \neg p$$

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

We have thus proved that $p \vee \neg p$ is a tautology.

2.
We prove the compound proposition:

$$\neg(p \wedge \neg p)$$

p	$\neg p$	$p \wedge \neg p$	$\neg(p \wedge \neg p)$
T	F	F	T
F	T	F	T

We have thus proved that $p \wedge \neg p$ is a contradiction and $\neg(p \wedge \neg p)$ is a tautology.

The above rules are also known as *rules of complementation*.

R. 6. (*Some Rules*)
Let p, q, and r be three propositions. Then we have the following rules:

1. *Commutative*

$$p \vee q \Leftrightarrow q \vee p$$
$$p \wedge q \Leftrightarrow q \wedge p$$

2. *Associative*

$$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

3. *Distributive*

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$
$$p \vee (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

4. *Idempotent*

$$(p \vee p) \Leftrightarrow p$$
$$(p \wedge p) \Leftrightarrow p$$

5. *Absorption*

$$p \vee (p \wedge q) \Leftrightarrow p$$
$$p \wedge (p \vee q) \Leftrightarrow p$$

6. *Involution*

$$\neg\neg p \Leftrightarrow p$$

7. De Morgan

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$
$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

Exercises

1.
Identify each of the following as a proposition or not a proposition. In the case of a proposition, determine its value:

1. Lagrange was a mathematician.
2. $5 \cdot 2 = 11$.
3. How will the US economy develop in the next decade?
4. Every integer greater than 5 can be written as the sum of three primes.

2.
Three companies C_1, C_2, and C_3 have been suspected of having started a “price war”.
An expert believes:

a) At least one of the companies started the „price war“.
b) If C_1 and C_2 were not responsible for the “price war”, then C_3 should be excluded as suspect.
c) If C_1 was responsible or C_3 not, then C_2 should be excluded as suspect.

Denote by:

p_1 :“ C_1 is responsible“.
p_2 :“ C_2 is responsible“.
p_3 :“ C_3 is responsible“.

1. Use p_1, p_2, and p_3 to describe the expert´s belief.
2. Formulate a propositional connector including *all* three expert’s propositions.
3. Try to find the company truly responsible for having started the “price war”.

3.
Let x be a real number. The validity of the inequality

$$e^x \geq 2$$

should be investigated.
Are the following conditions sufficient, necessary or necessary and sufficient for the above inequality to hold?

1. $x \geq 0$,
2. $x > 4$,
3. $x \geq \ln 2$,
4. x is integer.

Solutions

1.

1. Proposition (True)
2. Proposition (False)
3. No proposition
4. Proposition Truth unknown value (known as The Goldbach Conjecture).

2.

1.

a) $p_1 \vee p_2 \vee p_3$

b) $\neg(p_1 \wedge p_2) \Rightarrow \neg p_3$

c) $(p_1 \vee \neg p_3) \Rightarrow \neg p_2$

2.

$$p := (p_1 \vee p_2 \vee p_3) \wedge (\neg(p_1 \wedge p_2) \Rightarrow \neg p_3) \wedge ((p_1 \vee \neg p_3) \Rightarrow \neg p_2)$$

3.

$p_1 \vee p_2 \vee p_3$:

p_1	p_2	p_3	$p_1 \vee p_2 \vee p_3$
T	T	T	T
T	T	F	T
T	F	T	T
F	T	T	T
T	F	F	T
F	T	F	T
F	F	T	T
F	F	F	F

$\neg(p_1 \wedge p_2) \Rightarrow \neg p_3$

p_1	p_2	p_3	$p_1 \wedge p_2$	$\neg(p_1 \wedge p_2)$	$\neg p_3$	$\neg(p_1 \wedge p_2) \Rightarrow \neg p_3$
T	T	T	T	F	F	T
T	T	F	T	F	T	T
T	F	T	F	T	F	F
F	T	T	F	T	F	F
T	F	F	F	T	T	T
F	T	F	F	T	T	T
F	F	T	F	T	F	F
F	F	F	F	T	T	T

$(p_1 \vee \neg p_3) \Rightarrow \neg p_2$

p_1	p_2	p_3	$\neg p_3$	$p_1 \vee \neg p_3$	$\neg p_2$	$(p_1 \vee \neg p_3) \Rightarrow \neg p_2$
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	F	T	F	T	T	T
F	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	F	T	T	F	F
F	F	T	F	F	T	T
F	F	F	T	F	T	T

p_1	p_2	p_3	$p_1 \vee p_2 \vee p_3$	$\neg(p_1 \wedge p_2) \Rightarrow \neg p_3$	$(p_1 \vee \neg p_3) \Rightarrow \neg p_2$	p
T	T	T	T	T	F	F
T	T	F	T	T	F	F
T	F	T	T	F	T	F
F	T	T	T	F	T	F
T	F	F	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	F	T	F
F	F	F	F	T	T	F

$\therefore$ C_1 is responsible.

3.

1. Necessary: $e^x \geq 2 \Rightarrow x \geq 0$
   Not sufficient $x = 0 \Rightarrow e^x < 2$;

2. Sufficient: $x > 4 \Rightarrow e^x \geq 2$
   Not necessary: $x = 2 \Rightarrow e^x \geq 2$;

3. Necessary and sufficient: $e^x \geq 2 \Leftrightarrow x \geq \ln 2$;

4. Neither sufficient nor sufficient: $x = -10$ does not imply $e^x \geq 2$,

$$x = 2.5 \;\Rightarrow\; e^x \geq 2\,.$$

Introduction to the Sets Theory

<u>D. 1</u>. (*Set, Elements of a Set*)
A *set* is any well-defined collection of "objects". The *elements* of a set are the object in a set.

<u>R. 1.</u> (*Notation*)
Usually, we denote sets with upper-case letters, elements with lower-case letters.
$x \in S$ means the element x belongs to the set S; $x \notin S$ means the element x does not belong to the set S.

<u>D. 2.</u> (*Empty or Null Set*)
An *empty* or a *null set* has no members at all. It is denoted by $\varnothing$.

<u>D. 3.</u> (*Universal Set*)
The set of all elements currently under consideration is called a *universal set*. It is symbolized by Ω.

<u>D. 4.</u> (*Singleton*)
A set with only one element is called a *singleton*.

<u>R. 2.</u> (*Specification of Sets*)
There are three main ways to specify a set:

1. by listing all its elements (*list notation*)
2. by listing properties of its elements (*predicate notation*)
3. by defining a set of rules which defines its elements (*recursive rules notation*)

<u>R. 3.</u> (*Operations on Sets*)

1. Subset
2. Union
3. Intersection
4. Difference
5. Complement
6. Cartesian product (will be dealt with a little later).

<u>D. 5.</u> (*Subset, Proper Subset*)
Given two sets A and B, B is a *subset* of A, denoted by $B \subseteq A$, if every element in B belongs also to A:

$$B \subseteq A := \{x \mid x \in B \Rightarrow x \in A\}.$$

Set B is a *proper set* of A, denoted by $B \subset A$, if there exists an element in A that does not belong to B.

D. 3. (***Equality***)

$$A = B \Leftrightarrow \left(A \subseteq B \wedge B \subseteq A \right).$$

D. 6. (***Union***)
Given two sets A and B, the *union* of A and B, denoted by $A \cup B$, is the set that contains elements belonging to *at least* of the two sets:

$$A \cup B := \left\{ x \mid x \in A \vee x \in B \right\}.$$

D. 7. (***Intersection***)
Given two sets A and B, the *intersection* of A and B, denoted by $A \cap B$, is the set that contains elements belonging to *both* sets:

$$A \cap B := \left\{ x \mid x \in A \wedge x \in B \right\}.$$

D. 8. (***Disjoint Sets***)
Two sets A and B are called *disjoint*, if

$$A \cap B = \varnothing.$$

D. 9. (***Difference***)
Given two sets A and B, the *difference* of A and B, denoted by $A \setminus B$, is the set of elements of A which are not also elements of B:

$$A \setminus B := \left\{ x \mid x \in A \wedge x \notin B \right\}.$$

D. 10. (***Complement***)

Given a set A, the *complement* of A, denoted by $\bar{A}$ is the set of all elements in the universal set Ω, but not in A:

$$\bar{A} := \left\{ x \in \Omega \mid x \notin A \right\}.$$

Th. 1. (***Some Laws***)
For any arbitrary sets A, B, C, we have:

1. The distributive laws:

$$A \cup \left(B \cap C \right) = \left(A \cup B \right) \cap \left(A \cup C \right)$$

$$A \cap \left(B \cup C \right) = \left(A \cap B \right) \cup \left(A \cap C \right)$$

2. The de Morgan's laws:

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

<u>Ex. 1.</u>
Given the sets

$$A = \{x \in R^1 \mid -1 < x < 3\}$$
$$B = \{x \in R^1 \mid -8 < x\}$$
$$C = \{x \in R^1 \mid -8 < x < 1\},$$

construct the following sets:

$$A \cap B, \qquad A \cup C, \qquad \bar{B}, \qquad \bar{A} \cup \bar{C}, \qquad \bar{A} \cap \bar{C}, \qquad B \setminus A, \qquad C \setminus B.$$

Solution:

$$A \cap B = A,$$

$$A \cup C = \{x \in R^1 \mid -8 < x < 3\},$$

$$\bar{B} = \{x \in R^1 \mid x \le -8\},$$

$$\bar{A} \cup \bar{C} = \{x \in R^1 \mid x \le -1 \vee x \ge 1\}, \quad \bar{A} \cap \bar{C} = \{x \in R^1 \mid x \le -8 \vee x \ge 3\},$$

$$B \setminus A = \{x \in R^1 \mid -8 < x \le -1 \vee x \ge 3\},$$

$$C \setminus B = \varnothing$$

<u>D. 11.</u> (*Cardinality*)
If a set S has $n \in N$ distinct elements, n is the *cardinality* of S and S is a *finite* set. The cardinality of S is denoted by $|S|$.

<u>Ex. 2.</u>
The set $S = \{1, 3, 5, 7, 9\}$ has the cardinality $|S| = 5$.

<u>Th. 2.</u>
For an arbitrary set S the following assertions hold:

1. $S \subseteq \Omega$
2. $\varnothing \in S$
3. $S \subseteq S$.

<u>D. 12.</u> (*Power Set*)
The set of all subsets of a set S is called the *power set* of S and is denoted by 2^S or $P(S)$.

<u>Ex. 3.</u>
Give a list notation of the following set and its power set:

$$S = \{x \in \mathbb{R}^1 \mid (x-1)\cdot(x+2)\cdot(x-3) = 0\}.$$

Solution:

$$S = \{1, -2,\ 3\}$$

$$2^S = \{\{\ \}, \{1\}, \{-2\}, \{3\}, \{1,-2\}, \{1,3\}, \{-2,3\}, \{1,-2,3\}\}$$

<u>Th. 3.</u>
For an arbitrary set S, the number of subsets of S is $2^{|s|}$.

<u>D. 13.</u> (*Ordered Pair*)
An *ordered pair* is the set of two elements of two elements a and b in which the order of elements matters: (a,b).

<u>D. 14.</u> (*Cartesian Product*)
Given two sets A and B, the *(Cartesian) product* of A and B, denoted by AxB, is the set of all ordered pairs such that $a \in A$ and $b \in B$:

$$\{(a,b) \mid a \in A \ \wedge \ b \in B\} =: AxB\ .$$

<u>Ex. 4.</u>
Given the sets $A = \{a,\ \ b\},\ \ B = \{c,\ \ d\}$, we have:

$$AxB = \{(a,\ c),\ (a,\ d),\ (b,\ c),\ (b,\ d)\},.$$

$$BxA = \{(c,\ \ a),\ \ (d,\ \ a),\ \ (c,\ \ b),\ \ (d,\ \ b)\}.$$

<u>Th. 4.</u>

$$AxB \neq BxA\ .$$

<u>R. 4.</u>
Given the elements $a_1, a_2, ..., a_n$, the Cartesian product can be generalized to:

$$\{(a_1,\ a_2,\ ..., a_n) \mid a_1 \in A,\ a_2 \in A,\ ..., a_n \in A\} =: A^n\ .$$

For <u>real</u> numbers $x_i,\ i = 1,2,...,n,$ we have

$$R^n = \{(x_1,\ x_2,\ ..., x_n)\ \mid x_i \in R^1\}.$$

<u>D. 15.</u> (*Mapping*)
Given the sets X and Y, the set $A \subseteq XxY$ is called a *mapping of* X in Y. In an arbitrary ordered pair $(x,y) \in A,\ y$ is the *image* of x and x is the *preimage* or *inverse image* of y.

<u>D. 16.</u> (*Domain, Range*)
Let $A \subseteq XxY$ be a mapping of X in Y.

The set

$$D(A) := \{x \in X \mid \exists y \in Y \text{ with } (x,y) \in A\}$$

is called the *domain of A.*

The set

$$R(A) := \{y \in Y \mid \exists x \in X \text{ with } (x,y) \in A\}$$

is called the *range of A.*

D. 17. (***Function)***

A *function* is a special mapping where every element of X is associated with a unique element of Y.

We write

$$f : X \to Y \quad (\text{or } f(x) = y)$$

D. 18. (***Injective, Surjective and Bijective Functions***)**:**

1.

A function is *injective* (*one-to-one*) if

$$\langle \forall x_1, x_2 \in X, f(x_1) = f(x_2) \rangle \;\Rightarrow\; x_1 = x_2$$

2.

A function is *surjective* (*onto)* if

$$\forall y \in Y,\ \exists x \in X : f(x) = y$$

3.

A function is *bijective* if it is both injective and surjective. A bijective function is also called a *bijection* (a *one-to-one correspondence*.):

$$\forall y \in Y,\ \exists \text{ unique } x \in X : f(x) = y$$

D. 19. (***Inverse Mapping***)**:**

Let $A \subseteq XxY$ be a bijective mapping of X in Y. The set

$$A^{-1} := \{(y,x) \mid (y,x) \in YxX \wedge (x,y) \in A\}$$

is called the *inverse mapping of A* .

Th. 5.
Let $A \subseteq XxY$ be a bijective mapping of X in Y.

1.
$$D(A^{-1}) = R(A) \text{ and } D(A) = R(A^{-1}).$$

2.
$$(A^{-1})^{-1} = A.$$

Ex. 5.
1.
$$X = \{a,b,c\}, \qquad Y = \{1,2,3,4\}$$

$$A = \{(a,1),(a,2),(b,3),(c,4)\} \subseteq AxB$$ no function

2.
$$X = \{a,b,c,d\}, \qquad Y = \{1,2,3,4,5\}$$

$$A = \{(a,2),(b,1),(c,3),(d,4)\} \subseteq AxB$$ injective (not surjective)

$$D(A) = \{a,b,c\} = X; \quad R(A) = \{1,2,3\} \subset Y$$

3.
$$X = \{a,b,c,d,e\}, \qquad Y = \{1,2,3,4\}$$

$$A = \{(a,2),(b,1),(c,3),(d,3),(e,4)\} \subseteq AxB$$ surjective (not injective)

$$D(A) = \{a,b,c,d,e\} = X; \quad R(A) = \{1,2,3,4\} = Y$$

4.
$$X = \{a,b,c,d,e\}, \qquad Y = \{1,2,3,4,5\}$$

$$A = \{(a,2),(b,1),(c,5),(d,3),(e,4)\} \subseteq AxB$$ bijective (injective and surjective)

$$D(A) = \{a,b,c,d,e\} = X; \quad R(A) = \{1,2,3,4\} \subset Y$$

R. 5. ***(Real functions).***
For $X = R^n$, $n = 1,2...$ we speak of a real function.

D. 20. (***Graph of a function of a single variable***)**:**
Consider $f : X \to R^1$. The following set is called the graph of the function f:

$$graph\, f := \{(x,y) \in XxY \,|\, f(x) = y\}$$

Ex. 6.

$$f(x) = \begin{cases} -1 & x \le -1 \\ x & -1 < x \le 1 \\ 1 & x > 1 \end{cases}$$

Ex. 7

1. Consider the function

   $f : R^1 \to R^1$ with $Y = f(x) = 1 + x.$

   The function is bijective with $x = f^{-1}(y) = y - 1$

2. Consider the function

   $f : R^1 \to R^1$ with $Y = f(x) = x^2.$

   The function is not bijective.

   However, if we decompose the original domain in the following subdomains:

   $D_1 = \{x \in R^1 | x \ge 0\}$ and $D_2 = \{x \in R^1 | x < 0\}$,

   then both functions

   $y = f(x) = x^2,\ x \in D_1$ and $y = f(x) = x^2,\ x \in D_2$

   will be bijective with the inverse functions:

   $x = f_1^{-1}(y) = \sqrt{y},\ \forall x \in D_1$ and $x = f_2^{-1}(y) = -\sqrt{y},\ \forall x \in D_2$

D. 21. (***Relation***)
A binary relation on a set A is a mapping $R \subseteq AxA$. Instead of writing $a, b \in A$ it will be denoted by $a\ R\ b$

D. 22. (***Properties of Relations***)
Consider the set A and a relation $R \subseteq AxA$
The relation is called

- *reflexiv* if $x\ R\ x$ for all $x \in A$,

- *symmetric* if $x\ R\ y \ \Rightarrow \ y\ R\ x$ for all $x, y \in A$,
- *antisymmetric* if $[x\ R\ y \ \wedge \ y\ R\ x] \ \Rightarrow \ x = y, \ \forall x, y \in A$,
- *transitiv* if $[x\ R\ y \ \wedge \ y\ R\ z] \ \Rightarrow \ x\ R\ z, \ \forall x, y, z \in A$.

Ex. 8.

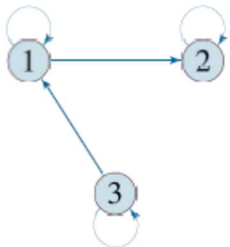
Consider the set $A = \{1,2,3\}$

1. The relation

$$R = \{(1,1),(1,2),(2,2),(3,3),(3,1)\}$$

is on set A is <u>reflexive</u>.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$



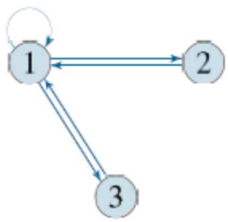
Reflexive relations are always represented by a matrix that has 1 on the main diagonal. The digraph of a reflexive relation has a loop from each node to itself.

2. The relation

$$R = \{(1,1),(1,2),(2,1),(1,3),(3,1)\}$$

is on set A <u>symmetric</u>.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

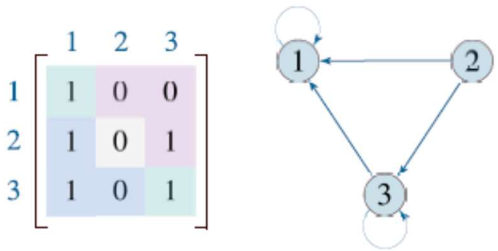


For a symmetric relation, the logical matrix is symmetric about the main diagonal. The transpose of the matrix is always equal to the original matrix. In a digraph of a symmetric relation, for every edge between distinct nodes, there is an edge in the opposite direction.

3. The relation

$$R = \{(1,1),(2,1),(2,3),(3,1),(3,3)\}$$

is on set A is <u>antisymmetric</u>.

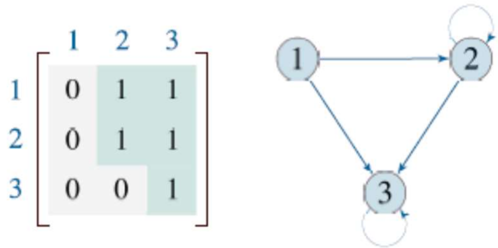


In a matrix representing an antisymmetric relation, all elements symmetric about the main diagonal are not equal to each other. The digraph of an antisymmetric relation may have loops, however connections between two distinct vertices can only go one way.

4. The relation

$$R = \{(1,2),(1,3),(2,2),(2,3),(3,3)\}$$

is on set A is <u>transitive</u>.



In a matrix of a transitive relation, for each pair of (i,j)- and (j,k) entries with value 1 there exists the (i,k)- entry with value 1. The presence of 1's on the main diagonal does not violate transitivity.

<u>Ex. 9.</u>
Let $A = \boldsymbol{N}$ (set of natural numbers).

a) Let $x\,R\,y \Leftrightarrow x \le y$. This relation is reflexive, antisymmetric und transitive, because

$x \le x,\ \forall x \in \boldsymbol{N}$ (reflexiv),
$(x \le y\ \wedge\ y \le x) \Rightarrow x = y,\ \forall x, y \in \boldsymbol{N}$ (antisymmetric),

$(x \le y \ \wedge \ y \le z) \ \Rightarrow \ x \le z, \ \forall x, y, z \in \boldsymbol{N}$ (transitiv).

b) Let $x \ R \ y \ \Leftrightarrow \ x < y$. This Relation is not reflexive, not symmetric but antisymmetric and transitive.

$(x < y \ \wedge \ y < z) \ \Rightarrow \ x < z, \ \forall x, y, z \in \boldsymbol{N}$ (transitiv).

c) Let $x \ R \ y$ if only if x is a divisor of y. This Relation is reflexive, antisymmetric und transitive.

d) $x \ R \ y$ if and only if $x = y$. This relation is reflexive, antisymmetric and transitive

D. 23. (*Equivalence Relation)*
The relation $R \subseteq AxA$ is called an *equivalence relation* if it is reflexive, transitive and symmetric.

Ex. 10.
The relation $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,2),(3,2),(1,3),(3,1)\}$ on set $A = \{1,2,3\}$ is an equivalence relation since it is reflexive, symmetric, and transitive

D. 24. (*Equivalence Classes)*
Let R be an equivalence relation on a set A, and let $a \in A$. The *equivalence class* of a is called the set of all elements of A which are equivalent to a. The equivalence class of an element a is denoted by $[a]$. Thus, by definition,

$$[a] = \{b \in A | aRb\} = \{b \in A | a \sim b\}$$

If $b \in [a]$ then the element b is called a *representative of the equivalence* class. Any element of an equivalence class may be chosen as a representative of the class $[a]$.

The set of all equivalence classes of A is called the *quotient set* of A by the relation R. The quotient set is denoted as A / R:

$$A / R = \{[a] | a \in A\}$$

Ex. 11.
A well-known sample equivalence relation is congruence modulo n.Two integers a and b are equivalent if they have the same remainder after dividing by n.

Exercises

1.
Given

$$A = \left\{ x \in R^1 \mid -1 < x < 3 \right\}$$
$$B = \left\{ x \in R^1 \mid -8 < x \right\}$$
$$C = \left\{ x \in R^1 \mid -8 < x < 1 \right\},$$

1. What are

   $A \cap B, \quad A \cup C, \quad \bar{B}, \quad \bar{A} \cup \bar{C}, \quad \bar{A} \cap \bar{C}, \quad B \setminus A, \quad C \setminus B.$

2. Show the validity of the following relations:

   $$A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$$

   and

   $$A \cap (B \setminus C) \neq (A \cup B) \setminus (A \cap C).$$

Solutions

1.

1.

$$A \cap B = A,$$

$$A \cup C = \{x \in R^1 \mid -8 < x < 3\},$$

$$\bar{B} = \{x \in R^1 \mid x \le -8\},$$

$$\bar{A} \cup \bar{C} = \{x \in R^1 \mid x \le -1 \vee x \ge 1\}, \quad \bar{A} \cap \bar{C} = \{x \in R^1 \mid x \le -8 \vee x \ge 3\},$$

$$B \setminus A = \{x \in R^1 \mid -8 < x \le -1 \vee x \ge 3\},$$

$$C \setminus B = \varnothing$$

2.

$$A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A) = \{x \in R^1 \mid -1 < x\}$$

$$A \cap (B \setminus C) = \{x \in R^1 \mid 1 \le x < 3\} \ne \{x \in R^1 \mid -8 < x \le -1 \vee x \ge 1\} = (A \cup B) \setminus (A \cap C)$$

Linear Algebra
Matrix Algebra

D. 1. 1. (*Matrix*)
A *matrix* is a rectangular two-dimensional array

$$A := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j}. & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

of $m \cdot n$ *element:* $a_{ij} \in R,\ i = 1,\ 2, ..., m;\ j = 1,\ 2, ..., n$ The ordered pair (m, n) is called the *dimension* of the matrix A.

(When we need to be explicit about a matrix's dimension, we often append it as a subscript: $A_{(m,n)}$)

R. 1. 1.
The matrix A will often be denoted by its *general element* a_{ij}:

$$A := (a_{ij}), i = 1, 2, ..., m;\ \ j = 1, 2, ..., n\ .$$

Ex. 1. 1.
The following table represents the distances between the two sources S_1 and S_2 and the three destinations D_1, D_2 and D_3 in a "transportation problem":

	D_1	D_2	D_3
S_1	40	50	34
S_2	70	23	80

The table could just as well be represented by the following matrix:

$$A := \begin{pmatrix} 40 & 50 & 34 \\ 70 & 23 & 80 \end{pmatrix}$$

Ex. 1. 2.
An enterprise uses three raw materials R_1 and R_2 to produce the final products P_1, P_2 and P_3.
The following table shows the raw material consumption per product unit:

	P_1	P_2
R_1	4	2
R_2	0	5
R_3	3	4

The table can also be represented by a matrix:

$$B := \begin{pmatrix} 4 & 2 \\ 0 & 5 \\ 3 & 4 \end{pmatrix}.$$

<u>D. 1. 2.</u> (*Matrix Types*)
The matrix A is called

1.
a *zero matrix* if $a_{ij} = 0, \ \forall i, j$. A zero matrix will be denoted as 0.

2.
a *square matrix* if $m = n$.
(In this case we speak of a matrix of the *order* m or n)

2. 1.
a *diagonal matrix* if $m = n$ and $a_{ij} = 0, \ \forall i, j$. The elements $a_{ij} = 0, \ \forall i, j$ are called *diagonal elements*.

2. 1. 1.
A *scalar matrix* if $m = n$ and

$$a_{ij} := \begin{cases} a = const & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}.$$

2.1.1.1.
an *identity matrix* if $m = n$ and

$$a_{ij} := \begin{cases} 1 & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}.$$

The identity matrix is usually written as

$$I := (\delta_{ij}), i, j = 1, 2, ..., n$$

$$\delta_{ij} := \begin{cases} 1 & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}$$

The symbol $\delta_{ij}, \ i, j = 1, \ 2, ..., n$ is called the *Kronecker delta*.

2.2.
a *lower triangular matrix*, if $m = n$ and $a_{ij} = 0$ for $i < j$. It is called an *upper triangular matrix* if $m = n$ and $a_{ij} = 0$ for $i > j$.

2.3.
a *symmetric matrix*, if $a_{ij} = 0$ and $a_{ij} = a_{ji}$, $i, j = 1, 2, ..., n$. It is called a *skew-symmetric matrix* if $m = n$ and $a_{ij} = -a_{ji}$, $i, j = 1, 2, ..., n$.

3.
a *row vector* if $a_{ij} = -a_{ji}$, $i, j = 1, 2, ..., n$ and a *column vector*, if $n = 1$.
(Vectors will be usually denoted by small letters.)

4.
a *scalar* if $m = n = 1$.

D. 1. 3.
Let A and B be matrices of the same dimension.

$$\begin{aligned} A := B \quad &\Leftrightarrow \quad a_{ij} = b_{ij}, \quad \forall i, j, \\ A :< B \quad &\Leftrightarrow \quad a_{ij} < b_{ij}, \quad \forall i, j, \\ A :\le B \quad &\Leftrightarrow \quad a_{ij} \le b_{ij}, \quad \forall i, j. \end{aligned}$$

D. 1. 4. (*Transpose of a Matrix*)
Let $A := (a_{ij}):\ i = 1, 2, ..., m;\ j = 1, 2, ..., n$. The matrix $A^T := (a_{ji}),\ \ j = 1, 2, ..., n;\ i = 1, 2, ..., m$ is called the *transpose* of the matrix A.

Ex. 1. 3.
Transpose the matrix in the example Ex.1.1.

Solution:

$$A := \begin{pmatrix} 40 & 50 & 34 \\ 70 & 23 & 80 \end{pmatrix}, \qquad A^T = \begin{pmatrix} 40 & 70 \\ 50 & 23 \\ 34 & 80 \end{pmatrix}.$$

D. 1. 5. (*Sum of Matrices*)
Let $A := (a_{ij}):\ i = 1, 2, ..., m;\ j = 1, 2, ..., n$ and $B := (b_{ij}):\ i = 1, 2, ..., m;\ j = 1, 2, ..., n$ be matrices.

$$C := A + B \quad \text{with} \quad C := (c_{ij}) := (a_{ij} + b_{ij}), \quad i = 1, 2, ..., m; \quad j = 1, 2, ..., n.$$

Ex. 1. 4.
Find the sum of the matrices

$$A := \begin{pmatrix} 13 & -50 & 17 \\ 10 & 11 & -80 \end{pmatrix}, \qquad B := \begin{pmatrix} -10 & 90 & 0 \\ 15 & -8 & 92 \end{pmatrix}.$$

Solution:

$$C := A + B = \begin{pmatrix} 3 & 40 & 17 \\ 25 & 3 & 12 \end{pmatrix}.$$

<u>D. 1. 6.</u> (*Multiplication of a Matrix by a Real Number*)

Let $A := (a_{ij}):\ i = 1,\ 2,...,m;\ j = 1,\ 2,...,n$ be a matrix and $\alpha \in R$.

$$B := \alpha \cdot A := (\alpha \cdot a_{ij}),\ i = 1,\ 2,...,m;\ j = 1,\ 2,...,n\,.$$

<u>Ex. 1. 5.</u>

Multiply the number $\alpha := -2$ by the matrix

$$A := \begin{pmatrix} 0 & -1 & 2 \\ 3 & 2 & 0 \\ -3 & 1 & 1 \end{pmatrix}.$$

Solution:

$$B := \alpha \cdot A = \begin{pmatrix} 0 & 2 & -4 \\ -6 & -4 & 0 \\ 6 & -2 & -2 \end{pmatrix}.$$

<u>D. 1. 7.</u> (*Conformable Matrices*)

The matrices $A := (a_{ij}) : i = 1,\ 2,...,m;\ j = 1,\ 2,...,n$ and $B := (b_{kl}):\ k = 1,\ 2,...,p;\ l = 1,\ 2,...,q$ are said to be conformable, if

$$\boxed{n = p}$$

<u>D. 1. 8.</u> (*Matrix Multiplication*)

Let $A := (a_{ij}):\ i = 1,\ 2,...,m;\ j = 1,\ 2,...,n$ and $B := (b_{kl}):\ k = 1,\ 2,...,p;\ l = 1,\ 2,...,q$ be given with $n = p$ (conformability).

$$A \cdot B =: C := (c_{il}),\ \ i = 1, 2,..., m;\ \ l = 1, 2,..., q\ ,$$

$$c_{il} := \sum_{j=1}^{n} a_{ij} . b_{jl}\,, \qquad i = 1, 2,..., m;\ \ l = 1, 2,..., q\ .$$

<u>Ex. 1. 6.</u>

Determine:

1.

$$C := A \cdot B$$

with

$$A := \begin{pmatrix} -2 & 4 & 0 \\ 1 & -5 & 3 \end{pmatrix}, \qquad B := \begin{pmatrix} -1 & 3 & 7 \\ 2 & -4 & 7 \\ 0 & 1 & -2 \end{pmatrix}$$

Solution:

$$C := A \cdot B = \begin{pmatrix} -2\cdot(-1)+4\cdot 2+0\cdot 0 & -2\cdot 3+4\cdot(-4)+0\cdot 1 & -2\cdot 7+4\cdot 7+0\cdot(-2) \\ 1\cdot(-1)-5\cdot 2+3\cdot 0 & 1\cdot 3-5\cdot(-4)+3\cdot 1 & 1\cdot 7-5\cdot 7+3\cdot(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -22 & 14 \\ -11 & 26 & -34 \end{pmatrix}$$

2.

$$c := A \cdot b$$

with

$$A := \begin{pmatrix} -2 & 4 & 0 \\ 1 & -1 & 3 \\ 6 & 0 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Solution:

$$c := A \cdot b = \begin{pmatrix} -2\cdot 2+4\cdot(-1)+0\cdot 3 \\ 1\cdot 2+(-1)\cdot(-1)+3\cdot 3 \\ 6\cdot 2+0\cdot(-1)+1\cdot 3 \end{pmatrix} = \begin{pmatrix} -8 \\ 12 \\ 15 \end{pmatrix}$$

Ex. 1. 7.
Given

$$a := \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

determine the following products:

1. $a \cdot b$
2. $a^T \cdot b$
3. $a \cdot b^T$

Solution:

1.
The product is not defined, since the matrices are not conformable.

2.

$$a^T \cdot b = \begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 2 \cdot 3 + 1 \cdot 4 = 10 \, .$$

(This kind of product is called *scalar product*.)

3.

$$a \cdot b^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 & 2 \cdot 4 \\ 1 \cdot 3 & 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 3 & 4 \end{pmatrix}$$

(This kind of product is called *dyadic* or *dot product*.)

<u>R. 1. 2.</u> (*Some Laws of Matrix Algebra*)
Let A, B, C be matrices.

- Associative:

$$(A + B) + C = A + (B + C)$$
$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

- Commutative:

$$A + B = B + A$$

- Distributive:

$$A \cdot (B + C) = A \cdot B + A \cdot C$$
$$(A + B) \cdot C = A \cdot C + B \cdot C$$

<u>T. 1. 1.</u> (*Some Properties of Matrix Operations*)
Let A, B, C be matrices. It can be shown:

1. $(A. + B)^T = A^T + B^T$
2. $(A^T)^T = A$
3. $A. \cdot B \neq B \cdot A$
4. $(A. \cdot B)^T = B^T \cdot A^T$

<u>Ex. 1. 8.</u> (*An Economic Application of Matrix Multiplication*)
Consider a two-stage production process. In the first stage the raw materials R_1, R_2 and R_3 will be needed to produce the semi-products S_1, S_2. In a second stage the semi-products will be used to produce the final products P_1, P_2.
The following tables contain the necessary informations concerning the input-output-coefficients:

Raw Material Consumption per Semi-product Unit

	S_1	S_2
R_1	2	4
R_2	0	1
R_3	3	2

Semi Product Consumption per Final Product Unit

	P_1	P_2
S_1	4	1
S_2	0	2

1. Determine the raw material consumption per final product unit.
2. How much raw material will be needed in order to produce 100 units of P_1 and 80 units of P_2?
3. The raw materials R_1, R_2 and R_3 cost 2 €, 4 € and 5 € per unit respectively. Calculate the raw materials costs for the above production programme.

Solution:
Let the above tables of technical coefficients be represented by the following matrices:

$$M_{RS} := \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}, \qquad M_{SP} := \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}.$$

1.
Let M be the matrix, whose elements give the raw material consumption needed to produce one unit of the final products. Then we have

$$M = M_{RS} \cdot M_{SP} = \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 0 & 2 \\ 12 & 7 \end{pmatrix}.$$

2.
Let r denote the raw material vector. We obtain

$$r = M \cdot p = \cdot \begin{pmatrix} 8 & 10 \\ 0 & 2 \\ 12 & 7 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 80 \end{pmatrix} = \begin{pmatrix} 1600 \\ 160 \\ 1760 \end{pmatrix}.$$

3.
The costs of the raw material, denoted by C_R, will be calculated as follows:

$$C_R = \begin{pmatrix} 2 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1600 \\ 160 \\ 1760 \end{pmatrix} = 12640 \text{ €}.$$

D. 1. 9 (***Determinant of a Matrix***)
The determinant of a (square) matrix $A := (a_{ij}),\ i, j = 1,\ 2, \ldots, n$, denoted by $\det(A)$, will be defined as follows:

$$\det(A) := \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det\left(A_{ij}\right)$$

(cofactor expansion of A by the i - th row)

or

$$\det(A) := \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det\left(A_{ij}\right)$$

(cofactor expansion of A by the $j-$ th column)

Here is A_{ij} a *submatrix* of A obtained by eliminating the $i-$ th row and the $j-$th column.
$(-1)^{i+j}.A_{ij}$ are called *adjoints* or *adjuncts*.

For $n = 1$ we obtain:

$$\det(a) := a$$

For $n = 2$ we obtain:

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := a_{11} \cdot a_{22} - a_{12}.a_{21}$$

<u>Ex. 1. 9.</u>
Find the determinant of the matrix

$$A := \begin{pmatrix} 1 & 2 & -3 \\ -2 & -1 & 8 \\ 5 & 17 & -11 \end{pmatrix}$$

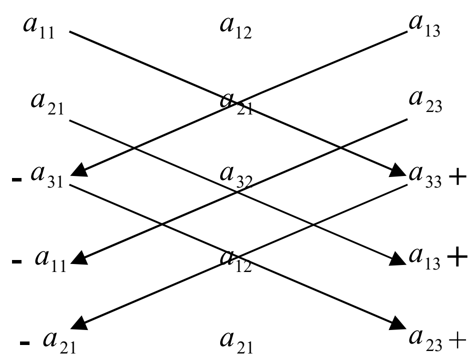
Solution:
Let us expand the matrix according to the second row. We obtain:

$$\det(A) = (-1)^{2+1} \cdot (-2) \cdot \det\begin{pmatrix} 2 & -3 \\ 17 & -11 \end{pmatrix} + (-1)^{2+2} \cdot (-1) \cdot \det\begin{pmatrix} 1 & -3 \\ 5 & -11 \end{pmatrix} + (-1)^{2+3} \cdot 8 \cdot \det\begin{pmatrix} 1 & 2 \\ 5 & 17 \end{pmatrix}$$

$$= 2 \cdot (2 \cdot (-11) - (-3) \cdot 17) - (1 \cdot (-11) - (-3) \cdot 5) - 8 \cdot (1 \cdot 17 - 2 \cdot 5)$$

$$= -2$$

<u>R. 1. 3.</u> (*Sarrus Scheme*)
There is a convenient scheme, the so-called Sarrus scheme to find the determinant of a three-dimensional matrix.
This is illustrated below:



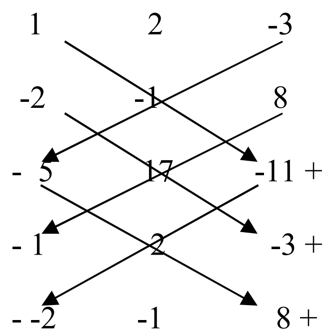
$$\det A = +(a_{11}\cdot a_{21}\cdot a_{33} + a_{21}\cdot a_{33}\cdot a_{13} + a_{31}\cdot a_{12}\cdot a_{23}) - (a_{13}\cdot a_{21}\cdot a_{31} + a_{23}\cdot a_{33}\cdot a_{11} + a_{33}\cdot a_{12}\cdot a_{21})$$

<u>Ex. 1. 11.</u>
Find the determinant of the matrix

$$A := \begin{pmatrix} 1 & 2 & -3 \\ -2 & -1 & 8 \\ 5 & 17 & -11 \end{pmatrix}$$

by applying the Sarrus scheme.

Solution :



$$\det(A) = (11 + 102 + 80) - (15 + 136 + 44) = -2$$

<u>R. 1. 4.</u> (*Properties of Determinants*)

1. If a row (or column) of a matrix is multiplied by a constant c, its determinant is also multiplied by c.
2. If a multiple of one row (or column) is added to another the determinant is unchanged.
3. If two rows are interchanged, the determinant is multiplied by -1.
4. $\det I = 1$.
5. $\det A.B = \det A + \det B$.

<u>D. 1. 10</u> (*Singular and Regular Matrices*)
The matrix $A := (a_{ij}),\ i, j = 1,\ 2, \ldots, n$ is called *singular*, if $\det(A) = 0$; otherwise, it is called *regular*.

<u>Ex. 1. 12.</u>
It can be shown that the matrix

$$A := \begin{pmatrix} 1 & 2 & -3 \\ -2 & -1 & 8 \\ 0 & 3 & 2 \end{pmatrix}$$

is singular (show it!)

<u>D. 1. 11</u> (*Inverse of a Matrix*)
Let $A := (a_{ij}),\ i, j = 1,\ 2, \ldots, n$ be a given matrix. If there is a matrix $B := (b_{ij}),\ i, j = 1,\ 2, \ldots, n$ such that

$$A \cdot B = B \cdot A = I$$

then B is called the *inverse* of A, written A^{-1}. A is then said to be *invertible*.

<u>T. 1. 1.</u>
Let $A := (a_{ij}),\ i, j = 1,\ 2, \ldots, n$ be a regular matrix. Then

$$A^{-1} = \frac{1}{\det(A)} A_{adj}^{T},$$

$$A_{adj} := ((-1)^{i+j} \cdot \det(A_{ij})).$$

<u>Ex. 1. 13.</u>
Invert the following matrix:

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 2 \\ 1 & 5 & 3 \end{pmatrix}.$$

Solution:

$\det A = -28$,

$A_{11} = -7, \qquad A_{12} = -7, \qquad A_{13} = 14,$

$A_{21} = -7, \qquad A_{22} = 5, \qquad A_{23} = -6,$

$A_{31} = 7, \qquad A_{32} = -1, \qquad A_{33} = -10$

$$A^{-1} = -\frac{1}{28}\begin{pmatrix} -7 & -7 & 7 \\ -7 & 5 & -1 \\ 14 & -6 & -10 \end{pmatrix}.$$

<u>R. 1. 5.</u>

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

<u>D. 1. 12.</u>(*Trace of a Matrix)*
The *trace* of a (square) matrix is the sum of the main diagonal elements:

$$trA = \sum_{i=1}^{n} a_{ii} \ .$$

<u>Ex. 1. 14.</u>

$$trA := \begin{pmatrix} 1 & 2 & -3 \\ -2 & -1 & 8 \\ 5 & 17 & -11 \end{pmatrix} = 1 - 1 - 11 = -11.$$

<u>R. 1. 5.</u>

1. $trA \cdot B = trB \cdot A$
2. $trA \cdot B \cdot C = trC \cdot A \cdot B = trB \cdot C.A$

Exercises

1. 1.
Consider the vectors

$$x^T = (1, \ -2), \qquad y^T = (1, \ 0, \ -1)$$

and the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix}.$$

1.Which of the following operations are allowed:

1) $y^T xA$, 2) $y^T Ax$, 3) Ayx^T, 4) Axy, 5) $\left(A^T y\right)^T x$
6) $xy^T A^T$.

2. Find the results for those operations which make sense.

1. 2.
Consider the matrices

$$A = \begin{pmatrix} 4 & 5 & a \\ 6 & 7 & 8 \\ b & 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1.5 & 6 & b \\ 3 & 4 & 6 \\ b & 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 13.5 & 39 & 34.5 \\ 54 & 80 & 84 \\ 22.5 & 38 & 39 \end{pmatrix}.$$

Determine the values of a and b for which $C = A \cdot B$.

1. 3.
Consider a two-stage production process. In the first stage the raw materials R_1, R_2 and R_3 will be needed to produce the semi-products S_1, S_2. In a second stage the semi-products will be used to produce the final products P_1, P_2.
The following tables contain the necessary informations concerning the input-output-coefficients:

Raw Material Consumption
per Semi-product Unit

	S_1	S_2
R_1	1	2
R_2	3	0
R_3	0	5

Semi-Product Consumption
per Final Product Unit

	P_1	P_2
S_1	3	0
S_2	1	2

1. Determine the raw material consumption per final product unit.
2. How much raw material will be needed in order to produce 200 units of P_1 and 180 units of P_2?
3. The raw materials R_1, R_2 and R_3 cost 3 €, 2 € and 5 € per unit respectively. Calculate the raw materials costs for the above production programme.

1. 4.
Find the determinant of the matrices

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 2 & 2 \\ 2 & 5 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 & -1 & 4 \\ 3 & -2 & 5 & 2 \\ 4 & 3 & -1 & 3 \\ 5 & 2 & 3 & 1 \end{pmatrix}.$$

1. 5.
For which values of a is the matrix

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 3 & a \end{pmatrix}.$$

a regular matrix?

1. 6.
The following table shows the amounts of Articles A_1 and A_2 produced by a firm in the 4 quarters Q_i, $i = 1, 2, 3, 4$ of a year.

	A_1	A_2
Q_1	40	10
Q_2	30	30
Q_3	20	20
Q_4	10	30

The production of A_1 and A_2 requires three accessory parts P_1, P_2 and P_3 produced on the machines M_1 and M_2. The following table shows the machine hours needed to produce each unit of the accessory parts

	P_1	P_2	P_3
M_1	1	2	2
M_2	2	3	2

The next table shows the amounts of the accessory parts P_1, P_2 and P_3 needed to produce one unit of the two articles A_1 and A_2

	A_1	A_2
P_1	1	2
P_2	0	3
P_3	2	1

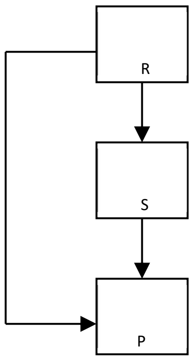
Answer the following questions using *matrix calculations*.

1. How many accessory parts will be needed per quarter?
2. How many machine hours will be needed per quarter in order to produce the accessory parts?

1. 7.

The following chart represents the production process of a firm. Denote by

R: the raw material block,
S: the semi-product block
P: the final product block.



The final products are produced partially *directly* and partially *indirectly* via the semi-products. Following input-output tables are available

Direct input-output coefficients

	P_1	P_2
R_1	2	0
R_2	1	3

	S_1	S_2
R_1	4	3
R_2	2	5

Final input-output coefficients

	P_1	P_2
R_1	8	16
R_2	11	25

1. Describe the above tables as matrices.
2. Represent the production process as a matrix equation.
3. How many units of semi-products will be required to produce one unit of the final outputs?
4. How many units of raw materials will be required for the following production programme?

$$P_1: \text{ 10 units}; \quad P_2: \text{ 18 units}.$$

Solutions

1. 1.

1.

1) Not defined.
2) Not defined
3) Defined
4) Not defined.
5) Not defined
6) Defined.

2.

3) $$Ayx^T = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix}$$

6)

$$xy^T A^T = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}$$

1. 2.

$$\begin{pmatrix} 4 & 5 & a \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} = 39$$

$24 + 20 + 2a = 39$, $\qquad a = -\frac{5}{2}$;

$$\begin{pmatrix} 6 & 7 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1.5 \\ 3 \\ b \end{pmatrix} = 54$$

$9 + 21 + 8b = 54$, $\qquad b = 3$.

1. 3.

Let the above tables of technical coefficients be represented by the following matrices:

$$M_{RS} := \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 5 \end{pmatrix}, \qquad M_{SP} := \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

1.

Let M be the matrix, whose elements give the raw material consumption needed to produce one unit of the final products. Then we have

$$M = M_{RS} \cdot M_{SP} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 9 & 0 \\ 5 & 10 \end{pmatrix}.$$

2.
Let r denote the raw material vector. We obtain

$$r = M \cdot p = \cdot \begin{pmatrix} 5 & 4 \\ 9 & 0 \\ 5 & 10 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 180 \end{pmatrix} = \begin{pmatrix} 1720 \\ 1800 \\ 2800 \end{pmatrix}.$$

3.
The costs of the raw material, denoted by C_R, will be calculated as follows:

$$C_R = \begin{pmatrix} 3 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1720 \\ 1800 \\ 2800 \end{pmatrix} = 22760\,€ .$$

1. 4.

$$\det A = 3 \cdot \det\begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix} - 4 \cdot \det\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + 2 \cdot \det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = -6$$

$$\det B = 2 \cdot \det\begin{pmatrix} -2 & 5 & 2 \\ 3 & -1 & 3 \\ 2 & 3 & 1 \end{pmatrix} - 3 \cdot \det\begin{pmatrix} 3 & 5 & 2 \\ 4 & -1 & 3 \\ 5 & 3 & 1 \end{pmatrix} - \det\begin{pmatrix} 3 & -2 & 2 \\ 4 & 3 & 3 \\ 5 & 2 & 1 \end{pmatrix} - 4 \cdot \det\begin{pmatrix} 3 & -2 & 5 \\ 4 & 3 & -1 \\ 5 & 2 & 3 \end{pmatrix} = -146$$

1. 5.

$$\det A = 4 \cdot \det\begin{pmatrix} 1 & 4 \\ 3 & a \end{pmatrix} - 2 \cdot \det\begin{pmatrix} 2 & 4 \\ 3 & a \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix} = -15 \neq 0$$

The matrix A is regular for all a.

1. 6.
Let

$$M_{QA} = \begin{pmatrix} 40 & 10 \\ 30 & 30 \\ 20 & 20 \\ 10 & 30 \end{pmatrix}, \qquad M_{MP} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 2 \end{pmatrix}, \quad M_{PA} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 1 \end{pmatrix}.$$

1.

$$M_{QP} = M_{QA} \cdot M_{PA}^T = \begin{pmatrix} 40 & 10 \\ 30 & 30 \\ 20 & 20 \\ 10 & 30 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 60 & 30 & 90 \\ 90 & 90 & 90 \\ 60 & 60 & 60 \\ 70 & 90 & 50 \end{pmatrix}.$$

2.

$$M_{QM} = M_{QP} \cdot M_{MP}^T = \begin{pmatrix} 60 & 30 & 90 \\ 90 & 90 & 90 \\ 60 & 60 & 60 \\ 70 & 90 & 50 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 300 & 390 \\ 450 & 630 \\ 300 & 420 \\ 350 & 510 \end{pmatrix}.$$

1. 7.

1.

$$M_{RP} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \qquad M_{RS} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}, \qquad M = \begin{pmatrix} 8 & 16 \\ 11 & 25 \end{pmatrix}.$$

2.

$$M = M_{RP} + M_{RS} \cdot M_{SP}.$$

3.

$$M - M_{RP} = M_{RS} \cdot M_{SP}$$

$$M_{SP} = M_{RS}^{-1} \cdot \left(M - M_{RP}\right)$$

$$M_{RS}^{-1} = \frac{1}{14} \cdot \begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}, \qquad M - M_{RP} = \begin{pmatrix} 6 & 16 \\ 10 & 22 \end{pmatrix}$$

$$M_{SP} = \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix}.$$

4.

Denote by $r = \begin{pmatrix} r_1 & r_2 \end{pmatrix}^T$: the raw material vector.

$$r = \begin{pmatrix} 8 & 16 \\ 11 & 25 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 18 \end{pmatrix} = \begin{pmatrix} 368 \\ 560 \end{pmatrix}.$$

Linear Algebra
System of linear Equations

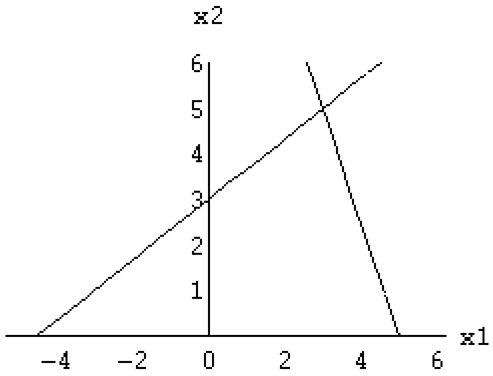
Ex. 2. 1
Solve the following system of linear equations

$$\begin{cases} 2x_1 - 3x_2 = -9 \\ 5x_1 + 2x_2 = 25 \end{cases}$$

Solution:

$$x_1 = 3, \qquad x_2 = 5$$

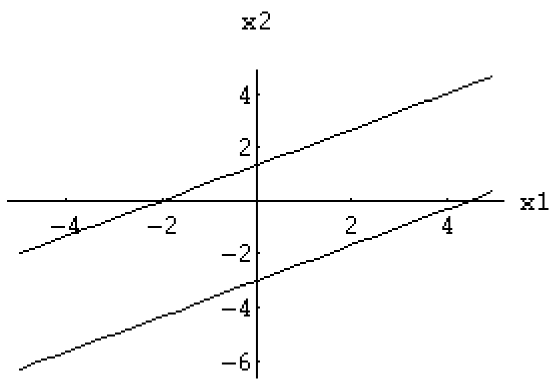
$\therefore$ the system has a unique solution.



Ex. 2. 2
Solve the following system of linear equations

$$\begin{cases} 2x_1 - 3x_2 = 6 \\ -4x_1 + 6x_2 = 8 \end{cases}$$

Solution:
$\therefore$ the system has no solution.

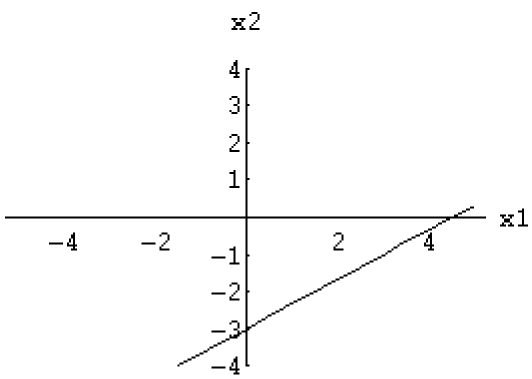


Ex. 2. 3
Solve the following system of linear equations

$$\begin{cases} 2x_1 - 3x_2 = 6 \\ -4x_1 + 6x_2 = -12 \end{cases}$$

Solution:

$\therefore$ the system has infinitely many solutions.



The *general solution* of the system will be:

$$x_1 = 3 + \frac{3}{2}x_2 .$$

x_1 is called the *dependent variable* and x_2 the *independent variable*.

The following solutions will be called *special* or *particular solutions*:

$$x_2 := 6 \quad \Rightarrow \quad x_2 = 12 ,$$

$$x_2 := -8 \quad \Rightarrow \quad x_2 = -9$$

Setting the independent variable x_2 equal to zero, we obtain the so-called *basic solution*:

$$x_1 = 3, \quad x_2 = 0 .$$

x_1 is called the *basic variable* and x_2 the *non-basic variable*.

D. 2. 1. (***System of Linear Equations***)
A *system of linear equations* can be defined as follows:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ \dots\ + a_{1j}x_j + \ \dots\ + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ \dots\ + a_{2j}x_j + \ \dots\ + a_{2n}x_n &= b_2 \\
&\ \, \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \ \dots\ + a_{ij}x_j + \ \dots\ + a_{in}x_n &= b_i \\
&\ \, \vdots \\
a_{m1}x_1 + a_{mi2}x_2 + \ \dots\ + a_{mj}x_j + \ \dots\ + a_{mn}x_n &= b_m
\end{aligned}
$$

Let

$$
\begin{aligned}
&A := \left(a_{ij}\right), \quad i = 1,2,\dots,m;\ j = 1,2,\dots,n\,, \\
&x := \left(x_j\right), \quad j = 1,2,\dots,n\,, \\
&b := \left(b_i\right), \quad i = 1,2,\dots,m\,.
\end{aligned}
$$

The above system of linear equations in the *matrix form* can be written as follows:

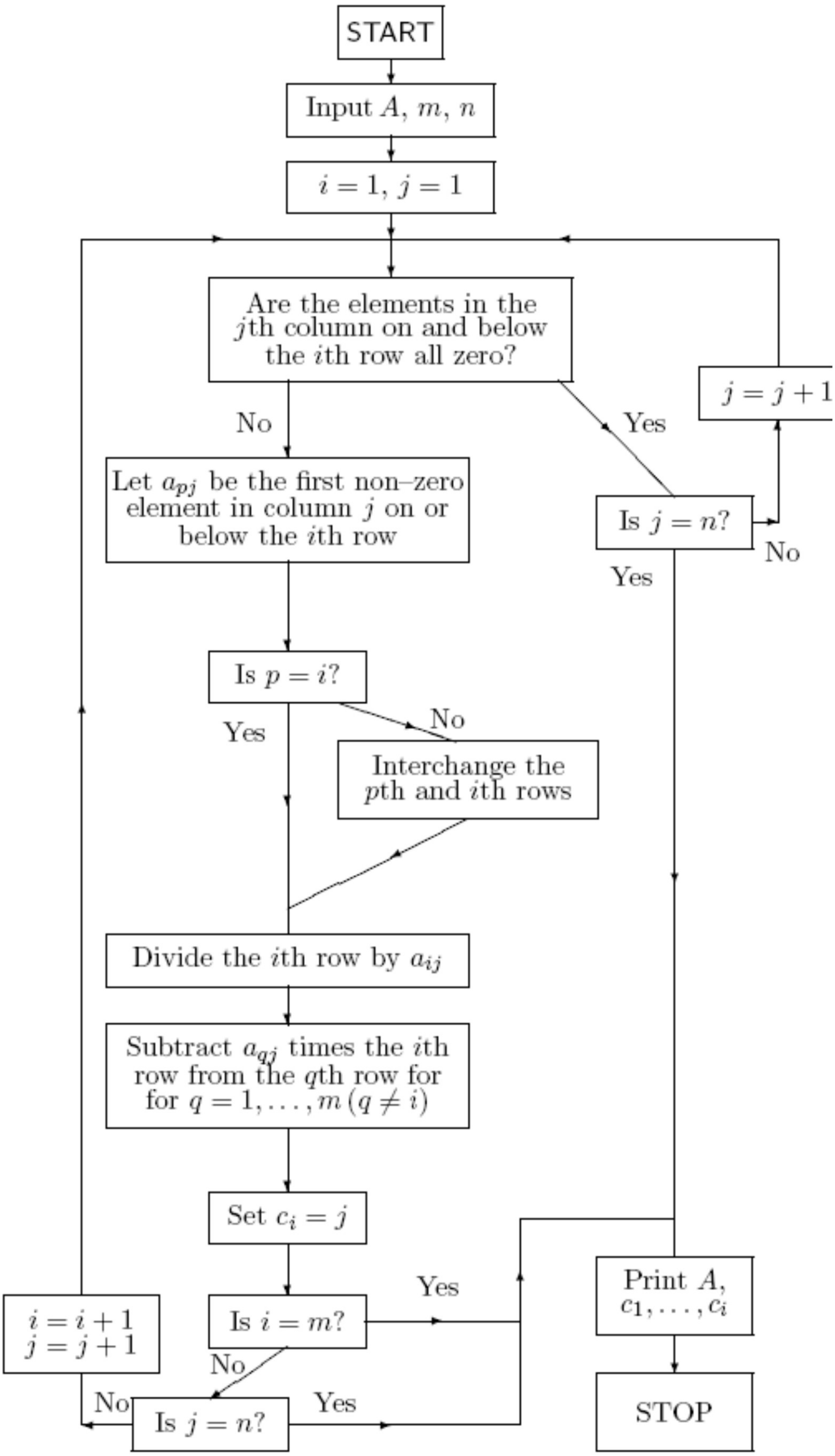
$$Ax = b$$

A system of linear equations is called *homogeneous*, if $b = 0$.

R. 2. 1.
There are three different possibilities for the solution of a system of linear equations:

1. The system has a unique solution.
2. The system has infinitely many solutions
3. The system has no solutions.

R. 2. 2.
We shall choose the Gauss-Jordan algorithm to solve simultaneous linear equations and to invert matrices. A flow-chart for this algorithm is presented below:


START
Input A, m, n
$i = 1$, $j = 1$
Are the elements in the jth column on and below the ith row all zero?
No
Yes
$j = j + 1$
Let a_{pj} be the first non–zero element in column j on or below the ith row
Is $j = n$?
No
Yes
Is $p = i$?
Yes
No
Interchange the pth and ith rows
Divide the ith row by a_{ij}
Subtract a_{qj} times the ith row from the qth row for for $q = 1, \ldots, m \,(q \neq i)$
Set $c_i = j$
Yes
Print A, $c_1, \ldots, c_i$
$i = i + 1$
$j = j + 1$
Is $i = m$?
No
No
Is $j = n$?
Yes
STOP

Ex. 2. 4.
Let

$$A=\begin{pmatrix}-1 & 6 & 2\\ 2 & -2 & -1\\ 3 & -4 & -2\end{pmatrix},\quad b=\begin{pmatrix}4\\ 2\\ 1\end{pmatrix}.$$

Using the Gauss-Jordan method

1. find A^{-1}, if it exists,
2. calculate the determinant of the matrix A.
3. Let $x=\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}^T$. Solve the system of equations $Ax=b$.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_0
$\leftarrow x_4$	**-1**	6	2	1	0	0	4
x_5	2	-2	-1	0	1	0	2
x_6	3	-4	-2	0	0	1	1
$\rightarrow x_1$	1	-6	-2	-1	0	0	-4
$\leftarrow x_5$	0	**10**	3	2	1	0	10
x_6	0	14	4	3	0	1	13
x_1	1	0	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	0	2
$\rightarrow x_2$	0	1	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	0	1
$\leftarrow x_6$	0	0	$-\frac{1}{5}$	$\frac{1}{5}$	$-\frac{7}{5}$	1	-1
x_1	1	0	0	0	2	-1	3
x_2	0	1	0	$\frac{1}{2}$	-2	$\frac{3}{2}$	$-\frac{1}{2}$
$\rightarrow x_3$	0	0	1	-1	7	-5	5

1.

$$A^{-1}=\begin{pmatrix}0 & 2 & -1\\ \frac{1}{2} & -2 & \frac{3}{2}\\ -1 & 7 & -5\end{pmatrix}.$$

2.

$$\det A = -1 \cdot 10 \cdot \left(-\frac{1}{5}\right) = 2 .$$

3.

$$x = \begin{pmatrix} 3 & -1/2 & 5 \end{pmatrix}^T .$$

Ex. 2. 5.

Using the Gauss-Jordan method, solve the following system of linear equations:

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 2 \\ 3x_1 - x_2 + 5x_3 - 3x_4 - x_5 = 6 \\ 2x_1 + x_2 + 2x_3 - 2x_4 - 3x_5 = 8 \end{cases}$$

Solution:

x_1	x_2	x_3	x_4	x_5	x_0
1	-2	3	-1	2	2
3	-1	5	-3	-1	6
2	1	2	-2	-3	8
1	-2	3	-1	2	2
0	**5**	-4	0	-7	0
0	5	-4	0	-7	4
1	0	$\frac{7}{5}$	-1	$-\frac{4}{5}$	2
0	1	$-\frac{4}{5}$	0	$-\frac{7}{5}$	0
0	0	0	0	0	4

The last row leads to the contradiction: 0 =4. Therefore, the system is inconsistent and has no solution.

<u>Ex. 2. 6.</u>
Using the Gauss-Jordan method, solve the following system of linear equations:

$$\begin{cases} x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 - x_2 + x_3 + x_4 = 8 \\ 3x_1 + x_2 + 3x_3 - x_4 = 16 \end{cases}$$

Solution:

x_1	x_2	x_3	x_4	x_0
1	1	1	-1	4
1	-1	1	1	8
3	1	3	-1	16
1	1	1	-1	4
0	**-2**	0	2	4
0	-2	0	2	4
1	1	1	-1	4
0	**-2**	0	2	4
1	0	1	0	6
0	1	0	-1	-2

We write the last tableau as linear equations:

$$\begin{cases} x_1 + x_3 = 6 \\ x_2 - x_4 = -2 \end{cases}.$$

The system has the *general solution*:

$$x_1 = 6 - x_3$$
$$x_2 = -2 + x_4.$$

Choosing $x_3 := 5, \quad x_4 := 6$, we obtain a *special solution* $x = (1 \quad 4 \quad 5 \quad 6)^T$. There is an infinite number of such solutions.

Choosing $x_3 = x_4 := 0$, we have the *basic solution* $x = (6 \quad -2 \quad 0 \quad 0)^T$. x_1, x_2 are called *basic variables,* x_3, x_4 *nonbasic variables.*

Exercises

2. 1.

In a two-stage production process three raw materials R_1, R_2 and R_3 will be used to produce the final products P_1, P_2 and P_3 with the help of the semi-products S_1, S_2 and S_3.
The corresponding input-output coefficients are given in the following tables

	S_1	S_1	S_3
R_1	2	1	0
R_2	5	2	4
R_3	3	1	0

	P_1	P_2	P_3
R_1	5	5	10
R_2	12	31	28
R_3	7	6	14

1. Draw a chart representing the production process.
2. Determine the unit consumption of the semi-products per unit of each final product...
3. It is planned to produce 100 units of P_1, 180 units of P_2 and 110 units of P_3. Find the total consumption of each raw material for the given production programme.

2. 2.

Invert the following matrices:

$$A = \begin{pmatrix} 4 & 2 & -1 \\ 5 & 3 & -2 \\ 3 & 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

2. 3.

A firm uses three raw materials R_1, R_2, R_3 to produce the products P_1, P_2, P_3. The following table shows the raw material quantities needed to produce one unite of each product (the so-called "technical coefficients") as well as the availability of the raw materials:

	P_1	P_2	P_3	availability
R_1	2	1	2	51
R_2	1	2	1	48
R_3	2	0	3	44

Determine a production programme exhausting the available raw materials.

2. 4.
Given the system of linear equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 - 3x_2 \qquad &= -3 \\ x_1 + 3x_2 + ax_3 &= a, \end{aligned}$$

determine such values of a for which the system has

1. infinitely many solutions
2. a unique solution.

2. 5.
The products $P_i,\ i = 1,\ 2,\ 3,$ will be produced on the machines $M_i,\ i = 1,\ 2,\ 3$. The following table shows the time needed to produce one unit of each product on the individual machines as well as the available machine time:

	P_1	P_2	P_3	Machine Time
M_1	1	2	2	100
M_2	4	6	2	200
M_3	9	14	6	500

The machine times are to be fully exhausted.

1. Formulate the mathematical model
2. Determine all feasible solutions of the model, assuming that the production of P_3 can be varied.
3. Is it possible to produce 55 units of P_3? Justify your answer.

2. 6.
A firm uses the raw materials R_1, R_2 and R_3 to produce the products P_1, P_2 and P_3. The following table contains the necessary informations about the raw material consumption per product unit as well as the availability of the raw materials.

	P_1	P_2	P_3	Availability
R_1	1	2	1	130
R_2	0	1	1	80
R_3	1	0	1	p

Assuming that the raw materials are to be fully exhausted, determine the parameter p for which there exists at least one feasible production programme.
Give two such programmes.

2. 7.

A firm produces four products P_1, P_2, P_3, and P_4 on four machines M_1, M_2, M_3, and M_4. The following table shows the times needed to process one unit of each product on each machine in minutes:

	P_1	P_2	P_3	P_4
M_1	2	3	4	2
M_2	6	1	2	1
M_3	1	3	6	3
M_4	3	1	8	4

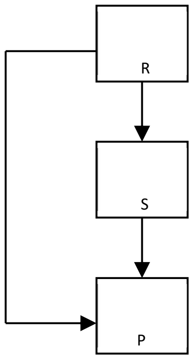
1. Given a time capacity of 85 minutes for each machine, determine the production programme.

2. Find a special solution which maximises the production of P_3.

2. 8.

The following chart represents the production process of a firm. Denote by

R: the raw material block,
S: the semi-product block
P: the final product block.



The final products are produced partially *directly* and partially *indirectly* via the semi-products. Following input-output tables are available

Direct input-output coefficients

	P_1	P_2
R_1	2	0
R_2	1	3

	S_1	S_2
R_1	4	3
R_2	2	5

Final input-output coefficients

	P_1	P_2
R_1	8	16
R_2	11	25

1. Describe the above tables as matrices.
2. Represent the production process as a matrix equation.
3. How many units of semi-products will be required to produce one unit of the final outputs?

4. How many units of raw materials will be required for the following production programme?

$$P_1: \text{ 10 units;} \qquad P_2: \text{ 18 units}.$$

2. 9.

A factory uses the raw materials R_1, R_2 and R_3 to produce the final products P_1, P_2, P_3 and P_4. The following table shows the raw material consumption per final product unit and the availabilities of the raw materials.

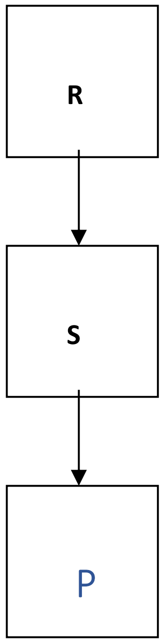
	P_1	P_2	P_3	P_4	Availability
R_1	3	6	3	6	120
R_2	1	4	5	6	100
R_3	2	5	6	8	130

Assuming that the raw materials are to be fully exhausted, find all feasible production programmes for which the output P_4 can be varied.

Solutions

2. 1.

1.



2.

Let

$$M_{RS} = \begin{pmatrix} 2 & 1 & 0 \\ 5 & 2 & 4 \\ 3 & 1 & 0 \end{pmatrix}, \quad M_{RP} = \begin{pmatrix} 5 & 5 & 10 \\ 12 & 31 & 28 \\ 7 & 6 & 14 \end{pmatrix}.$$

$$M_{RP} = M_{RS} \cdot M_{SP}$$

$$M_{SP} = M_{RS}^{-1} \cdot M_{RP}$$

.

Gauß-Jordan Tableau

x_1	x_2	x_3	x_4	x_5	x_6
2	1	0	1	0	0
5	2	4	0	1	0
3	1	0	0	0	1
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
0	$-\frac{1}{2}$	4	$-\frac{5}{2}$	1	0
0	$-\frac{1}{2}$	0	$-\frac{3}{2}$	0	1
1	0	4	-2	1	0
0	1	-8	5	-2	0
0	0	-4	1	-1	1
1	0	0	-1	0	1
0	1	0	3	0	-2
0	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

$$M_{RS}^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -2 \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}, \qquad M_{SP} = M_{RS}^{-1} \cdot M_{RP} = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 0 & 5 & 1 \end{pmatrix}$$

3.

$$\begin{pmatrix} 5 & 5 & 10 \\ 12 & 31 & 28 \\ 7 & 6 & 14 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 180 \\ 110 \end{pmatrix} = \begin{pmatrix} 2500 \\ 9860 \\ 3320 \end{pmatrix}.$$

2. 2.

a)

Gauss-Jordan Tableau

x_1	x_2	x_3	x_4	x_5	x_6
4	2	-1	1	0	0
5	3	-2	0	1	0
3	2	-1	0	0	1
1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	0
0	$\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{5}{4}$	1	0
0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{3}{4}$	0	1
1	0	$\frac{1}{2}$	$\frac{3}{2}$	-1	0
0	1	$-\frac{3}{2}$	$-\frac{5}{2}$	2	0
0	0	$\frac{1}{2}$	$\frac{1}{2}$	-1	1
1	0	0	1	0	-1
0	1	0	-1	-1	3
0	0	1	1	-2	2

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 3 \\ 1 & -2 & 2 \end{pmatrix}$$

b)

Gauss-Jordan Tableau

x_1	x_2	x_3	x_4	x_5	x_6
1	1	-1	1	0	0
-2	1	1	0	1	0
1	1	1	0	0	1
1	1	-1	1	0	0
0	3	-1	2	1	0
0	0	2	-1	0	1
1	0	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0
0	1	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0
0	0	2	-1	0	1
1	0	0	0	$-\frac{1}{3}$	$\frac{1}{3}$
0	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
0	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$

$$B^{-1} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

2. 3.

Let x_i, $i = 1, 2, 3$: the produced amount of product P_i.

The Model

$$\begin{aligned} 2x_1 + \quad x_2 + 2x_3 &= 51 \\ x_1 + 2x_2 + \quad x_3 &= 48 \\ 2x_1 \qquad\quad + 3x_3 &= 44 \end{aligned} \qquad x_i \geq 0 : \quad i = 1, 2, 3.$$

Gauss-Jordan Tableau

x_1	x_2	x_3	x_0
2	1	2	51
1	2	1	48
2	0	3	44
1	$\frac{1}{2}$	1	$\frac{51}{2}$
0	$\frac{3}{2}$	0	$\frac{45}{2}$
0	-1	1	-7
1	0	1	18
0	1	0	15
0	0	1	8
1	0	0	10
0	1	0	15
0	0	1	8

$x_1 = 10,\qquad x_2 = 15,\qquad x_3 = 8$

2. 4.

Gauss-Jordan Tableau

x_1	x_2	x_3	x_0
1	2	-1	0
2	-3	0	-3
1	3	a	a
1	2	-1	0
0	-7	2	-3
0	1	$1+a$	a
1	0	$-\frac{3}{7}$	$-\frac{6}{7}$
0	1	$-\frac{2}{7}$	$\frac{3}{7}$
0	0	$\frac{9}{7}+a$	$-\frac{3}{7}+a$

1. For $a=-\frac{9}{7} \wedge a=\frac{3}{7}$, i. e. for no a.
2. For $a \neq -\frac{9}{7}$

2. 5.

Let x_i, $i=1,2,3$: the produced amount of product P_i.

The Model:

$$
\begin{aligned}
x_1 + 2x_2 + 2x_3 &= 100 \\
4x_1 + 6x_2 + 2x_3 &= 200 \\
9x_1 + 14x_2 + 6x_3 &= 500 \\
x_1,\ x_2,\ x_3 &\geq 0
\end{aligned}
$$

2.

Gauss-Jordan-Tableau

x_1	x_2	x_3	x_0
1	2	2	100
4	6	2	200
9	14	6	500
1	2	2	100
0	-2	-6	-200
0	-4	-12	-400
1	0	-4	-100
0	1	3	100
0	0	0	0

$$x_1 = -100 + 4x_3 \geq 0$$
$$x_2 = 100 - 3x_3 \geq 0$$

i. e.

$$25 \leq x_3 \leq \frac{100}{3}$$

3.

No, since $55 \notin \left[25,\ {}^{100}\!/_{3}\right]$

2. 6.

Let x_i, $i = 1, 2, 3$: the produced amount of product P_i.

The Model:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 130 \\ x_2 + x_3 &= 80 \\ x_1 \quad\quad + x_3 &= p \end{aligned} \qquad x_i \geq 0,\ i = 1,\ 2,\ 3;\quad p > 0\,.$$

Gauß-Jordan-Tableau

x_1	x_2	x_3	x_0
1	2	1	130
0	1	1	80
1	0	1	p
1	2	1	130
0	**1**	1	80
0	-2	0	$-130+p$
1	0	-1	-30
0	1	1	80
0	0	**2**	$30+p$
1	0	0	$-15+\frac{p}{2}$
0	1	0	$65-\frac{p}{2}$
0	0	1	$15+\frac{p}{2}$

$$\begin{cases} -15+\dfrac{p}{2}\geq 0 \\ 65-\dfrac{p}{2}\geq 0 \\ 15+\dfrac{p}{2}\geq 0 \end{cases} \quad\Rightarrow\quad 30\leq p\leq 130.$$

2. 7.

Denote by

x_i, $i = 1,2,3,4$: the number of product P_i, $i = 1, 2, 3, 4$.

<u>*The model*</u>:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 2x_4 &= 85 \\ 6x_1 + x_2 + 2x_3 + x_4 &= 85 \\ x_1 + 3x_2 + 6x_3 + 3x_4 &= 85 \\ 3x_1 + x_2 + 8x_3 + 4x_4 &= 85 \end{aligned}, \qquad x_i \ge 0: \quad \text{integer}, \quad i = 1, 2, 3, 4$$

Gauss-Jordan-Tableau

x_1	x_2	x_3	x_4	x_0
2	3	4	2	85
6	1	2	1	85
1	3	6	3	85
3	1	8	4	85
1	$\frac{3}{2}$	2	1	$\frac{85}{2}$
0	-8	-10	-5	-170
0	$\frac{3}{2}$	4	2	$\frac{85}{2}$
0	$-\frac{7}{2}$	2	1	$-\frac{85}{2}$
1	0	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{85}{8}$
0	1	$\frac{5}{4}$	$\frac{5}{8}$	$\frac{85}{4}$
0	0	$\frac{17}{8}$	$\frac{17}{16}$	$\frac{85}{8}$
0	0	$\frac{51}{8}$	$\frac{51}{16}$	$\frac{255}{8}$
1	0	0	0	10
0	1	0	0	15
0	0	1	$\frac{1}{2}$	5
0	0	0	0	0

1.

$$x_1 = 10, \quad x_2 = 15, \quad x_3 = 5 - \frac{1}{2}x_4, \quad \text{or} \quad x_4 = 10 - 2x_3 \geq 0, \quad \text{d.h.} \quad 0 \leq x_3 \leq 5$$

2.

$$x_1 = 10, \qquad x_2 = 15, \qquad x_3 = 5, \qquad x_4 = 0$$

2. 8.

1.

Denote by

$$M_{RP} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \qquad M_{RS} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}, \qquad M = \begin{pmatrix} 8 & 16 \\ 11 & 25 \end{pmatrix}.$$

2.

$$M = M_{RP} + M_{RS} \cdot M_{SP}.$$

3.

$$M_{RS} \cdot M_{SP} = M - M_{RP},$$

$$M_{RS}^{-1} \cdot M_{RS} \cdot M_{SP} = M_{RS}^{-1}\left(M - M_{RP}\right),$$

$$M_{SP} = M_{RS}^{-1}\left(M - M_{RP}\right),$$

$$M_{RS}^{-1} = \frac{1}{14}\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix},$$

$$M_{SP} = \frac{1}{14}\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 6 & 16 \\ 10 & 22 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix}.$$

	P_1	P_2
S_1	8	16
S_2	11	25

4.

$$\begin{pmatrix} 8 & 16 \\ 11 & 25 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 18 \end{pmatrix} = \begin{pmatrix} 368 \\ 560 \end{pmatrix}.$$

2. 9.
Denote by

$x_i, \quad i = 1, 2, 3, 4$: the number of product $P_i, \quad i = 1,\ 2,\ 3,\ 4$.

<u>*The model*</u>:

$$
\begin{aligned}
3x_1 + 6x_2 + 3x_3 + 6x_4 &= 120 \\
x_1 + 4x_2 + 5x_3 + 6x_4 &= 100\,, \qquad\qquad x_i \geq 0,\ \ i = 1,\ 2,\ 3,\ 4. \\
2x_1 + 5x_2 + 6x_3 + 8x_4 &= 130
\end{aligned}
$$

Gauss-Jordan-Tableau

x_1	x_2	x_3	x_4	x_0
3	6	3	6	120
1	4	5	6	100
2	5	6	8	130
1	2	1	2	40
0	2	4	4	60
0	1	4	4	50
1	0	-3	-2	-20
0	1	2	2	30
0	0	2	2	20
1	0	0	1	10
0	1	0	0	10
0	0	1	1	10

$x_1 + x_4 = 10 \geq 0$

$x_2 = \ 10$

$x_3 + x_4 = 10 \geq 0$

i. e.

$$0 \leq x_4 \leq 10\,.$$

Analysis in Economics
Part I

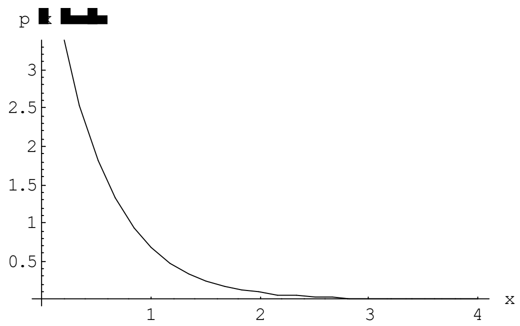
<u>D. 1. 1.</u> (*Function*)
A *function* f from a set A into a set B, denoted by $f: A \to B$, is a correspondence that assigns to each element $x \in A$ exactly one element $y \in B$. We call y the *image* of x under f and denote it by $f(x)$. The *domain* of f is the set $f(A) = \{f(x) \mid x \in A\} \subseteq B$

<u>R. 1. 1.</u>
In this part, we only work with functions whose domains and ranges are sets of real numbers. We call such functions *real functions of one real variable*.

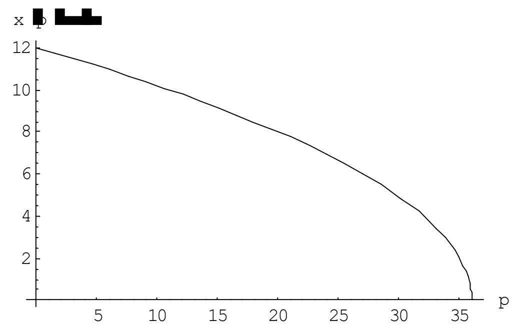
<u>Ex. 1. 1.</u>
Some important economic functions:

1. (Demand functions)

i) $p(x) = 5e^{-0.2x}$



ii) $x(p) = 2\sqrt{36-p}$.

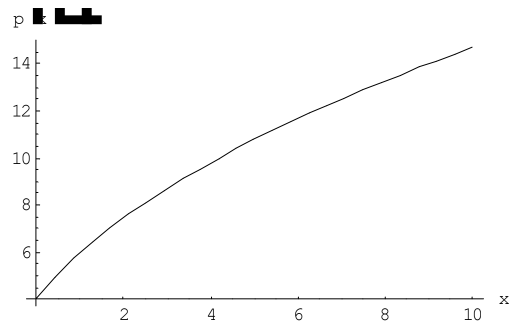


Here are:

p: the price per unit of product
x: the demand.

2. (Supply functions)

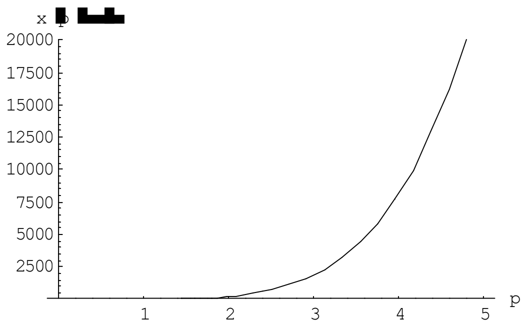
i) $p(x) = 2\sqrt{5x+4}$



ii) $x(p) = -50 + 8p^5$.

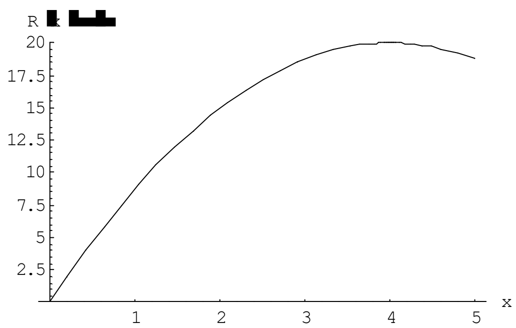
Here are:

p : the price per unit of product
x : the supply.



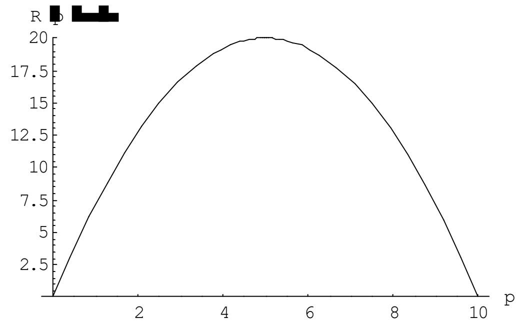
3. (Revenue functions)

i) $R(x) := x \cdot p(x) = x \cdot (10 - 1.25x)$
$= 10x - 1.25x^2$



ii) $R(p) = p \cdot x(p) = p \cdot (8 - 0.8p)$

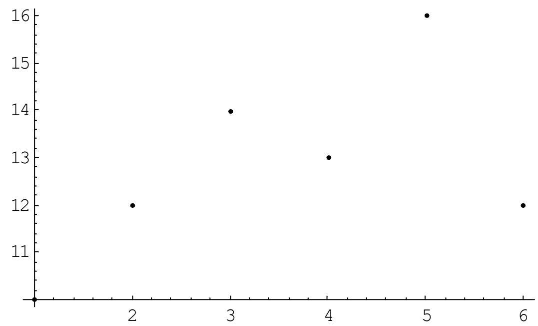
$$= 8p - 0.8p^2$$



iii) (A discrete revenue function)

The following table shows the revenue [thousand €] of a firm in the first six month of a year:

Month	1	2	3	4	5	6
Revenue	10	12	14	13	16	12



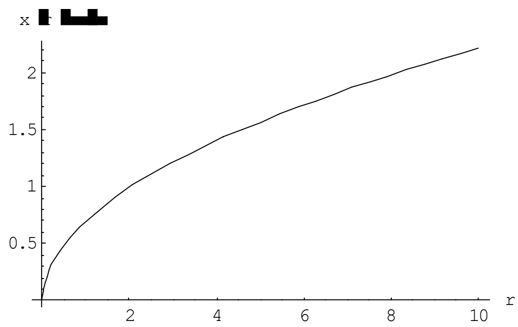
4. (Production functions)

Let

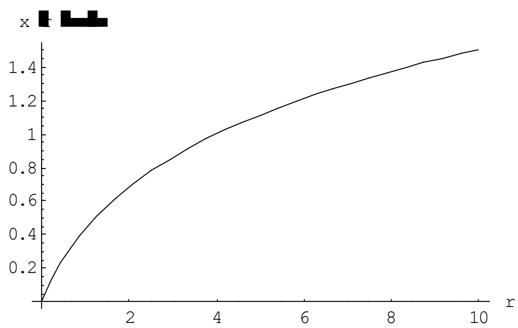
r: Factor input

x: Output.

i) (Cobb-Douglas production function): $x(r) = 0.7r^{0.5}$



ii) (CES production function): $x(r) = \left(r^{-0.5} + 0.5\right)^{-2}$



iii) (Limitational production function):

$$x(r) = \begin{cases} 0.75r & \text{für} \quad r \le 20 \\ 15 & \text{für} \quad r > 20 \end{cases}$$

5. (Total cost functions)
Es sei

x: Production
C: Total costs
C_f: Fixed costs
C_v: Variable costs
$C(x) = C_f + C_v(x)$

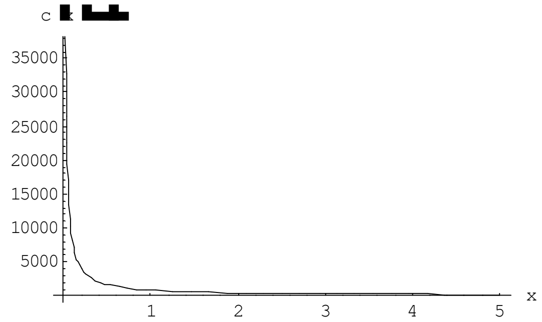
i) (Neoclassical production function): $C(x) = 2001 + 36 \cdot e^{0.01x}$

ii) (Piecewise defined cost function):

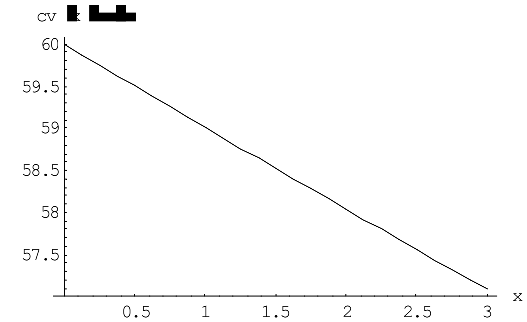
$$C(x) = \begin{cases} 0.25x + 3 & \text{für} \quad 0 < x \le 4 \\ 0.2x + 5 & \text{für} \quad 4 < x \le 8 \\ 0.5x + 3 & \text{für} \quad 8 < x \le 12 \\ 0.12x^2 - 2.5x + 21 & \text{für} \quad 12 < x \le 16 \end{cases}$$

6. (Average, average variable and average cost functions)

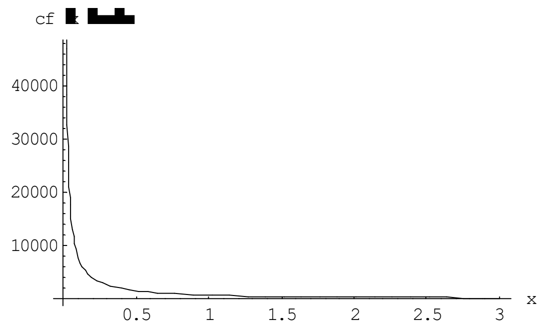
$$c(x) := \frac{C(x)}{x}, \quad x > 0, \qquad c(x) = \frac{800 + 0.01x^3 - x^2 + 60x}{x} = \frac{800}{x} + 0.01x^2 - x + 60$$



$$c_v(x) := \frac{C_v(x)}{x}, \quad x > 0, \qquad c_v(x) = \frac{0.01x^3 - x^2 + 60x}{x} = 0.01x^2 - x + 60$$

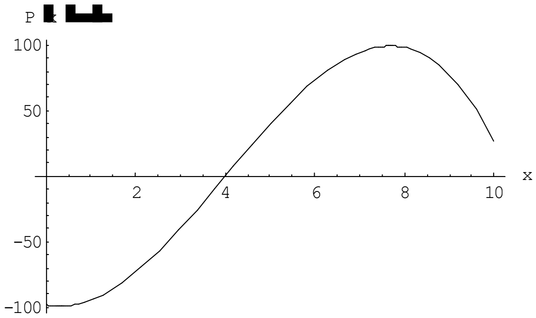


$$c_f := \frac{C_f}{x}, \quad x > 0, \qquad c_f(x) = \frac{800}{x}, \quad x > 0$$



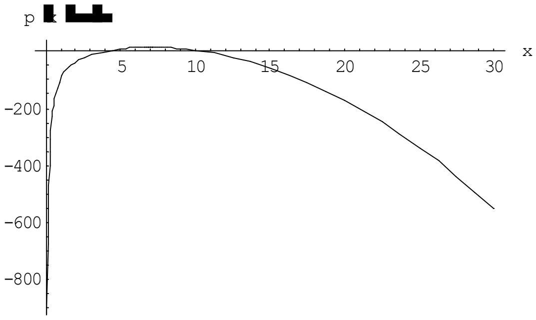
7. (Profit functions)

$$P(x) := R(x) - C(x), \qquad P(x) = 52.50x - \left(x^3 - 12x^2 + 60x + 98\right)$$
$$= -x^3 + 12x^2 - 7.5x - 98$$



8. (Average profit function)

$$p(x) := \frac{P(x)}{x}, \quad x > 0, \qquad p(x) = \frac{-x^3 + 12x^2 - 7.5x - 98}{x} = -x^2 + 12x - 7.5 - \frac{98}{x}$$



<u>Ex. 1. 2.</u>
Find the domain of the

1. production function

$$x(r) = \sqrt{2r - 200}$$

2. demand function

$$p(x) = \frac{100}{\sqrt{x}} - 4\sqrt{x} + 20$$

3. function

$$E(Y) = 200 \cdot \ln(Y + 100) - 750,$$

(E : monthly expenditure on energy; Y : monthly family income)

Solution:
1. $r \geq 100$
2. $x > 0$
3. $Y > 0$.

<u>D. 1. 2.</u> (*Limit of a Function*)
Let $f : A \to B$ be a real function and $x_0,\ l \in R$. Then

$$\left\langle \lim_{x\to a} f(x) = b \right\rangle \quad \Leftrightarrow \quad \left\langle \forall \varepsilon > 0 \ \exists \delta > 0: \ \left(0 < |x-a| < \partial\right) \ \Rightarrow \ |f(x) - b| < \varepsilon \right\rangle.$$

D. 1. 3. **(*One-sided Limits*)**

1. Let $f:\ A \to B$ be a real function defined in $\left]x_0,\ x_0+\delta\right[,\ \delta > 0$.

   A number l_r is called the *limit of* $f(x)$ as x *approaches* x_0 *from the right* (or simply called the *right-hand limit of* f *at* x_0, symbolically

$$\left\langle \lim_{x\to x_0^+} f(x) = l_r \right\rangle, \text{ if } \left\langle \forall \varepsilon > 0 \ \exists \delta > 0: \ \left(x_0 < x < x_0 + \delta\right) \ \Rightarrow \ |f(x) - l_r| < \varepsilon \right\rangle.$$

2. Let $f:\ A \to B$ be a real function defined in $\left]x_0-\delta,\ x_0\right[,\ \delta > 0$.

   A number l_l is called the *limit of* $f(x)$ as x *approaches* x_0 *from the left* (or simply called the *left-hand limit of* f *at* x_0, symbolically

$$\left\langle \lim_{x\to x_0^-} f(x) = l_l \right\rangle, \text{ if } \left\langle \forall \varepsilon > 0 \ \exists \delta > 0: \ \left(x_0 - \delta < x < x_0\right) \ \Rightarrow \ |f(x) - l_l| < \varepsilon \right\rangle.$$

T. 1. 1.

A function $f:\ A \to B$ has a limit as x approaches x_0 if and only if the left-hand and right-hand limits at x_0 exist and are equal. In symbols, we write

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x).$$

T. 1. 2. (***Uniqueness of Limits***)

$$\left\langle \lim_{x\to x_0} f(x) = a \wedge \lim_{x\to x_0} f(x) = b \right\rangle \ \Rightarrow \ a = b$$

D. 1. 4. **(*Infinite Limits*)**

1.
$$\left\langle \lim_{x\to x_0^-} f(x) = \infty \right\rangle, \text{ if } \left\langle \forall n \in N \ \exists \delta > 0: \ \left(x_0 - \delta < x < x_0\right) \ \Rightarrow f(x) > n \right\rangle.$$

2.
$$\left\langle \lim_{x\to x_0^-} f(x) = -\infty \right\rangle, \text{ if } \left\langle \forall n \in Z \setminus (N \cup \{0\}) \ \exists \delta > 0: \ \left(x_0 - \delta < x < x_0\right) \ \Rightarrow f(x) < n \right\rangle.$$

3.
$$\left\langle \lim_{x\to x_0^+} f(x) = \infty \right\rangle, \text{ if } \left\langle \forall n \in N \ \exists \delta > 0: \ \left(x_0 < x < x_0 + \delta\right) \ \Rightarrow f(x) > n \right\rangle.$$

4.
$$\left\langle \lim_{x\to x_0^+} f(x) = -\infty \right\rangle, \text{ if } \left\langle \forall n \in Z \setminus (N \cup \{0\}) \ \exists \delta > 0: \ \left(x_0 < x < x_0 + \delta\right) \ \Rightarrow f(x) < n \right\rangle.$$

5.

$$\left\langle \lim_{x\to x_0} f(x)=\infty \right\rangle, \text{ if } \left\langle \forall n\in N \;\; \exists\delta>0:\; 0<\left|x\text{ - }x_0\right|<\delta \;\Rightarrow\; f(x)>n \right\rangle.$$

6.

$$\left\langle \lim_{x\to x_0} f(x)=-\infty \right\rangle, \text{ if } \left\langle \forall n\in Z\setminus(N\cup\{0\}) \;\exists\delta>0:\;\; \left(0<\left|x-x_0\right|<\delta\right) \;\Rightarrow f(x)<n \right\rangle.$$

Ex. 1. 3.

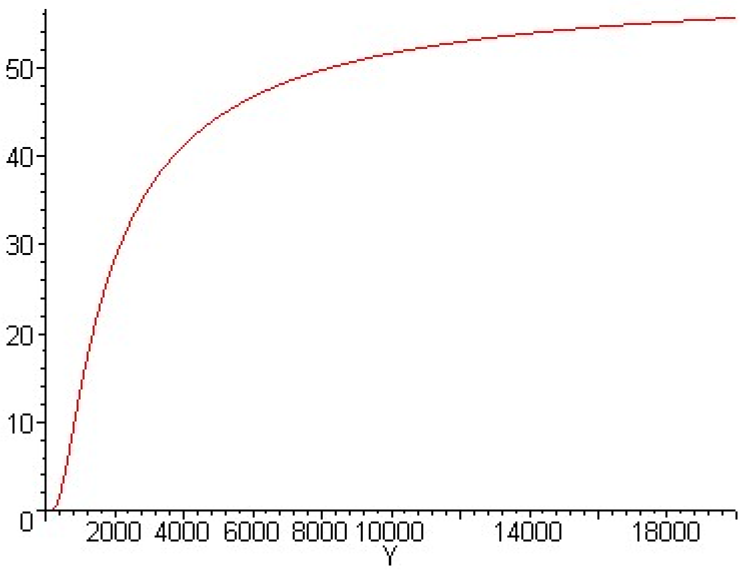
The following function expresses the dependency of the butter consumption B [€/month] of a family on its monthly income Y [€/month}:

$$B(Y)=60\cdot e^{-\frac{1500}{Y}}, \quad Y>0.$$

Investigate the change in butter consumption when the family income approaches zero?

Solution:

$$\lim_{Y\to 0^+} B(Y)=\lim_{Y\to 0^+} 60\cdot e^{-\frac{1500}{Y}}=60\cdot\lim_{Y\to 0^+}\frac{1}{e^{\frac{1500}{Y}}}=60\cdot 0=0.$$



D. 1. 5. (***Limit at Infinity***)

1.

$$\left\langle \lim_{x\to\infty} f(x)=l \right\rangle, \text{ if } \left\langle \forall\varepsilon>0 \;\; \exists n\in N:\; x>n \;\Rightarrow\; \left|f(x)\text{ - }l\right|<\varepsilon \right\rangle.$$

2.

$$\left\langle \lim_{x\to-\infty} f(x)=l \right\rangle, \text{ if } \left\langle \forall\varepsilon>0 \;\exists n\in Z\setminus(Z\cup\{0\}) \;\exists\delta>0:\;\; x<n \Rightarrow\; \left|f(x)-l\right|<\varepsilon \right\rangle.$$

Ex. 1. 4.
Given the function described in Ex. 1.3., find its point of satiation for the income approaching infinity.

Solution:

$$\lim_{Y\to\infty} B(Y) = \lim_{Y\to\infty} 60\cdot e^{-\frac{1500}{Y}} = 60\cdot e^{0} = 60\cdot 1 = 60$$

T. 1. 3.
Let $a, b, x_0 \in R$.

1. $\lim_{x\to x_0} b = b$.
2. $\lim_{x\to x_0} x = x_0$.
3. $\lim_{x\to x_0} \left(ax+b\right) = ax_0 + b$.
4. $\lim_{x\to x_0} |x| = x_0$.
5. $\lim_{x\to x_0} \sin x = \sin x_0$.
6. $\lim_{x\to x_0} \cos x = \cos x_0$.

T. 1. 4. (***The Algebra of Limits***)
Let f, g be two real functions. If $\lim_{x\to x_0} f(x) = a$ and $\lim_{x\to x_0} g(x) = b$, then the following rules hold:

1. $\lim_{x\to x_0} \left(f(x)+g(x)\right) = a+b$.
2. $\lim_{x\to x_0} \left(f(x)-g(x)\right) = a-b$.
3. $\lim_{x\to x_0} k\cdot f(x) = k\cdot a$ (k is any constant).
4. $\lim_{x\to x_0} f(x)\cdot g(x) = a\cdot b$.
5. $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{a}{b}$ if $b \neq `0$.

T. 1. 5. (***The Sandwich Theorem***)
Let $A \subseteq R$ and $f,\ g,\ h : A \to R$ be three real functions.

$$\left\langle g(x) \le f(x) \le h(x),\ \forall x \in A \quad \wedge \quad \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = a \right\rangle \quad \Leftrightarrow \quad \lim_{x\to x_0} f(x) = a \quad .$$

<u>T. 1. 6.</u> (*Elementary Limits of Circular Functions*)

1. $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$.
2. $\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$.
3. $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = 1$.

<u>T. 1. 7.</u> (*Limit Theorem for Composites*)

Let f and g be two real functions.

$$\left\langle \lim_{x \to a} f(x) = b \ \wedge \ \lim_{a \to b} g(u) = g(b) \right\rangle \quad \Leftrightarrow \quad \left\langle \lim_{x \to a} g\left(f(x)\right) = g(b) \right\rangle.$$

or, equivalently,

$$\lim_{x \to a} g \circ f(x) = g\left(\lim_{x \to a} f(x)\right).$$

<u>D. 1. 6.</u> (*Continuity*)

Let $f : A \to R$ be a real function. f is said to be continuous at $x = x_0$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

or, equivalently

$$\forall \varepsilon > 0,\ \exists \delta > 0: \ \ |x - x_0| < \delta \ \Rightarrow \ |f(x) - f(x_0)| < \varepsilon .$$

<u>R. 1. 2.</u>

Here and throughout, the symbol $f \in C(x_0)$ means „f is continuous at x_0“. Thus, we obtain a rule to test for continuity at a given Point:

<u>R. 1. 3.</u> (*Continuity Test*)

$$\begin{aligned} f \in C(x_0) \ \Leftrightarrow \ & (i)\ x_0 \in A, \\ & (ii)\ \lim_{x \to x_0} f(x) \text{ exists, and} \\ & (iii)\ \lim_{x \to x_0} f(x) = f(x_0). \end{aligned}$$

<u>Ex. 1. 5.</u>

A firm that produces some amount of output x has the following cost function:

$$C(x) = \begin{cases} (x-4)^2 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

1. What is the right-hand limit of this function as output approaches zero?
2. What is the function actually equal to when output is zero?
3. Is the function continuous? If not, explain why not.

Solution:

1.

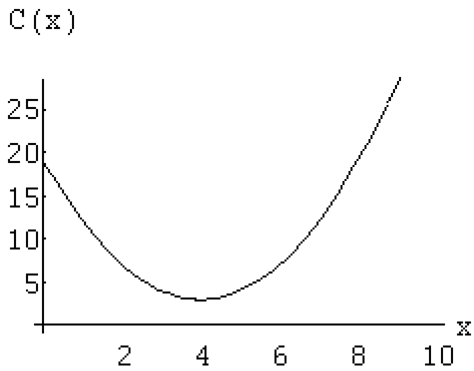
$$\lim_{x\to 0^+} C(x) = 19 .$$

2.

$$C(0) = 0 .$$

3.

No, since $\lim_{x\to 0} C(0) \neq C(0)$.



T. 1. 8.

If f is continuous at $x = x_0$, then the following combinations are also continuous at $x = x_0$:

$(i)\, f + g$,
$(ii)\, f - g$,
$(iii)\ k \cdot f \quad (k \in R)$,
$(iv)\, f \cdot g$,
$(v)\ \dfrac{f}{g}$, provided $g(x_0) \neq 0$.

T. 1. 9.

If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composite $g \circ f$ is continuous at x_0.

D. 1. 7. (One-sided Continuity)

1. A function f is called *continuous from the left at* x_0, if

$$\lim_{x\to x_0^-} f(x) = f(x_0) .$$

2. A function f is called *continuous from the right at* x_0, if

$$\lim_{x\to x_0^+} f(x) = f(x_0) .$$

3. If f is defined on $[a,\ b]$, continuous on $]a,\ b[$ and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$, then f is said to be *continuous on* $[a,\ b]$.

T. 1. 10.

$$f \in C(x_0) \quad \Leftrightarrow \quad \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

T. 1. 11. (***Fundamental Theorem on Continuous Real Functions***)
If $f:\ [a,\ b] \to R$ is a continuous function, then $f([a,\ b]) = [c,\ d]$ for some suitable $c, d \in R$. In addition, if a real function is continuous on $[a,\ b]$, then f attains an absolute maximum value M and an absolute minimum value m somewhere on this interval.

T. 1. 12. (***Min-Max Theorem for Continuous Functions***)
If $f:\ [a,\ b] \to R$ is a continuous function, then there exist $x_1, x_2 \in [a,\ b]$ such that

$$f(x_1) \le f(x) \le f(x_2), \quad \forall x \in [a,\ b].$$

T. 1. 13. (***Intermediate Value Theorem for Continuous Functions***)
If $f:\ [a,\ b] \to R$ is a continuous function and if $f(a) \neq f(b)$, then for any $k \in [f(a), f(b)]$ (or $[f(b), f(a)]$), there exists a number $c \in [a,\ b]$ such that $f(c) = k$.

T. 1. 14. (***Intermediate Zero Theorem***)
If $f:\ [a,\ b] \to R$ is a continuous function and $f(a) \cdot f(b) < 0$, then there exists $c \in [a,\ b]$ such that $f(c) = 0$

T. 1. 15. (***Boundedness Theorem***)
If $f:\ [a,\ b] \to R$ is a continuous function, then there exists a positive number M such that

$$|f(x)| \le M, \quad \forall x \in [a,\ b].$$

D. 1. 8. (***Uniform Continuity***)
If $f:\ [a,\ b] \to R$ be a real function. f is said to be *uniformly continuous on* f if

$$\forall \varepsilon > 0,\ \exists \delta > 0: \ \forall x, y \in A,\ |x - y| < \delta \ \Rightarrow \ |f(x) - f(y)| < \varepsilon.$$

R. 1. 4.

Uniform continuity $\Rightarrow$ continuity, but not the converse

D. 1. 9. (***Derivative***)
Let $f:\ A \to R$ and $x_0 \in]a,\ b[\subseteq A$. The *derivative* of f is a function f' whose value at x_0 is the number

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

or, equivalently

$$f'(x_0) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x_0)}{\Delta x}$$

with $x = x_0 + \Delta x$.

Generally speaking, the derivative of f at any $x_0 \in \left]a,\ b\right[\subseteq A$ is given by

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If the derivative $f'(x_0)$ exists, we say that f *has a derivative* (or, *is differentiable*) *at* x_0. If f has a derivative at every point of its domain, then f is said to be *differentiable*.

Ex. 1. 6.
A firm has the following revenue function

$$E(x) = 150x - 0.5x^2.$$

Find its derivative at $x_0 = 2$.

Solution:

$$E'(2) = \lim_{h \to 0} \frac{E(2+h) - E(2)}{h}$$

$$= \lim_{h \to 0} \frac{\left[150 \cdot (2+h) - 0.5 \cdot (2+h)^2\right] - \left[150 \cdot 2 - 0.5 \cdot 2^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{148h - h^2}{h}$$

$$= \lim_{h \to 0} (148 - h) = 148.$$

D. 1. 10. (*One-sided Derivative*)
Let $f:\ A \to R$ be a real function. Then f is said to be *differentiable on* $\left[a,\ b\right]$ if f' exists for all $x \in \left]a,\ b\right[$ and the limits

(i) $$f_+^{'}(a) := \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad (\textit{right-hand derivative at } a)$$

or, equivalently,

$$f_+^{'}(a) := \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

and

(ii) $$f_{-}^{'}(a) := \lim_{h \to 0^{-}} \frac{f(b+h) - f(b)}{h}$$ (*left-hand derivative at b*)

or, equivalently,

$$f_{-}^{'}(a) := \lim_{x \to b^{-}} \frac{f(x) - f(b)}{x - b}$$

exists at the endpoints a and b.

T. 1. 16.
A real function $f: A \to R$ is differentiable at $x = x_0$ if and only if

$$f_{+}^{'}(x_0) = f_{-}^{'}(x_0) = f'(x_0).$$

T. 1. 17. (***Differentiability-Continuity Theorem***)
If a real function f is differentiable at x_0, then f is continuous at x_0.

T. 1. 18. (***Algebra of Derivatives***)
If f and g are real functions that are differentiable at x, then $f \pm g, f \cdot g$ and f / g are differentiable at x (in the case of f / g provided that $g(x) \neq 0$). Moreover,

(i) $$\left(f \pm g\right)'(x) = f'(x) \pm g'(x)$$ (*Sum-Difference Rule*)

(ii) $$\left(f \cdot g\right)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$ (*Product Rule*)

(iii) $$\left(f / g\right)'(x) = \frac{f(x) \cdot g'(x) - g(x) \cdot f'(x)}{\left(g(x)\right)^2}$$ (*Quotient Rule*)

T. 1. 19. (***Chain Rule***)
Let the function F be defined as f composed with g, that is $F = f \circ g = f(g(x))$. Then F' is given by

$$F' = \frac{dF}{dx} = f'(g(x)) \cdot g'(x).$$

R. 1. 5.
Alternatively, in Leibniz notation, the chain rule can be expressed as

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Ex. 1. 7.
A firm's cost function is given by

$$C(x) = 200 \cdot e^{0.01x+400}, \quad x \geq 0 .$$

Find the marginal cost function.

Solution:

$$z = 0.01x + 400, \qquad \frac{dz}{dx} = 0.01$$

$$C(x) = 200 \cdot e^{z}, \qquad \frac{dC}{dz} = 200 \cdot e^{z}$$

$$C'(x) = \frac{dC(x)}{dx} = \frac{dz}{dx} \cdot \frac{dC(x)}{dz} = 0.01 \cdot 200 \cdot e^{z} = 2e^{0.01x+400} .$$

<u>D. 1. 11.</u> (*Elasticity Function*)

Let f be a differentiable function on A . The function

$$\varepsilon_{f,x}(x) := \frac{x}{f(x)} \cdot f'(x)$$

will be called the *elasticity function of f on A* .

<u>R. 1. 6.</u>

1. The elasticity $\varepsilon_{f,x}$ can be interpreted as follows: A percentage change of x leads to an approximate change of f by $\varepsilon_{f,x}$.

2. $\varepsilon_{f,x} > 0$ means that both x and f change in the same direction; $\varepsilon_{f,x} < 0$ means that x and f change in opposite directions.

3. Following cases can be distinguished:

$\lvert\varepsilon_{f,x}\rvert < 1$:	f is *inelastic*,
$\lvert\varepsilon_{f,x}\rvert > 1$:	f is *elastic*
$\lvert\varepsilon_{f,x}\rvert = 1$:	f is *proportional elastic*,
$\lvert\varepsilon_{f,x}\rvert \to \infty$:	f is *completely elastic*,
$\varepsilon_{f,x} \equiv 0$	f is *completely inelastic*.

4. Let f be denoted by $y = f(x)$ und its inverse by $x = g(y)$. Then

$$\varepsilon_{y,x} \cdot \varepsilon_{x,y} = 1 .$$

<u>Ex. 1. 8.</u>

A firm has the revenue function

$$R(x) = 300x - 2.5x^{2} .$$

Find the elasticity function $\varepsilon_{E,x}\left(x\right)$ at the point $x = 10$ and interpret your result

Solution:

$$\varepsilon_{R,x}(x) = \frac{x}{300x - 2.5x^2} \cdot (300 - 5x),$$

$$\varepsilon_{R,x}(10) \approx 0.91.$$

An increase of demand from 10 units by 1% will lead to an increase of the revenue of the firm by approximately 0.91%. The revenue function is inelastic at $x = 10$.

<u>D. 1. 12.</u> (*Differential*)
Let f be a differentiable function at x and $\Delta x \neq 0$. The difference between $f(x + \Delta x)$ and $f(x)$, denoted by Δf,

$$\Delta f := f(x + \Delta x) - f(x)$$

is called the *increment of* f *from* x *to* $x + \Delta x$.
The product $f'(x)\Delta x$ is called *differential of* f *at* x *with increment* Δx, and is denoted by df,

$$df := f'(x)\Delta x$$

or, equivalently,

$$dy := f'(x)\Delta x.$$

<u>R. 1. 7.</u>
In the above definition, Δx can be any nonzero value. However, in most applications of differentials, we choose $dx = \Delta x$. Thus, we also write

$$dy := f'(x)dx.$$

We have

$$\Delta f \approx df$$

or, equivalently

$$\Delta y \approx f'(x)\Delta x = f'(x)dx$$

<u>Ex. 1. 9.</u>
The total cost function of a firm is given by

$$C(x) = 0.06x^3 - 2x^2 + 60x + 200.$$

The firm would like to increase its production from 10 to 12 units. Find the approximate increase of costs using the differential of the cost function.

Solution:

$$dC(x) = C'(x) \cdot dx_{|x=10,\, dx=2} = \left(0.18x^2 - 4x + 60\right) \cdot dx_{|x=10,\, dx=2} = 76.$$

<u>D. 1. 13.</u> (*Maximum and Minimum of a Function*)
Let $f:\ A \to R$ be a function.
1. A number M is called the *maximum value of* f *over* A if

$$f(x) \le M,\ \forall x \in A\ \wedge\ f(c) = M \text{ for some } c \in A.$$

2. A number m is called the *minimum value of* f *over* A if

$$f(x) \ge m,\ \forall x \in A\ \wedge\ f(c) = m \text{ for some } c \in A.$$

<u>T. 1. 20.</u>
Let $f:\ A \to R$ be a function and $c \in A$. If $f'(c)$ exists and $f'(c) \neq 0$, then $f(c)$ is neither a maximum nor a minimum value of f in any neighbourhood of c.

Proof:
If $f'(c)$ and $f'(c) > 0$, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0 .$$

The there exists an Interval $\left]c-\delta,\ c+\delta\right[$ such that

$$\frac{f(x)-f(c)}{x-c} > 0,\ \forall x \in \left]c-\delta,\ c\right[\cup \left]c,\ c+\delta\right[.$$

This implies that $f(x) - f(c)$ and $x - c$ have the same sign in $\left]c-\delta,\ c\right[\cup \left]c,\ c+\delta\right[$, that is,

$$f(x) < f(c) \text{ if } x < c$$

and

$$f(x) > f(c) \text{ if } x > c .$$

Hence, $f(c)$ is neither a maximum nor a minimum value of f in any neighbourhood of c. A similar argument holds if $f'(c) < 0$.

<u>T. 1. 21.</u> (*Contrapositive of T. 1. 20*)
Let $f:\ A \to R$ be a function and $c \in A$. If $f(c)$ is either a maximum or a minimum value of f in some neighbourhood of c, then either $f'(c)$ does not exist or $f'(c) = 0$.

<u>D. 1. 14.</u> (*Critical Number*)
Let $f:\ A \to R$ be a function. A number $c \in A$ is called a *critical number of* f if either $f'(c)$ does not exist or $f'(c) = 0$.

<u>T. 1. 22.</u> (*Rolle*)
Let $[a,\ b] \subseteq A$ and $f:\ A \to R$ be a function. If
(i) f is continuous on $[a,\ b]$,

(ii) f is differentiable on $]a,\ b[$, and
(iii) $f(a) = f(b)$,

then there exists an element $c \in]a,\ b[$ such that $f'(c) = 0$

T. 1. 23. (***Mean Value Theorem***)
Let $[a,\ b] \subseteq A$ and $f:\ A \to R$ be a function. If f is continuous on $[a,\ b]$ and differentiable on $]a,\ b[$, then there exists a number $c \in]a,\ b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

D. 1. 15. (***Monotonic Functions***)
Let $f:\ A \to R$ be a function.
(i) f is said to be *monotonic increasing* if $f(x_1) \le f(x_2),\ \ \forall x_1, x_2 \in A : x_1 < x_2$.
(ii) f is said to be *monotonic decreasing* if $f(x_1) \ge f(x_2),\ \ \forall x_1, x_2 \in A : x_1 < x_2$.
(iii) f is said to be *strictly increasing* if $f(x_1) < f(x_2),\ \ \forall x_1, x_2 \in A : x_1 < x_2$.
(iv) f is said to be *strictly decreasing* if $f(x_1) > f(x_2),\ \ \forall x_1, x_2 \in A : x_1 < x_2$.
(v) f is said to be *monotonic* if f is either increasing in A or decreasing in A.
(vi) f is said to be *strictly monotonic* if f is either strictly increasing in A or strictly decreasing in A.

T. 1. 24. (***Monotonicity Theorem***)
Let $f:\ A \to R$ be a function. If f is continuous on $[a,\ b] \subseteq A$ and differentiable on $]a,\ b[$, then

(i) $f'(x) > 0,\ \forall x \in]a,\ b[\ \Rightarrow f$ is strictly increasing on $[a,\ b]$.
(ii) $f'(x) < 0,\ \forall x \in]a,\ b[\ \Rightarrow f$ is strictly decreasing on $[a,\ b]$.

Ex. 1. 10.
Find the intervals in which the profit function

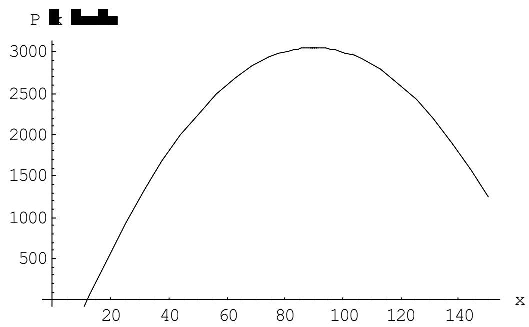
$$P(x) = -0.5x^2 + 90x - 1000$$

is strictly monotonic and sketch the graph of f.

Solution:

$P'(x) = -x + 90$,
$P'(x) > 0\ \Rightarrow\ x < 90$.

$\therefore\ f$ is strictly increasing in $]0,\ 90[$ and strictly decreasing for $x > 90$.



<u>D. 1. 16.</u> (*Extrema of a Function*)
Let $f: A \to R$ be a function and $x_0 \in A$. We say that

(i) $f(x_0)$ is a *relative maximum of* f *on* A if $f(x) \le f(x_0),\ \forall x \in A \cap U_\varepsilon(x_0)$.

(ii) $f(x_0)$ is a *relative minimum of* f *on* A if $f(x) \ge f(x_0),\ \forall x \in A \cap U_\varepsilon(x_0)$.

(iii) $f(x_0)$ is an *absolute maximum of* f *on* A if $f(x) \le f(x_0),\ \forall x \in A$.

(iii) $f(x_0)$ is an *absolute minimum of* f *on* A if $f(x) \ge f(x_0),\ \forall x \in A$.

<u>T. 1. 25.</u> (*A Necessary Condition for Relative Extrema*)
If a function f has a relative extremum at a number x_0, then $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

<u>T. 1. 26.</u>
If f is a differentiable function on a set A and has a relative extremum at x_0, then $f'(x_0) = 0$.

<u>T. 1. 27.</u> (*First Derivative Test for Extrema*)
Let x_0 be a critical number of f and f be continuous at x_0. If there exists an $\varepsilon > 0$ such that

(i)
$$\left\langle f'(x) < 0,\ \forall x \in \left]x_0 - \varepsilon,\ x_0\right[\ \wedge f'(x) > 0,\ \forall x \in \left]x_0,\ x_0 + \varepsilon\right[\right\rangle$$
$$\text{then}$$
$$\left\langle f(x_0) \text{ is a relative minimum} \right\rangle.$$

(ii)
$$\left\langle f'(x) > 0,\ \forall x \in \left]x_0 - \varepsilon,\ x_0\right[\ \wedge f'(x) < 0,\ \forall x \in \left]x_0,\ x_0 + \varepsilon\right[\right\rangle$$
$$\text{then}$$
$$\left\langle f(x_0) \text{ is a relative maximum} \right\rangle.$$

(iii)
$$\left\langle f'(x) \text{ keeps the same sign on } \left]x_0 - \varepsilon,\ x_0\right[\cup \left]x_0,\ x_0 + \varepsilon\right[\right\rangle$$
$$\text{then}$$
$$\left\langle f(x_0) \text{ is not an maximum} \right\rangle.$$

T. 1. 28. (***Second Derivative Test for Extrema***)
If x_0 is a critical number of a function f which is twice differentiable on an interval $]x_0 - \varepsilon,\ x_0 + \varepsilon[$ for some $\varepsilon > 0$, then

(i)

$$f''(x_0) < 0 \quad \Rightarrow \quad f(x_0) \text{ is a relative maximum of } f.$$

(ii)

$$f''(x_0) > 0 \quad \Rightarrow \quad f(x_0) \text{ is a relative minimum of } f.$$

R. 1. 8.
An algorithm for obtaining the extrema of a given continuous function f on the interval $[a,\ b]$ is given as follows:

1. Determine all critical numbers of f on $]a,\ b[$.
2. Calculate the value of f at all its critical numbers and also $f(a)$ and $f(b)$.
3. Compare all the value of f in 2. and the largest one is the maximum value of f on $[a,\ b]$ while the smallest value is the minimum value of f on $[a,\ b]$.

Ex. 1. 11.
The profit function of a firm depending on the price charged for its product is given by

$$P(p) = -p^3 + 4800p\text{-}119000, \qquad\qquad 0 \le p \le 45.$$

Find the price for which its profit will be maximised.

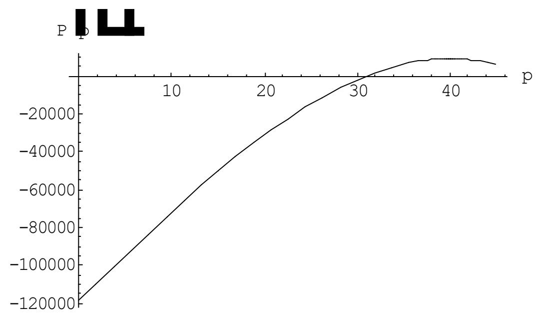
Solution:
1.
$P'(p) = -3p^2 + 4800 := 0,\quad 0 < p < 45 \qquad \Rightarrow \qquad p = 40$

$P''(p) = -6p < 0$.

Hence, the only critical numbers of P on $]0,\ 45[$ is $p = 40$ with $P(40) = 9000$.

2.
$P(0) = -119000$, $P(45) = 5875$.

3.
Because of $-119000 < 5875 < 9000$ the firm's profit will be maximised for $p = 40$:



D. 1. 17. (***Convex and Concave Function***)
Let f be a function which is continuous on $[a, b]$ and is differentiable on $]a, b[$. We say that

(i) f is convex if the graph of f lies above the tangent lines to f throughout $]a, b[$.
(ii) f is concave if the graph of f lies below the tangent lines to f throughout $]a, b[$.

T. 1. 29. (***Test for Convexity and Concavity***)
Let $f: A \to R$ be a function whose second derivative exists on $]a, b[$. Then we have:

(i) $f''(x) > 0, \ \forall x \in]a, b[\quad \Rightarrow \quad f$ is convex.
(ii) $f''(x) < 0, \ \forall x \in]a, b[\quad \Rightarrow \quad f$ is concave.

D. 1. 18. (***Inflection Points***)
The point $(x_0, f(x_0))$ is called a *point of inflection* of a function f if there exists an interval $]x_0 - \varepsilon, x_0 + \varepsilon[$ such that $f''(x) > 0, \ \forall x \in]x_0 - \in, x_0[$ and $f''(x) < 0, \ \forall x \in]x_0, x_0 + \varepsilon[$.

T. 1. 30. (***Test for Inflection Points***)
If $(x_0, f(x_0))$ is an inflection point of f, then either $f''(x_0) = 0$ or $f''(x_0)$ does not exist.

Ex. 1. 12.
A firm has the total cost function

$$C(x) = 8400x + \left(1008000x^2 - 3060x^3 + 3x^4\right) \cdot 10^{-4}, \quad x \geq 100.$$

Discuss the most important properties of its average cost function $c(x) := \dfrac{C(x)}{x}$.

Solution:
$c(x) = 8400 + 10^{-4}(1008000x - 3060x^2 + 3x^3)$

1. Domain

$D(c(x)) = \left[100, \ \bar{x}\right]$; $\quad \bar{x}$: maximum production capacity.

Let us assume that the firm has a maximum capacity of $\bar{x} = 800$.

1. *Continuity*

   $c(x)$ ist continuous on D.

3. *Points of Intersection with the Axes*

   a) *with the* $x-axis$

$c(x) := 0 \quad \Rightarrow \quad 8400 + 10^{-4}(1008000x - 3060x^2 + 3x^3) = 0$

It can be (numerically) shown that the graph of $c(x)$ does not cut the $x-$axis.

   b) *With the* $y-$ *axis*

Because of $x \geq 100$ there is also no point of intersection with the $y-$ axis.

4. *Monotonicity*

$$c'(x) = 10^{-4}(1008000 - 6120x + 9x^2)$$

$$c'(x) := 0 \qquad \Rightarrow \qquad x_1 = 280, \qquad x_2 = 400$$

$$(x-280)\cdot(x-400) \geq 0 \qquad \Rightarrow \qquad c(x) \text{ is non-decreasing.}$$

The solution of the above inequality leads to the following results:

$c(x)$ is non-decreasing for $\forall x \in]100, 280[\cup]400, 800[$
$c(x)$ is non-increasing for $\forall x \in]280, 400[$.

5. *Extrema*

$$c''(x) = 10^{-4}(-6120x + 18x); \qquad c''(280) = -0.108 < 0; \qquad c''(400) = 0.108 > 0.$$

Thus, $c(x)$ assumes a relative maximum at $x = 280$ with $c(280) = 19219.20$ and a relative minimum at $x = 400$ with $c(400) = 18960.00$.
Because of $c(100) = 15720.00$ and $c(800) = 46800.00$ the absolute minimum and the absolute maximum of $c(x)$ are $(100,\ 15720)$ and $(800,\ 46800)$, respectively.

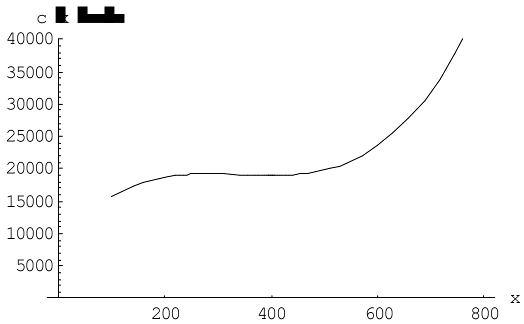
6. *Convexity and Concavity*

$$\left\langle c''(x) = 10^{-4}(-6120x + 18x) \geq 0 \right\rangle \quad \Rightarrow \quad \left\langle \begin{matrix} c(x) \text{ is convex}, \forall x \in]340,\ 800[\\ c(x) \text{ is cocave}, \forall x \in]100,\ 340[\end{matrix} \right\rangle, \text{ i.e. } x \geq 340$$

$\therefore$ $(340,\ 19089.60)$ is a point of inflection of $c(x)$.

The average costs increase progressively in $]340, 800[$ and regressively in $]100, 340[$.

7. *Graph of the Function:*



Exercises

1.

The production function of a firm is given by

$$x = \sqrt[3]{4r^2}, \quad r \geq 0,$$

Here are:

r: input,
x: output.

The firm has to pay 20 € for each unit of input and has 40 € fixed costs. Determine the firm's total cost function and plot it.

2.

A company has a profit function given by

$$P(x) = -x^2 + 20x + 312$$

where x denotes the quantity produced.

1. Find and interpret $x-$intercepts and $y-$intercepts of the curve $y = P(x)$.
2. What quantity produced gives the maximum profit? What is the maximum profit?
3. Use the above information to sketch the curve $y = P(x)$.
4. If the constant term in our expression for $P(x)$ is changed from 312 to 156, how does the answer to 2. change?
5. Given that the company has a linear cost function and that it costs 620 € to produce four units and 700 € to produce eight units, determine the cost function.

3.

The production function of a firm is given by

$$x(r) = -20 + 0.1r.$$

Here are:

r: input,
x: output.

The firm has to pay 2.00 € for each unit of input and has fixed costs amounting to 300.00 €. A marketing research has estimated the following price function:

$$p(x) = 220 - 4x.$$

1. Determine the total and the marginal cost functions of the firm.
2. At what level of production will the total costs be minimal?
3. Find the revenue and the marginal revenue functions.
4. How much should be produced so that the firm's profit will be maximised? Find the maximal profit.
5. Determine the price for which the profit will be maximal.

4.
Maximise profits P for a firm, given total revenue

$$R(x) = 4000x - 33x^2 \quad (x: \text{ sales})$$

and total cost

$$C(x) = 2x^3 - 3x^2 + 400x + 5000, \ (x: \text{ output}),$$

assuming $x > 0$.

5.
The total cost function of a firm is given by

$$C(x) = 8400x + \left(1008000x^2 - 3060x^3 + 3x^4\right) \cdot 10^{-4}.$$

Find the intervals in which the marginal costs are less than the average costs.

6.
Consider the demand function

$$x(p) = 100e^{-0.04p}, \ \ 0 \le p \le 100, \quad (x: \text{ demand; } p: \text{ price}).$$

Given the cost function

$$C(x) = 2000 + 30x$$

1. describe the revenue and the profit as functions of price.
2. determine the price for which the firm's profit will be maximal.

7.
The total costs of a firm are given by the function

$$C(x) = 30x + \frac{1450x^2}{400 + x^2}, \qquad 10 \le x \le 50.$$

1. Determine the marginal cost function. Calculate and interpret its value at $x = 20$.
2. Find the absolute minimum of the average cost function...
3. Investigate the monotonicity of the average costs function.

8.

Given the demand function

$$p(x) = 16 - 0.5x,$$

find and interpret

1. the elasticity of price with regards to a demand of $x_0 = 8$ units,
2. the elasticity of demand with regards to the corresponding price.

9.
The total cost function of a firm for a certain commodity is given by

$$C(x) = \frac{1}{240}x^3 - \frac{3}{8}x^2 + \frac{40}{3}x + 100$$

x: output,
$C(x)$: cost of producing x output units.

1. Determine the total cost of producing 40, 50, 60 and 70 units of output.
2. Sketch the total cost function.
3. Write down the revenue function, assuming that the firm charges 20 € for each unit of the commodity.
4. Draw the revenue function in the same sketch as the total cost function.
5. Which amount should be at least sold if no losses are to be incurred?
6. Find the firm's maximizing output as well as its maximum profit.
7. How much must be produced, if the firm's *variable average costs* are to be minimal?
8. The firm intends to increase its output from 70 to 71 units. Determine the increase in costs

   i) approximately,
   ii) exactly

9. Find the percentage increase of costs

   i) approximately,
   ii) exactly,

   if there is to be a one percent increase of output at the level of 48 units.
   How would you comment the results?

10.
Consider the following demand function:

$$x(p) = \frac{500}{p+5} - 10, \quad 0 \le p < 45 \qquad (x\text{: demand; } p\text{: price})$$

1. Find the marginal demand function.
2. Give a function describing approximately the percentage change of demand with regards to a percentage change of the price.
3. Calculate the elasticity of demand for prices for $x = 10$ and $x = 40$.
4. Determine the quantity of demand for which the price elasticity of demand is equal to –3.
5. Find the domains in which the demand function is elastic or inelastic.

11.
The following function gives the relation between the price of a commodity and the demand for it:

$$p(x) = 100 - 0.5x$$

The total cost of producing the commodity is given by the function

$$C(x) = 1000 + 10x$$

Find

1. the revenue function.
2. the profit function
3. the output guaranteeing maximal profit.

12.
The total cost function of a firm is given by

$$C(x) = 1000 + 20x$$

1. Determine the marginal cost function and interpret it at the point x_0.
2. Write the average cost function.
3. Investigate the behaviour of the total and the average cost functions for $x \to +\infty$.

13.
The total cost of a firm is given by the function

$$C(x) = 60 - 12x + 2x^2.$$

1. Determine the output which minimises the total costs.
2. Find the output for which the average costs will be minimised.
3. Argue why the marginal costs function cannot have a minimum.
4. Find the point of intersection of the marginal and the average costs curves.
5. Interpret the values of the marginal and average costs function at an arbitrary point x_0.

14.
The production function of a firm is given by

$$y = \sqrt{10r}\,, \quad (r : \text{input}; \ y : \text{output})$$

1. Is the marginal production function increasing or decreasing?
2. Find the total cost function, assuming that each unit of input costs 2 €.

15.
The production function of a firm is given by

$$x(r) = -\frac{1}{3}r^3 + 2r^2.$$

1. Find the input for which the production will be at its maximum level.
2. In which interval(s) will the production be „negative“?

3. Give the average production function.
4. Determine the amount of input which maximises the production per unit of input.
5. Interpret the first derivative of the production function.

16.
The costs C of producing a certain product are given by the function

$$C(x) = 0.06x^3 - 2x^2 + 60x + 200 .$$

C : total costs ,
x : output .

1. At present, we have the production level $x = 10$. Calculate the *approximate* change of costs if the production will be

   i) increased by 2 units,
   ii) decreased by 1 unit.

2. Find the *exact* change in costs.

17.
A firm that produces some amount of output x has the following cost function:

$$C(x) = \begin{cases} (x-4)^2 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases} .$$

1. What is the right-hand limit of this function as output approaches zero?
2. What is the function actually equal to when output is zero?
3. Is the function continuous? If not, explain why not.

18.
The total cost of a firm is given by the function

$$C(x) = 200 + 5x - 0.01x^2 + 0.01x^3 , \quad (x\text{: output; } C\text{: costs}).$$

1. Find the marginal cost function and interpret it for an output of 5 units.
2. Given the price function
$$p(x) = 50 - 0.01x ,$$

   determine the profit function of the firm.

3. Find the output for which the firm's profit will be maximised. How high will be this profit?
4. Producing at a level of 40 units, the firm increases its output by one percent. Find the resulting

   i) approximate
   ii) exact

   percentage change of profit.

19.

The total cost of a firm is given by the function

$$C(x)=\frac{1}{12}x^3-\frac{3}{4}x^2+\frac{13}{4}x,\quad x\in[1,\ 8],\qquad (x\text{: output; } C\text{: costs}).$$

1. Find the marginal cost function and interpret it for an output of 5 units.
2. Given the price function

$$p(x)=6-\frac{x}{2},\quad x\in[1,\ 8]\qquad (p\text{: price}),$$

   determine the profit function of the firm.

   Find the output for which the firm's profit will be maximised. How high will be this profit?
3. Producing at a level of 3 units, the firm increases its output by one percent. Find the resulting

   i) approximate

   ii) exact

   percentage change of profit.

20.

A company is manufacturing and selling insulated mugs. The company has monthly fixed costs of 1500 € and there is a total monthly cost of 1800 € when producing 100 mugs. Each mug sells for 7 €,

1. Find the cost, revenue, and profit functions for the mug manufacturer, assuming each is a linear function.
2. How many mugs must the company make and sell in order to break even?

21.

A monopolist's demand function is given as

$$p=15-q\quad (p\text{: price; } q\text{: demand}).$$

The firm's cost function is

$$C(q)=q^2+3q+2\,.$$

Find the level of output and the price level that maximises profit.

22.
The demand function for a commodity is given as

$$p = 10 - 0.5q \quad , \quad q \in [0,\ 20] \quad (p: \text{ price; } q: \text{ demand}).$$

1. Express the price elasticity of demand as a function of price
2. Determine the intervals in which the demand as a function of price is elastic or inelastic.

23.

A monopoly's demand function is given as

$$p = 15 - 6q$$

where q is output, p is price, and its total cost function is given by

$$C(q) = 2q^3 - 3q^2 + 3q + 2 .$$

Find the firm's maximising level of output, the price it will charge and the amount of profit it will make.

24.
A firm has the following production function:

$$Q(L) = 3L^2 - 0.1L^3 \quad (Q: \text{ Production; } L: \text{ Labour})$$

1. Find the firm's marginal product of labour.
2. Find the values of Q for which the marginal product of labour and the average product of labour are maximised.
3. Show that, when average product of labour is at a maximum, marginal product of labour equals average product of labour.

25.
The average cost function of a firm is

$$c(x) = 12 - 4x + x^2 \qquad (x: \text{ level of output}).$$

1. Derive the total and marginal cost functions.
2. Sketch the average and marginal cost curves in the same diagram.
3. If the firm takes the prices as given and price is 16.00 €, what quantity will it sell to maximise profit? What are the firm's profits at this output?

26.
A monopolist's cost function is

$$C(x) = \frac{x}{2500} \cdot (x - 100)^2 + x \qquad (x: \text{ level of output}).$$

It faces the demand function

$$p(x) = 4 - \frac{x}{25}.$$

1. The monopolist would like to maximise his profit. Find its output, the associated price, and its profit.
2. Sketch the marginal cost and the marginal revenue functions in the same diagram.

Solutions

1.

$$x = \sqrt[3]{4r^2}, \ \ r \geq 0 \quad \Rightarrow \quad r = \frac{1}{2}\sqrt{x^3}$$

$$C(x) \ = 40 + 10\sqrt{x^3} \ .$$

2.

1.

$x-$ *intercept*:

$$\left(-x^2 + 20x + 312 = 0 \ \wedge \ x \geq 0\right) \ \Rightarrow \ x = 30.29778313 \approx 30.30 \, ,$$

i. e. for an output of 20.30 units the company's profit will be equal to zero.

$$y - intercept\text{: } x = 0 \ \Rightarrow \ P(x) = 312,$$

i. e. for an output of zero units the company's profit will be equal to 312.

2.

$$P'(x) = -2x + 20 \, , \quad P'(x) = 0 \ \Rightarrow \ -2x + 20 = 0 \ \Rightarrow \ x = 10 \, ,$$

$$P''(x) = -2 < 0 \, .$$

Hence, $x = 10$ gives the maximum profit. This will be equal to $P(10) = 412$.

3.

Changing the value of the constant simply moves the profit curve up or down and, in particular, it does not change the $x-$ coordinate of the maximum. In this case, changing the original constant 312 to 156 means that the profit function is now

$$\tilde{P}(x) = -x^2 + 20x + 156 \, ,$$

and so the curve will move down by 312-156 = 156 which means that the maximum value will now be 412 – 156 = 256 and this will still occur when $x = 10$.

The effect of changing the constant in this way is illustrated in the following figure:

5.

$$x = 4 \ \Rightarrow \ P(4) = 620, \quad x = 8 \ \Rightarrow \ P(8) = 700 \, ,$$

$$C(x) - 620 = \frac{700 - 620}{8 - 4}\left(x - 4\right),$$

$$C(x) = 20x + 540$$

3.

1.

$$x(r) = -20 + 0.1r \quad \Rightarrow \quad r(x) = 10x + 200$$

$$C(x) = C_{\text{var}} + C_{fix}$$
$$= (10x + 200)\cdot 2 + 300 = 20x + 700$$

$$C'(x) = 20.$$

2.

The cost function has because of $C'(x) = 20 \neq 0$ no minimum.

3.

$$R(x) = p(x)\cdot x$$
$$= 220x - 4x^2,$$

$$R'(x) = 220 - 8x.$$

4.

$$P(x) = R(x) - C(x), \qquad P'(x) = R'(x) - C'(x) = 220 - 8x - 20.$$

$$P'(x) = 0 \quad \Rightarrow \quad x = 25,$$

$$P''(x) = -8 < 0, \qquad P(25) = 1800 \text{ €}.$$

The firm has, therefore, to produce 25 units in order to gain a maximum profit of 1800 €.

5.

$$p(25) = 120 \text{ €}$$

4.

$$P(x) = R(x) - C(x)$$
$$= 4000x - 33x^2 - \left(2x^3 - 3x^2 + 400x + 5000\right)$$
$$P(x) = -2x^3 - 30x^2 + 3600x - 5000$$
$$\left(P'(x) = -6x^2 - 60x + 3600 \ \wedge \ x > 0\right) \Rightarrow x = 20$$
$$P''(x) = -12x - 60, \quad P''(20) = -300 < 0.$$

$\therefore$ Profit is maximised at $x = 20$ where $P(20) = 39000.$

5.

$$C'(x) = 8400 + 10^{-4}(2016000x - 9180x^2 + 12x^3)$$
$$c(x) = 8400 + 10^{-4}(1008000x - 3060x^2 + 3x^3)$$

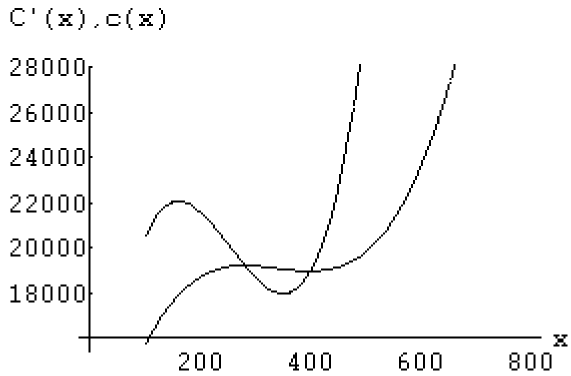
$$C'(x) < c(x)$$

$$\Leftrightarrow$$

$$8400+10^{-4}(2016000x-9180x^2+12x^3)<8400+10^{-4}(1008000x-3060x^2+3x^3)$$

$$\Leftrightarrow$$

$$x^3-680x^2+112000\text{x}<0 \Leftrightarrow \quad 280<x<400$$



6.

1.

$$R(p)=x(p)\cdot p$$
$$=100pe^{-0.04p}, \quad 0\le p\le 100$$

$$P(p)=R(p)-C(p)$$
$$=100pe^{-0.04p}-2000-3000e^{-0.04p}, \quad 0\le p\le 100$$

2.

$$P'(p)=e^{-0.04p}(220-4p), \quad 0<p<100$$

$$P''(p):=0 \quad \Rightarrow \quad p=55$$

$$P''(p)=-0.04e^{-0.04p}(220-4p)-4e^{-0.04p} \quad \Rightarrow \quad P''(p)<0.$$

The profit function $P(p)$ has because of $P(100)<P(55)$ its absolute maximum at $p=55$

7.

1.

$$C'(x)=30+\frac{1160000x}{(400+x^2)^2}, \quad 10<x<50$$

$$C'(20)=66.25\,€$$

An increase of the output by one unit leads to an approximate increase of the total costs by 66.25 €.

$$c(x) := \frac{C(x)}{x}$$

$$= 30 + \frac{1450x}{400 + x^2}, \qquad 10 \le x \le 50$$

$$c'(x) = \frac{-1450x^2 + 580000}{(400 + x^2)^2}, \qquad 10 < x < 50$$

$$c'(x) := 0 \qquad \Rightarrow \qquad x = 20$$

$$c''(x) = \frac{-2900x \cdot (400 + x^2) - 4x \cdot (580000 - 1450x^2)}{(400 + x^2)^3}, \qquad 10 < x < 50$$

$$c''(20) < 0$$

Consequently, at $x = 20$ the function $c(x)$ has over the interval $]10, \ 50[$ a relative maximum. Because of $c(10) = 59, \ c(50) = 55$ the function $c(x)$ has in $x = 50$ its absolute minimum.

3.
From 2. it follows that $c(x)$ increases on $[10, \ 20]$ and decreases on $[20, \ 50]$.

8.
1.

$$\varepsilon_{p,x}(x) = \frac{-0.5x}{16 - 0.5x}$$

$$\varepsilon_{p,x}(8) = -\frac{1}{3}.$$

2.

$$p(x) = 16 - 0.5p \ \Rightarrow \ x(p) = 32 - 2p$$

$$\varepsilon_{x,p}(p) = \frac{-2p}{32 - 2p}$$

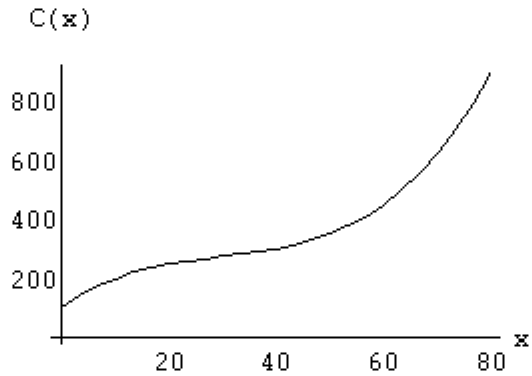
$$p(8) = 12$$

$$\varepsilon_{x,p}(12) = -3$$

9.
1.

$$C(40) = 300, \ C(50) = 350, C(60) = 450, C(70) = 625$$

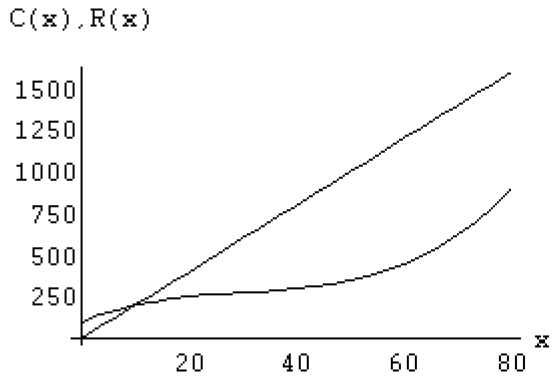
2.



3.

$R(x) = 20x$

4.



5.

$C(x) = R(x) = 10$ (See the curves above.)

6.

$$P(x) = R(x) - C(x) = -\frac{x^3}{240} + \frac{3x^2}{8} + \frac{20x}{3} - 100$$

$$P'(x) = -\frac{x^2}{80} + \frac{3x}{4} + \frac{20}{3} = 0$$

$x = 67.86$

$P''(x) = -\frac{x}{40} + \frac{3}{4}$, $\qquad P''(67.86) < 0$,

Maximum profit: $P(67.86) = 777.21$€

7.

$$c_v(x) = \frac{C_v(x)}{x} = \frac{x^2}{240} - \frac{3x}{8} + \frac{40}{3}$$

$$c_v'(x) = \frac{x}{120} - \frac{3}{8} := 0 \;\Rightarrow\; x = 45, \qquad c_v''(x) = \frac{1}{120} > 0$$

8.

i)

$$C'(70) = 22.0833$$

ii)

$$C(71) - C(70) = 647.5875 - 625 = 22.5875$$

9.

i)

$$\varepsilon_{K,x}(x) = \frac{\left(\frac{x^2}{80} - \frac{3x}{4} + \frac{40}{3}\right)x}{\left(\frac{x^3}{240} - \frac{3x^2}{8} + \frac{40x}{3} + 100\right)}$$

$$\varepsilon_{K,x}(48) = 0.874$$

The cost function is inelastic at $x = 48$.

ii)

$$C(48) = 336.8$$

$$C(48 \cdot 1.01) = C(48.48) = 339.79$$

$$\frac{339.79}{336.8} \cdot 100 - 100 = 0.89$$

10.

1.

$$x'(p) = \frac{-500}{(p+5)^2}$$

2.

$$\varepsilon_{x,p}(p) = \frac{\frac{-500p}{(p+5)^2}}{\frac{500}{p+5} - 10} = -\frac{50p}{(5+p)(45-p)}$$

3.

$$10 = \frac{500}{p+5} - 10 \;\Rightarrow\; p = 20$$

$$40 = \frac{500}{p+5} - 10 \;\Rightarrow\; p = 5$$

$$\varepsilon_{x,p}(20) = -1.6, \qquad \varepsilon_{x,p}(5) = -0.625$$

4.

$$\frac{-50p}{(5+p)(45-p)} = -3 \qquad \Rightarrow\; 3p^2 - 70p - 675 = 0 \quad \Rightarrow\; p = 30.67\,€$$

$$x(30.67) \approx 4.02$$

5.

$$\frac{-50p}{(5+p)(45-p)} = -1 \quad \Rightarrow \quad p^2 + 10p - 225 = 0 \quad \Rightarrow \quad p \approx 10.81\,€$$

i. e. the demand function is elastic for $p > 10.81$ and inelastic for $p < 10.81$.

11.

1.

$$R(x) = p(x) \cdot x$$
$$= -0.5x^2 + 100x$$

2.

$$P(x) = R(x) - R(x)$$
$$= -0.5x^2 + 90x - 1000$$

3.

$$P'(x) = -x + 90 \quad , \quad P'(x) := 0 \quad \Rightarrow \quad x = 90$$

$$P''(x) = -1 < 0 .$$

There will, therefore, be a maximum profit of 3050 € for $x = 90$. The corresponding price will be $p(90) = 55$ €.

12.
1.

$$C'(x) = 20.$$

A change of output by one unit from any level x_0 will result in a change of total costs by 20 €.

2.

$$c(x) := \frac{C(x)}{x} = \frac{1000}{x} + 20, \qquad x > 0$$

3.

$$\lim_{x \to +\infty} C(x) = \lim_{x \to +\infty} (1000 + 20x) = +\infty$$

$$\lim_{x \to +\infty} c(x) = \lim_{x \to +\infty} \left(\frac{1000}{x} + 20\right) = 20 = C'(x)$$

13.
1.

$$C'(x) = -12 + 4x$$

$$C'(x) := 0 \qquad \Rightarrow \qquad x = 3$$

$$C''(x) = 4 > 0.$$

Thus, the total costs function has its (absolute) minimum at $x = 3$ where $C(3) = 42$

2.

$$c(x) := \frac{C(x)}{x}$$
$$= \frac{60}{x} - 12 + 2x, \quad x > 0$$

$$c'(x) = -\frac{60}{x^2} + 2, \quad x > 0,$$

$$c'(x) := 0 \qquad \Rightarrow \qquad x = \sqrt{30},$$

$$c''(x) = \frac{120}{x^3} > 0$$

Thus, the average costs function $c(x)$ has its (absolute) minimum at $x = \sqrt{30}$.

3.
Since

$$C'(x) = -12 + 4x, \qquad C''(x) = 4 \neq 0$$

4.

$$-12 + 4x = \frac{60}{x} - 12 + 2x \quad \Leftrightarrow \quad x = \sqrt{30}\,.$$

5.
$C'(x_0)$ gives the approximate change of costs after a change of the output x_0 by one unit.
$c(x_0)$ gives the costs per unit for an output x_0 .

14.

$$y' = \frac{\sqrt{10}}{2\sqrt{r}}, \qquad y'' = -\frac{\sqrt{10}}{4\sqrt{r^3}} < 0,$$

Thus, the marginal production function is decreasing.

15.
1.

$$\begin{aligned} x'(r) &= -r^2 + 4r \\ &= r(4-r) \end{aligned}$$

$$x'(r) := 0 \;\Rightarrow\; r = 4$$

Because of

$$x''(r) = -2r + 4, \quad x''(4) < 0$$

has $x(r)$ a relative (its absolute) maximum for $r = 4$.

2.

$$x(r) = -\frac{1}{3}r^3 + 2r^2 = r^2(2 - \frac{1}{3}r) < 0 \;\Leftrightarrow\; 2 - \frac{1}{3}r < 0 \;\Leftrightarrow\; r > 6$$

3.

$$\frac{x(r)}{r} = -\frac{1}{3}r^2 + 2r$$

4.
Because of

$$\left(\frac{x(r)}{r}\right)' = -\frac{2}{3}r + 2 := 0 \;\Rightarrow\; r = 3, \quad \left(\frac{x(r)}{r}\right)'' = -\frac{2}{3} < 0$$

the function $\frac{x(r)}{r}$ has a relative (its absolute) maximum at $r = 3$.

5.
$x'(r_0)$ means: the production will approximately change by $x'(r_0)$ units, if the input r_0 will be changed by one unit.

16.

1.

$$dC(x) = C'(x) \cdot dx$$
$$C'(x) = 0.18x^2 - 4x + 60$$

3. $dC(x = 10, dx = 2) = 76$ €
4. $dC(x = 10, dx = -1) = -38$ €

2.

6. $\Delta C = C(12) - C(10) = 75.68$ €
7. $\Delta C = C(9) - C(10) = -38.26$ €

8. $\Delta C = C(12) - C(10) = 75.70$ €
9. $\Delta C = C(9) - C(10) = -38.30$ €

17.

1.

$$\lim_{x \to 0^+} C(x) = 16.$$

2.

$$C(0) = 0.$$

3.
No, since $\lim_{x \to 0} C(0) \neq C(0)$.

18.

1.

$$C'(x) = 5 - 0.02x + 0.03x^2$$

An increase of output from 5 to 6 units will lead to an approximate increase of costs by 5.65 units.

2.

$$P(x) = R(x) - C(x)$$

$$= p(x) \cdot x - C(x)$$

$$= (50 - 0.01x) \cdot x - \left(200 + 5x - 0.01x^2 + 0.01x^3\right)$$

$$= -200 + 45x - 0.01x^3$$

3.

$$P'(x) = 45 - 0.03x^2$$

$$P'(x) = 0 \ \wedge \ x \geq 0 \ \Rightarrow \ x = 38.73$$

$$P''(x) = -0.06x, \quad P''(38.73) < 0.$$

$\therefore$ The firm will maximise its profit for an output of 38.73 units. The maximum profit will amount to $P(38.73x) = 961.90$.

4.

i)

$$\varepsilon_{P,x}(x) = \frac{x}{P(x)} \cdot P'(x)$$

$$\varepsilon_{P,x}(x) = \frac{x}{-0.01x^3 + 45x - 200} \cdot \left(-0.03x^2 + 45x\right)$$

$$\varepsilon_{P,x}(40) = \frac{40}{-0.01 \cdot 40^3 + 45 \cdot 40 - 200} \cdot \left(-0.03 \cdot 40^2 + 45 \cdot 40\right) \approx -0.125\ \%.$$

ii)

$$P(40 \cdot 1.01) = P(40.4) = 958.61, \qquad P(40) = 960.00.$$

$$\frac{958.61}{960.00} \cdot 100 - 100 = -0.145\%\ \%$$

19.

$$C(x) = \frac{1}{12}x^3 - \frac{3}{4}x^2 + \frac{13}{4}x, \quad x \in [1,\ 8], \qquad (x\text{: output; } C\text{: costs}).$$

1.

$$C'(x) = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}, \quad x \in [1,\ 8]$$

$$C'(2) = 2.$$

An increase of output from 5 to 6 units will lead to an approximate increase of costs by 2 units.

2.

$$P(x) = R(x) - C(x)$$

$$= \left(6 - \frac{x}{2}\right) \cdot x - \left(\frac{1}{12}x^3 - \frac{3}{4}x^2 + \frac{13}{4}x\right)$$

$$P(x) = -\frac{1}{12}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x, \quad x \in [1,\ 8]$$

$$P'(x) = -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{11}{4}, \quad x \in [1,\ 8]$$

$$P'(x) = 0, \quad -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{11}{4} = 0, \quad x \in [1,\ 8] \quad \Rightarrow \quad x \approx 4.46$$

$$P''(4.46) < 0.$$

$$P(4.46) \approx 9.85, \quad P(1) \approx 2.92, \quad P(8) \approx -4.67\,.$$

Hence, the firm's profit will be maximal for an output of $x \approx 4.46.$ It will be approximately equal to 9.85.

3.

i)

$$\varepsilon_{P,x}(x) = \frac{x}{P(x)} \cdot P'(x)$$

$$= \frac{x}{-\frac{1}{12}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x} \cdot \left(-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{11}{4}\right).$$

$$\varepsilon_{P,x}(3) = \frac{3}{8.25} \cdot 2 \approx 0.73$$

ii)

$$P(3) = 8.25, \qquad P(3.03) \approx 8.31$$

$$\frac{8.31}{8.25} \cdot 100 - 100 \approx 0.73$$

20.

1.

$$C(x) - 1500 = \frac{1800 - 1500}{100 - 0} \cdot (x - 0)$$

$$C(x) = 3x + 1500$$
$$R(x) = 7x$$

$$P(x) = R(x) - C(x) = 4x - 1500.$$

2.
The company will break even when $P(x) = 0$. Therefore,

$$4x - 1500 = 0, \qquad x = 375.$$

So, by making and selling 375 mugs, the company will neither gain nor lose money.

21.

$$R(q) = 15q - q^2$$

$$P(q) = R(q) - C(q)$$

$$= -2q^2 + 12q + 2$$

$$P'(q) = -4q + 12, \qquad -4q + 12 = 0, \qquad q = 3.$$

Because of $P''(q) = -4 < 0$, $q = 3$ maximises the firm's profit. The corresponding price level is $p(3) = 12$.

22.
1.

$$p = 10 - 0.5q \quad \Rightarrow \quad q = 20 - 2p, \quad p \in [0,\ 10]$$

$$\varepsilon_{q,p}(p) = \frac{-2p}{20 - 2p}$$

2.

$$\frac{-2p}{20 - 2p} < -1, \qquad \frac{-p}{10 - p} + 1 < 0, \qquad \frac{-2p + 10}{10 - p} < 0.$$

Case 1:

$$\begin{cases} -2p + 10 > 0 \\ -p + 10 < 0 \end{cases} \Rightarrow \ p \in \varnothing$$

Case 2:

$$\begin{cases} -2p + 10 < 0 \\ -p + 10 > 0 \end{cases} \Rightarrow \ 5 < p < 10.$$

The demand is elastic for $p \in]5,\ 10[$ and inelastic for $p \in]0,\ 5[$.

23.

$$R(q) = p(q) \cdot q, \qquad R(q) = 15q - 6q^2,$$

$P(q) = R(q) - C(q)$, $\qquad P(q) = -2q^3 - 3q^2 + 12q - 2$,

$P'(q) = -6q^2 - 6q + 12$

$\left(P'(q) = 0 \ \wedge \ q \geq 0\right) \ \Rightarrow \ q = 1$, $\quad P''(1) = -12 < 0$,
$p(1) = 9, \quad P(1) = 5.$

24.
1.

$C(x) = 12x - 4x^2 + x^3$

$C'(x) = 12 - 8x + 3x^2$

2.
(Do it yourselves.)

3.

$R(x) = 16x$

$P(x) = R(x) - C(x) = 16x - \left(12x - 4x^2 + x^3\right)$
$P(x) = 4x + 4x^2 - x^3$, $\quad P'(x) = 4 + 8x - 3x^2$

$\left(4 + 8x - 3x^2 = 0 \ \wedge \ x \geq 0\right) \ \Rightarrow \ x \approx 3.10$,

Because of

$$P''(x) = 8x - 6x, \ \ P''(3.10) < 0$$

the firm should sell 3.10 units in order to maximise profit. The maximum profit will be $P(3.10) \approx 21.05$ €.

25.
1.

$C(x) = 12x - 4x^2 + x^3$

$C'(x) = 12 - 8x + 3x^2$

2.
(Do it yourselves.)
3.

$R(x) = 16x$

$P(x) = R(x) - C(x) = 16x - \left(12x - 4x^2 + x^3\right)$

$$P(x) = 4x + 4x^2 - x^3, \qquad P'(x) = 4 + 8x - 3x^2$$

$$\left(4 + 8x - 3x^2 = 0 \quad \wedge \quad x \geq 0\right) \quad \Rightarrow \quad x \approx 3.10,$$

Because of

$$P''(x) = 8x - 6x, \quad P''(3.10) < 0$$

the firm should sell 3.10 units in order to maximise profit. The maximum profit will be $P(3.10) \approx 21.05$ €.

26.

1.

$$R(x) = p(x) \cdot x = 4x - \frac{x^2}{25}$$

$$P(x) = R(x) - C(x) = 4x - \frac{x^2}{25} - \left(\frac{x}{2500} \cdot (x - 100)^2 + x\right)$$

$$= 4x - \frac{x^2}{25} - \left(\frac{x^3}{2500} - \frac{2x^2}{25} + 5x\right)$$

$$= -\frac{x^3}{2500} + \frac{x^2}{25} - x,$$

$$P'(x) = -\frac{3x^2}{2500} + \frac{2x}{25} - 1 = 0 \quad \Rightarrow \quad x_1 = 50, \quad x_2 = \frac{50}{3} \approx 16.67,$$

$$P''(x) = -\frac{6x}{2500} + \frac{2}{25}, \qquad P''(50) = -0.04 < 0, \quad P''(16,67) = 0.039992 > 0.$$

Therefore, the monopolist's profit will be maximised at $x = 50$, with $P(50) = 0.$ The corresponding price will be $p(50) = 2$.

2.

(Do it yourselves.)

Part II
Analysis in Economics

<u>D. 2. 1.</u> (*Function*)
Let $D \subseteq R^n$. A *function* f of *n variables*, denoted by $f : D \to R$, is a rule that assigns to each n-tuple $\left(x_1, x_2, ..., x_n\right) \in D$ a unique real number denoted by $f\left(x_1, x_2, ..., x_n\right)$. The set D is called the *domain* of f and its *range* is the set of values that f takes on, that is,
$f(D) = \left\{f(x_1, x_2, ..., x_n) \mid (x_1, x_2, ..., x_n) \in D\right\}$

<u>Ex. 2. 1.</u> (*Some Examples of Economic Functions*)
1. A Cobb-Douglas Function:

$$y(L, K) = 2 \cdot L^{0.4} \cdot K^{0.6} .$$

Here are:

L: Labour,
K: Capital.

2. A total cost function:

$$C(r_1, r_2) = r_1 + 4r_2 + r_1 \cdot r_2$$

Here are:

C: Factor costs,
r_1, r_2: Inputs.

3. A utility function:

$$U(x_1, x_2) = 128x_1 - 10x_2^2 .$$

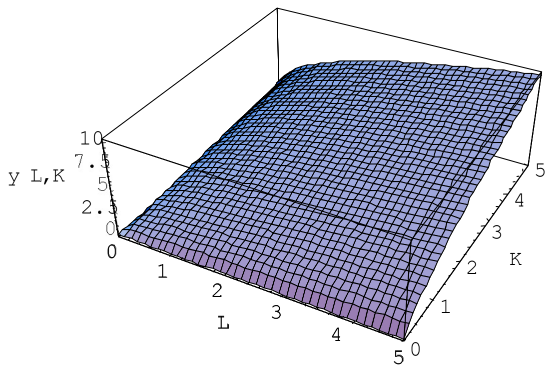
Here are:

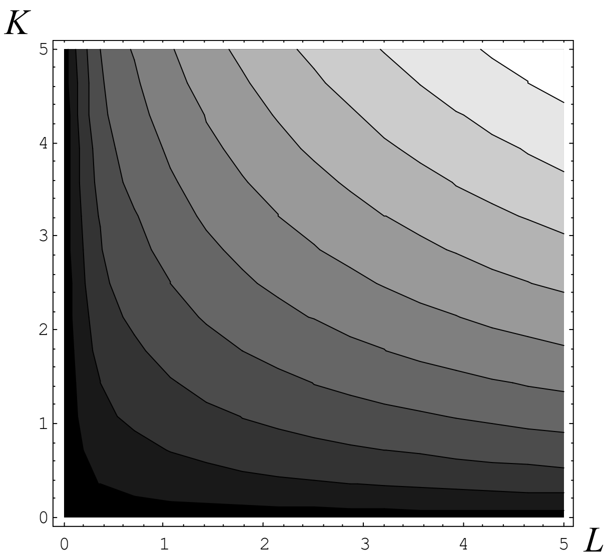
U: Utility,
x_1, x_2: Consumed amounts of two products.

<u>Ex. 2. 2.</u>
Sketch the graph and the contour map of the functions 1-3 in Ex. 2. 1.:

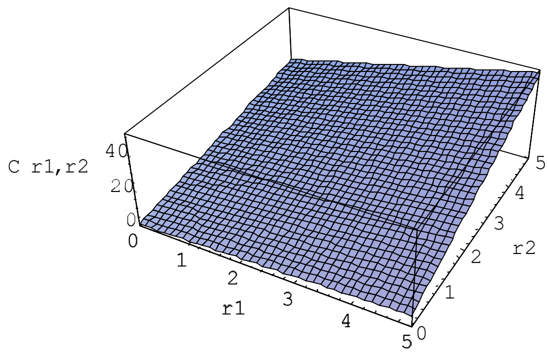
Solution:

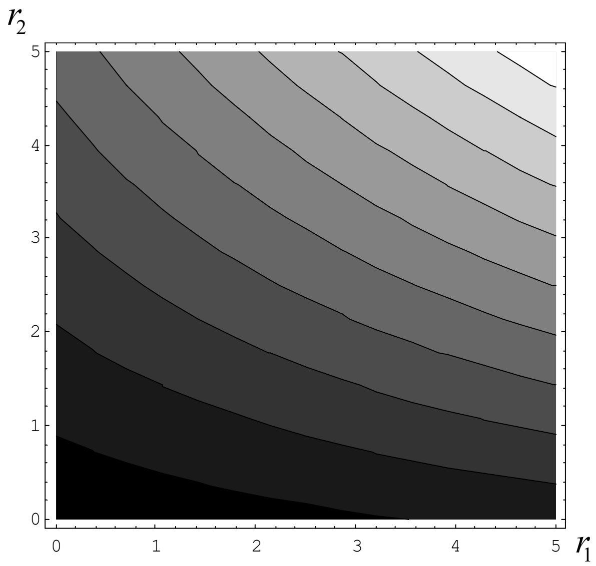
1. $y(L, K) = 2 \cdot L^{0.4} \cdot K^{0.6}$



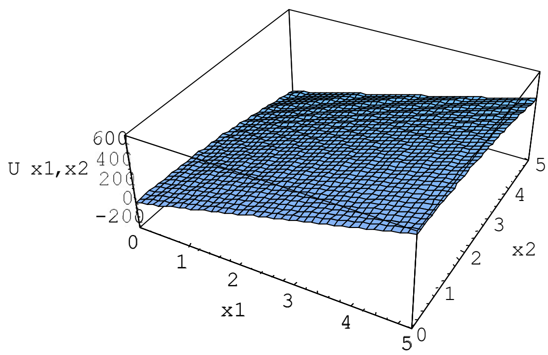


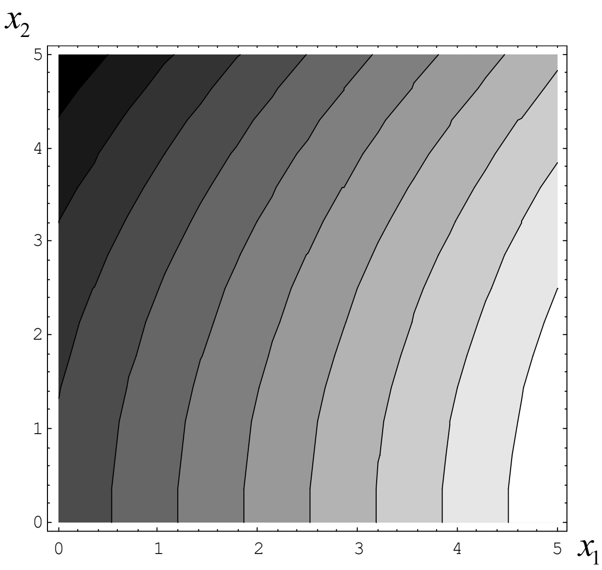
2. $C(r_1, r_2) = r_1 + 4r_2 + r_1 \cdot r_2$





3. $U(x_1, x_2) = 128x_1 - 10x_2^2$





D. 2. 2. (***Partial Derivatives of a Function of Two Variables***)
If f is a function of two variables, then its *partial derivatives* are the functions f_x and f_y defined by

$$f_{x(x,y)} := \lim_{h \to o} \frac{f(x+h,y) - f(x,y)}{h},$$
$$f_{y(x,y)} := \lim_{h \to o} \frac{f(x,y+h) - f(x,y)}{h}.$$

R. 2. 1. (***Notations for Partial Derivatives***)
Let $z = f(x,y)$. Then

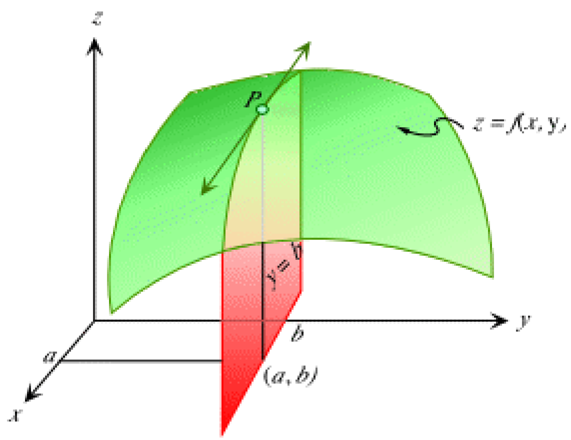
$$f_{x(x,y)} = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x,y) = \frac{\partial z}{\partial x},$$
$$f_{y(x,y)} = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x,y) = \frac{\partial z}{\partial y}.$$

R. 2. 2. (***Rule for Finding*** f_x **and** f_y)

(i) To find f_x keep y as constant and differentiate $f(x,y)$ with respect to x.

(ii) To find f_y keep x as constant and differentiate $f(x,y)$ with respect to y.

R. 2. 3. (***Geometric Interpretation of Partial Derivatives***)
Let $z = f(x,y)$. Then, taking the partial derivative f_x and evaluating it at $(a,\ b)$ amounts to holding y constant at $y = b$ and finding the rate of change of f at $x = a$. Thus, the partial derivative is the slope of the tangent line to this curve at the point where $x = a$ and $y = b$, along the plane $y = b$. (See the figure below.)



Ex. 2 . 3.
Find the partial derivatives of the function

$$y(L,K) = 2 \cdot A^{0.4} \cdot K^{0.6}$$

at $L = 100$ and $K = 80$ and interpret the results.

Solution:

$y_L(L,K) = 0.8 \cdot A^{-0.6} \cdot K^{0.6}$; $\qquad y_L(100,80) = 0.699751727 \approx 0.70$.

Increasing $L = 100$ by one unit while keeping $K = 80$ constant will lead to an approximate increase of the output by 0.70 units.

$y_K(L,K) = 1.2 \cdot A^{0.4} \cdot K^{-0.4}$; $\qquad y_K(100,80) = 0.502791789$.

Increasing $K = 80$ by one unit while keeping $L = 100$ constant will lead to an approximate increase of the output by 0.50 units.

D. 2. 3. (***Partial Derivatives of a Function of*** *n* ***Variables***)
Let $(x_1, x_2, ..., x_n) \in R^n$ and f be a function of n variables $x_1, x_2, ..., x_n$. The partial derivative of f with respect to x_i, $i = 1,2,...,n,$ is hat function denoted by f_{x_i}, such that its function value at any point P in the domain of f is given by

$$f_{x_i}(x_1,...,x_n) := \lim_{h \to 0} \frac{f(x_1,...,x_{i-1}, x_i, x_{i+1},...,x_n) - f(x_1,...,x_n)}{h}$$

if this limit exists, and it is called the $i-th$ *partial derivative of f.*

R. 2. 4. (***Higher Order Partial Derivatives***)
If $z = f(x, y)$ is a function of two variables x and y, then f_x and f_y are also functions of two variables and we hall call them *first order partial derivatives* (or simply *first partial derivatives*). If it is possible to differentiate each of these partial derivatives with respect to x or y, then this will result in four *second partial derivatives* (or simply *second partial derivatives*), namely,

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(f_x(x,y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2},$$
$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(f_x(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x},$$
$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(f_x(x,y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y},$$
$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(f_x(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y^2}.$$

The two second partial derivatives $\frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y)$ and $\frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y)$ are called *mixed partial derivatives* of f (or simply *mixed partials* of f).

T. 2. 1.
Let f be a function of two variables x and y. If f, f_x, f_y, f_{xy} and f_{yx} are continuous on an open Region, then $f_{xy} = f_{yx}$ throughout this region.

Ex. 2 . 4.
Find the partial derivatives of the function f if

$$f(L,K) = 2 \cdot A^{0.4} \cdot K^{0.6}.$$

Solution:

$f_L(L,K) = 0.8 \cdot A^{-0.6} \cdot K^{0.6}$, $\qquad f_K(L,K) = 1.2 \cdot A^{0.4} \cdot K^{-0.4}$,

$f_{LL}(L,K) = -0.48 \cdot A^{-1.6} \cdot K^{0.6}$, $\qquad f_{KK}(L,K) = -0.48 \cdot A^{0.4} \cdot K^{-01.4}$

$$f_{LK}(L,K) = 0.48 \cdot A^{-0.6} \cdot K^{-0.4} = f_{KL}(L,K)$$

D. 2. 4. (***Limit of*** $f(x,y)$)
Let f be a function of two variables defined throughout the interior of a circle with centre (a,b), except possibly at (a,b) itself. The expression

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

means that $\forall \varepsilon > 0,\ \exists \delta > 0$ such that

$$|f(x,y) - l| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

R. 2. 5. (***Two-Path Rule***)
If $\lim_{(x,y)\to(a,b)} f(x,y) = l_1$ along a path C_1 and $\lim_{(x,y)\to(a,b)} f(x,y) = l_2$ along a path C_2 such that $l_1 \neq l_2$, then $\lim_{(x,y)\to(a,b)} f(x,y) = l$ does not exist.

T. 2. 2. (***The Algebra of Limits***)
Let f and g be functions of two variables x and y for which

$$\lim_{(x,y)\to(a,b)} f(x,y) = l_f \qquad \text{and} \qquad \lim_{(x,y)\to(a,b)} g(x,y) = l_g$$

where $l_f,\ l_g \in R$. Then

1. $\lim_{(x,y)\to(a,b)} [f(x,y) + g(x+y)] = l_f + l_g$
2. $\lim_{(x,y)\to(a,b)} [f(x,y) - g(x+y)] = l_f - l_g$

3. $\lim\limits_{(x,y)\to(a,b)} k\cdot\left[f(x,y)\right] = k\cdot l_f$ (k is constant)

4. $\lim\limits_{(x,y)\to(a,b)} f(x,y)\cdot g(x,y) = l_f \cdot l_g$

5. $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{l_f}{l_g}$ (provided $l_g \neq 0$).

<u>D. 2. 5.</u> (*Continuity*)
Let f be a function of two variables x and y defined on a disc with centre (a,b). Then f is said to be *continuous at the point* (a,b) if

1. f is defined at (a,b),
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

A function is said to be *continuous* if it is continuous at every point of its domain.

<u>T. 2. 3.</u>
If f and g are functions which are continuous at the point (a,b), then

1. $f+g$ is continuous at (a,b),
2. $f-g$ is continuous at (a,b),
3. $f\cdot g$ is continuous at (a,b),
4. $\dfrac{f}{g}$ is continuous at (a,b) provided that $g(a,b)\neq 0$.

<u>T. 2. 4.</u>
Let f be a function of two variables and g a function of one variable. If f is continuous at (a,b) and g is continuous at $f(a,b)$, then the composite function $h = g\circ f$ defined by $h(x,y) = g(f(x,y))$ is continuous at (a,b).

<u>D. 2. 6.</u> (*Partial Differentials*)
Let $(x_1,x_2,...,x_n)\in R^n$ and f be a function of n variables $x_1,x_2,...,x_n$. The *partial differential of* f with respect to x_i, $i=1, 2,...,n,$ is defined by

$$df_{x_i} := f_{x_i}\cdot dx_i,\quad i=1,\ 2,...,n.$$

<u>R. 2. 6.</u>

$$df_{x_i} \approx \Delta f_{x_i},\quad i=1,\ 2,...,n.$$

<u>Ex. 2. 5.</u>
Consider a firm that uses capital (K) and labour (L) to produce a good according to the following production function

$$Y(L,K) = \sqrt{L\cdot K}\ ,$$

where the current level of capital and labour are given by $K = 400$ and $L = 100$, respectively. Suppose that the management of the firm wants to increase only the total stock by 0.6 units. Approximate the change in production using the partial differential.

Solution:

$$dK_K\left(L,K,dK\right) = Y_K\left(L,K\right)\cdot dK\ ,$$

$$dK_K\left(L,K,dK\right) = \frac{1}{2}\sqrt{\frac{L}{K}}\cdot dK\ ,$$

$$dK_K\left(L=1,K=4\right) = \frac{1}{2}\sqrt{\frac{100}{400}}\cdot 0.6$$

$$= \frac{1}{2}\sqrt{\frac{1}{4}}\cdot 0.6 = 0.15$$

D. 2. 7. (*Total Differential*)

Let $(x_1, x_2, ..., x_n) \in R^n$ and f be a function of n variables $x_1, x_2, ..., x_n$. The *total differential of* f is defined by

$$df := \sum_{i=1}^{n} f_{x_i}\cdot dx_i.$$

R. 2. 7.

$$df \approx \Delta f.$$

Ex. 2. 6.

Consider the informations in Ex. 2. 5. Suppose that the management of the firm wants to increase the total stock by 0.6 units and the labour by 0.1 units. Approximate the change in production using the total differential.

Solution:

$$dY(L,K) = Y_A(L,K)\cdot dL + Y_K(L,K)\cdot dK$$

$$dY(L,K) = \left(\frac{1}{2}\sqrt{\frac{K}{L}}\cdot dA + \frac{1}{2}\sqrt{\frac{L}{K}}\cdot dK\right)_{|L=1,K=4,dL=0.1,dK=0.6}$$

$$= \frac{1}{2}\sqrt{\frac{1}{4}}\cdot 0.6 + \frac{1}{2}\sqrt{\frac{400}{100}}\cdot 0.1$$

$$0.15 + 0.10 = 0.175\,.$$

T. 2. 5. (*Implicit Differentiation*)

If an equation $F(x,y) = 0$ determines implicitly a differentiable function f of one variable x such that $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}.$$

D. 2. 8. (*Partial Elasticity*)

Let *f* be a partially differentiable function of n variables $x_1, x_2, ..., x_n$. The partial *elasticity of* f with respect to x_i is defined by

$$\varepsilon_{f,x_i}(x_1, x_2, ..., x_n) := \frac{x_i}{f(x_1, x_2, ..., x_n)} \cdot f_{x_i}, \quad i = 1,\ 2, ..., n.$$

R. 2. 8.

The elasticity ε_{f,x_i} gives the approximate percentage increase of production in reaction to a one percent increase of the factor of x_i.

Ex. 2. 7

Find and interpret the partial elasticity's of the production function

$$y(L,K) = 2 \cdot L^{0.2} \cdot K^{0.8}$$

at the point $(20,\ 10)$.

Solution:

$$\begin{aligned}\varepsilon_{y,L}(L,K) &= \frac{L}{y(L,K)} \cdot y_L \\ &= \frac{L}{2 \cdot L^{0.2} \cdot K^{0.8}} \cdot 0.4 \cdot L^{-0.8} \cdot K^{0.8} = 0.2\end{aligned}$$

An increase of labour by 1% (*irrespective of the production level*) leads to an approximate increase of production by 0.2%

$$\begin{aligned}\varepsilon_{y,K}(L,K) &= \frac{K}{y(L,K)} \cdot y_K \\ &= \frac{K}{2 \cdot L^{0.2} \cdot K^{0.8}} \cdot 1.6 \cdot L^{0.2} \cdot K^{-0.2} = 0.8\end{aligned}$$

An increase of capital by 1% (*irrespective of the production level*) leads to an approximate increase of production by 0.8%

Hence, the exponents in the Cobb-Douglas production are the partial elasticities of production with respect to labour and capital.

D. 2. 9. (*Homogeneity*)

Let *f* be a function of n variables for which $(tx_1, tx_2, ..., tx_n) \in D,\ t > 0$. f is called *homogeneous of degree k* if

$$f(tx_1, tx_2, ..., tx_n) = t^k \cdot f(x_1, x_2, ..., x_n),\ \forall (x_1, x_2, ..., x_n) \in D.$$

Ex. 2. 8.

1.

The production function

$$f(x_1, x_2) = 8x_1 + 9x_2$$

is homogeneous of degree 1 because:

$$f(tx_1, tx_2) = 8tx_1 + 9tx_2 = t\left(8x_1 + 9x_2\right) = t \cdot f\left(x_1, x_2\right).$$

2.

The production function

$$f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$$

is homogeneous of degree 2 because:

$$f(tx_1, tx_2) = (tx_1)^2 + t^2x_1x_2 + (tx_2)^2 = t^2\left(x_1^2 + x_1x_2 + x_2^2\right) = t^2 \cdot f\left(x_1, x_2\right).$$

3.

The production function

$$f(x_1, x_2) = x_1^3 + 2x_1x_2 + x_2^3$$

is not homogeneous because t cannot be completely factored out:

$$f(tx_1, tx_2) = (tx_1)^3 + 2t^2x_1x_2 + (tx_2)^3.$$

D. 10. 10. (***Returns to Scale***)

A production function exhibits *constant returns to scale* if when all inputs are increased by a given proportion k, output increases by the same proportion.

If output increases by a proportion greater than k, there are *increasing returns to scale*; and if output increases by a smaller proportion than k, there are *diminishing returns to scale*.

D. 2. 11. (***Relative Extrema of Functions of Two Variables***)

Let f be a function of two variables. We say that f has a *relative maximum* at the point (x_0, y_0) (or $f(x_0, y_0)$ is a *relative maximum* of f) if there is some disc D with centre (x_0, y_0) such that

$$f(x, y) \le f(x_0, y_0),\ \forall (x, y) \in D.$$

If

$$f(x, y) \ge f(x_0, y_0),\ \forall (x, y) \in D,$$

then $f(x_0, y_0)$ is called a *relative minimum* of f.

D. 2. 12. (***Absolute Extrema of Functions of Two Variables***)
If the inequality $f(x, y) \le f(x_0, y_0)$ holds for all points (x, y) in the domain of f, then $f(x_0, y_0)$ is called an *absolute maximum* of f. Likewise $f(x_0, y_0)$ is called an *absolute minimum* of f if $f(x, y) \ge f(x_0, y_0)$ holds for all points (x, y) in the domain of f.

T. 2. 11. (***A Necessary Condition for Relative Extrema***)
Let $z = f(x, y)$ be a function of two variables. If f has a relative extremum (either a relative maximum or a relative minimum) at (x_0, y_0) and $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist, then

$$f_x(x_0, y_0) = 0 \quad \wedge \quad f_y(x_0, y_0) = 0.$$

D. 2. 13. (***Critical Points***)
Let f be a function of two variables x and y. The point (x_0, y_0) is called a *critical point* of f if either

1. $f_x(x_0, y_0) = 0 \quad \wedge \quad f_y(x_0, y_0) = 0$, or
2. $f_x(x_0, y_0) = 0$ or $f_y(x_0, y_0) = 0$ does not exist.

D. 2. 14. (***Saddle Point***)
Let f be a function of two variables x and y. We say that the function f has a *saddle point* on its graph at $(x_0, y_0, f(x_0, y_0))$ if (x_0, y_0) is a critical point of f and f does not have a local extremum at (x_0, y_0).

T. 2. 7. (***Second Partials Test***)
Let f be a function of two variables x and y. Suppose f has continuous second partial derivatives in some open disc with centre (x_0, y_0) and $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Let

$$D = D(x_0, y_0) := f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - \left[f_{xy}(x_0, y_0) \right]^2.$$

Then

1. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, $f(x_0, y_0)$ is a relative maximum.
2. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, $f(x_0, y_0)$ is a relative minimum.
3. If $D < 0$, f has a saddle point at $(x_0, y_0, f(x_0, y_0))$.
4. If $D = 0$, the test fails.

Ex. 2. 9.
A firm's production Q depending on two input amounts r_1 and r_2 is given by the following function

$$Q(r_1, r_2) = 440 + 4r_1 + 10r_2 - r_1^2 + 3r_1 \cdot r_2 - 2.5r_2^2.$$

Determine the factor combination for which the production will be maximal. How much will it be for this combination?

Solution

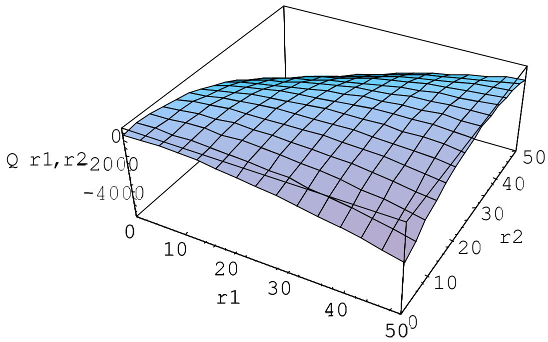
$$Q_{r_1}(r_1,r_2) = 4 - 2r_1 + 3r_2 := 0$$
$$Q_{r_2}(r_1,r_2) = 10 + 3r_1 - 5r_2 := 0$$

$$\Rightarrow \quad r_1 = 50, \quad r_2 = 32$$
$$Q_{r_1r_1}(r_1,r_2) = -2, \quad Q_{r_1r_2}(r_1,r_2) = 3, \quad Q_{r_2r_2}(r_1,r_2) = -5\,.$$

For the factor combination $r_1 = 50,\ r_2 = 32$ the production will be maximal, since

$$Q_{r_1r_1}(r_1,r_2)\cdot Q_{r_2r_2}(r_1,r_2) - \left[Q_{r_1r_2}(r_1,r_2)\right]^2 = (-2)\cdot(-5) - 3^2 = 1 > 0 \text{ and } Q_{r_1r_1}(r_1,r_2) = -2 < 0\,.$$

The firm will then have a maximum output oft of $Q(50,32) = 700$.



<u>R. 2. 9.</u> (*Optimisation with Constraints as Equations*)
Suppose $f:\ R^n \to R$ and $g:\ R^n \to R^m$. Consider the constrained optimisation problem:

$$\max_{x\in R^n} f(x) \text{ s.t. } g(x) = 0.$$

If x^* is a solution to this problem, then there exist *Lagrange multipliers* $(\lambda_1,\lambda_2,\ldots,\lambda_n) =: \lambda$ such that with the *Lagrangean*

$$L(x;\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i \cdot g_i(x)$$

the following conditions are fulfilled:

$$L_{x_j}\left(x^*;\lambda^*\right) = 0, \quad j = 1,2,\ldots,n$$
$$L_{\lambda_i}\left(x^*;\lambda^*\right) = 0, \quad i = 1,2,\ldots,m.$$

R. 2. 11. (***Sufficient Condition for*** $n = 2$)
We define the so-called *bordered Hessian*:

$$\bar{H} := \begin{pmatrix} 0 & g_{x_1} & g_{x_2} \\ g_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ g_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{pmatrix}.$$

Let x^* be a stationary point of $L(x_1, x_2, \lambda)$. Then

$$\det \bar{H} \begin{cases} < 0 & \text{then } L \text{ assumes a relative minimum at } x^* \\ = 0 & \text{then further investigations are needed} \\ > 0 & \text{then } L \text{ assumes a relative maximum at } x^* \end{cases}$$

R. 2. 11.
The Lagrange multiplier λ_i approximates the marginal impact on the objective function caused by a small change in the constant of the constraint i.

R. 2. 12.
Optimisation problems with constraints as equations can in the simples cases be also solved by the *elimination method*.

Ex. 2. 10.
What combination of goods G_1 and G_2 should a firm produce to minimise costs when the joint cost function is

$$C(x_1, x_2) = 6x_1^2 + 10x_2^2 - x_1x_2 + 30 \quad (x_i,\ i = 1,2: \text{ amount of } G_i)$$

and the firm has a production quota

$$x_1 + x_2 = 34\ ?$$

Estimate the effect on costs if the production quota is reduced by 1 unit.

1. Solution by the elimination method:

$$x_2 = 34 - x_1$$

$$\tilde{C}(x_1) = 6x_1^2 + 10(34 - x_1)^2 - x_1(34 - x_1) + 30$$

$$\tilde{C}(x_1) = 17x_1^2 - 714x_1 + 11590$$

$$\tilde{C}'(x_1) = 34x - 714$$

$$34x_1 - 714 = 0 \quad \Rightarrow \quad x_1 = 21, \qquad \tilde{C}''(x_1) = 34 > 0,$$

Hence $\tilde{C}(x_1)$ assumes its relative minimum at $x_1 = 21$.

$x_2 = 34 - x_1 = 34 - 21 = 13$.

$C(21,13) = 4093.$

2. Solution by the method of Lagrange multipliers:

$$L(x_1, x_2; \lambda) = 6x_1^2 + 10x_2^2 - x_1x_2 + 30 - \lambda(x_1 + x_2 - 34)$$

$$L_{x_1}(x_1, x_2; \lambda) = 12x_1 - x_2 - \lambda = 0$$
$$L_{x_2}(x_1, x_2; \lambda) = -x_1 + 20x_2 - \lambda = 0$$
$$L_{\lambda}(x_1, x_2; \lambda) = x_1 + x_2 - 33 = 0$$

$$\Rightarrow \quad x_1 = 21, \quad x_2 = 13, \quad \lambda = 239.$$

$$\bar{H} := \begin{pmatrix} 0 & g_{x_1} & g_{x_2} \\ g_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ g_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 12 & -1 \\ 1 & -1 & 20 \end{pmatrix}.$$

$$\det \bar{H} = -34 < 0 .$$

Therefore, the function $L(x_1, x_2, \lambda)$ assumes a relative minimum at

$$x_1 = 21, \quad x_2 = 13, \quad \lambda = 239$$

With $\lambda = 239$, a decrease in the constant (the production quota) will lead to a cost increase of approximately 239.

Exercises

1.
A local music hall is selling ticket for a certain concert. The matinee show costs 15 € per person and the evening show costs 25 € per person. If x_1 is the number of matinee tickets purchased and x_2 the number of evening tickets for the concert, find the revenue function $R(x_1, x_2)$ for the concert and evaluate and interpret $R(125, 200)$

2.
Suppose a metal manufacturing company has a Cobb-Douglas production function

$$f(L, K) = 10 \cdot L^{0.25} K^{0.75}$$

where L is the number of hours of labour and K is the Euro amount of capital invested.
If the company uses 2000 hours of labour and 1500 € in capital, how many units of metal will be produced?

3.
The production function of a firm is given by

$$y(L, C) = 2L^{0.4}C^{0.6}, \quad L, C > 0$$

Here are:

C: capital
L: labour
y: production .

Give the partial marginal productivity functions of labour and capital.

4.
The production function of a firm is given by

$$y(L, C) = 90L^{0.8}C^{0.2}, \; L, C > 0$$

Here are:

C: capital
L: labour
y: production .

Find and interpret the partial marginal productivity functions of labour and capital

i) for $L = 1000, \quad \mathrm{C} = 200$
ii) If each capital unit can be substituted by 8 units of labour.

5.
The production function of a firm is given by

$$y(L,C) = L^{0.8}C^{0.2}, \quad L,C > 0$$

Here are:

C: capital
L: labour
y: production .

Find and interpret the partial derivatives of first and second order of this function with respect to labour and capital.

6.
A firm has the production function

$$Q = \sqrt{C \cdot L} \, .$$

Here are:

C: quantity of capital
L: quantity of labour
Q: quantity of production .

Sketch and label the level curves for $Q = 1, 2, 3$ on a single diagram. What is the economic term for these curves? How can they be interpreted?

7.
A firm's production Q depending on two input amounts r_1 and r_2 is given by the following function:

$$Q(r_1, r_2) = 440 + 4r_1 + 10r_2 - r_1^2 + 3r_1 \cdot r_2 - 2.5r_2^2 \, .$$

Determine the factor combination for which the production will be maximal. How much will it be for this combination?

8.
Two producers Pr_1 and Pr_2 produce the same product. Denote by

$p_i, \ i = 1, \ 2$: the price dictated by Pr_i,
$x_i, \ i = 1, \ 2$: the amount of the product supplied by Pr_i

It has been estimated that the following relations hold between p_i and $x_i, \ i = 1, \ 2$.

$$x_1 = 100 - 2p_1 - \ \ p_2$$
$$x_2 = 120 - \ \ p_1 - 3p_2$$

Further, we have the following cost functions of the producers:

$$C_1(x_1) = 120 + 2x_1$$
$$C_2(x_2) = 120 + 2x_2 .$$

1. Find the profit functions of each producer as well as their common profit function depending on the prices p_1, p_2.
2. Determine the prices such as they will guarantee a maximum total profit. Calculate the maximum profit?
3. Following a „price war", producer, Pr_2 decides to set his price at the level $p_2 = 16$. Determine the level of p_1 maximising the profit of producer Pr_1.
4. Is it advantageous for the consumers if the two producers put an end to their „price war"?

9.

A company produces two types of scooters $S_1,\ S_2$. The demand for the two types is given by the following function:

$$p_1(x_1, x_2) = 35 - 3x_1 + 4x_2 ,$$
$$p_2(x_1, x_2) = 20 - 2x_1 + x_2 .$$

($p_i,\ i = 1,\ 2$: price of S_i; $x_i,\ i = 1,\ 2$: demand for S_i)

The company's cost function, in hundreds of €, is given by

$$C(x_1, x_2) = 8.5x_1 + 6x_2 + 400 .$$

1. Find the revenue function $R(x_1, x_2)$ of the company and evaluate $R(10,\ 18)$.
2. Determine the company's profit function $P(x_1, x_2)$ and evaluate $P(10,\ 18)$.
3. Find and interpret the first partial derivatives of the profit function for $x_1 = 10,\ x_2 = 18$.
4. Determine the partial elasticities of the revenue function for $x_1 = 10,\ x_2 = 18$ and interpret them.

10.

An online fitness company sells two types of shoes, aerobic and running. The store pays 30 € for a pair of aerobic shoes and 45 € for a pair of running shoes. The daily demand equations for each type of shoe are as follows:

$x_1 = 850 - 36p + 15q$: demand equation for aerobic shoes

$x_2 = 1075 + 20p - 25q$: demand equation for running shoes

where p is the selling price for each pair of aerobic shoes and q is the selling price for each pair of running shoes.
What price should the store charge for each model of shoes if it wants to maximise its profit.

11.
Draw the isoquants for the production function

$$x(r_1, r_2) = r_1^2 + r_2^2 .$$

12.
Given the profit function

$$P(x_1, x_2) = -3x_1^2 - 2x_2^2 - 2x_1x_2 + 160x_1 + 120x_2 - 18$$

for a firm producing two products, maximise profit.

13.
In monopolistic competition producers must determine the price that will maximise their profit. Assume that a producer offers two different brands of products, for which the demand functions are:

$$x_1 = 14 - 0.25p_1$$
$$x_2 = 24 - 0.5p_2$$

and the joint cost function

$$C(x_1, x_2) = x_1^2 + 5x_1x_2 + x_2^2 .$$

Determine the profit-maximising level of output, the price that should be charged for each brand of the product, and the profit.

14.
A monopolistic firm has the following demand functions for each of its products P_1 and P_2:

$$x_1 = 72 - 0.5p_1 ,$$
$$x_2 = 120 - p_2 .$$

The combined cost function is

$$C(x_1, x_2) = x_1^2 + x_2^2 + x_1x_2 + 35 ,$$

The total production is to be equal to 40.

Find the profit-maximising level of

(1) output
(2) price

(3) profit.

15.
Given a budget constraint

$$4L + 3K = 108,$$

optimise the generalised Cobb-Douglas production function

$$Q(L,K) = L^{0.5} \cdot K^{0.4}.$$

16.
1. Minimise costs for a firm with the cost function

$$C(x_1, x_2) = 5x_1^2 + 2x_1x_2 + 3x_2^2 + 800$$

subject to the production quota

$$x_1 + x_2 = 39.$$

2. Estimate additional costs if the production quota is increased to 40

17.
A rancher faces the profit function

$$P(x_1, x_2) = 110x_1 - 3x_1^2 - 2x_1x_2 - 2x_2^2 + 140x_2.$$

where

x_1 : sides of beef

x_2 : hides.

The output must be in the proportion

$$x_1 = 2x_2.$$

Using the *method of Lagrange multiplier*, determine the level of output that will maximise the rancher's profit.

18.
Maximise the following utility function

$$u(x_1, x_2) = x_1^{0.60} \cdot x_2^{0.25}$$

subject to the budget constraint

$$8x_1 + 5x_2 = 680,$$

using the method of Lagrange multipliers. Interpret your results.

19.

1. Maximise the utility function

$$u(x_1, x_2) = x_1^2 \cdot x_2^2$$

subject to the budget constraint

$$x_1 + 2x_2 = 20 .$$

2. Estimate the effect of changing the right-hand side of the budget constraint by 1 unit on the objective function.

20.

Suppose that a consumer buys two goods G_1 and G_2 and has a utility function

$$U(x_1, x_2) = x_1^{0.5} x_2^5 .$$

G_1 costs \$2, G_2 \$3 per unit. The consumer has a budget of \$100.
How much of the two goods should the consumer purchase to maximise her/his utility?

21.

A producer has the possibility of discriminating between the domestic and foreign markets for a product where the demands, respectively, are

$$x_1 = 21 - 0.1p_1$$
$$x_2 = 50 - 0.4p_2$$

The producer has the following total cost function:

$$C(x) = 2000 + 10x$$

where $$x = x_1 + x_2$$

1. What price will the producer charge in order to maximizes profit
   a) with discrimination between markets
   b) without discrimination?

2. Compare the profit differential between discrimination and non-discrimination.

Solution

1.

Since revenue is the selling price times the number of tickets sold, the revenue function for the concert is

$$R(x_1, x_2) = 15x_1 + 25x_2.$$

This means that when 125 tickets are sold for the matinee show and 200 tickets are sold for the evening show, the music hall generates revenue of

$$R(125, 200) = 15 \cdot 125 + 25 \cdot 200 = 6875 \text{ €}$$

2.

$$f(2000,1500) = 10 \cdot 2000^{0.25} \cdot 1500^{0.75} \approx 16119.$$

3.

$$\frac{\partial y}{\partial L}(L, C) = 0.8L^{-0.6}C^{0.6}, \qquad \frac{\partial y}{\partial C}(L, C) = 1.2L^{0.4}C^{-0.4}$$

4.

i)

$$\frac{\partial y}{\partial L}(L, C) = 72L^{-0.2}C^{0.2} \qquad \frac{\partial y}{\partial C}(L, C) = 18L^{0.8}C^{-0.8}$$

$$\frac{\partial y}{\partial L}(1000;\ 200) = 52.1841; \qquad \frac{\partial y}{\partial C}(1000;\ 200) = 65.2302$$

An increase of labour from 1000 units to 1001 units, while keeping the capital at 200 units, will lead to an approximate increase of production by 52.1841 units.
An increase of capital from 200 units to 201 units, while keeping the labour at 1000 units, will lead to an approximate increase of production by 65.2302 units.

ii)

$$C = 8L \quad \text{or} \quad L = \frac{1}{8}C$$

$$\frac{\partial y}{\partial L}(1,\ 8) = 72 \cdot 8^{0.2} = 109.1316, \qquad \frac{\partial y}{\partial C}\left(\frac{1}{8}, 1\right) = 18 \cdot \left(\frac{1}{8}\right)^{0.8} = 3.4104$$

An increase of labour by 1 unit, while holding the capital constant, will lead to an approximate increase of production by 109.1316 units.
An increase of capital by 1 unit, while holding the labour constant, will lead to an approximate increase of production by 3.4104 units.

5.

$$\frac{\partial y}{\partial L} = 0.8L^{-0.2}C^{0.2} > 0;\ \frac{\partial y}{\partial C} = 0.2L^{0.8}C^{-0.8} > 0\ .$$

Holding the capital (labour) constant, the output increases with increasing labour (capital).

$$\frac{\partial^2 y}{\partial L^2} = -0.16L^{-1.2}C^{0.2} < 0; \qquad \frac{\partial^2 y}{\partial C^2} = -0.16A^{0.8}C^{-1.8} < 0$$

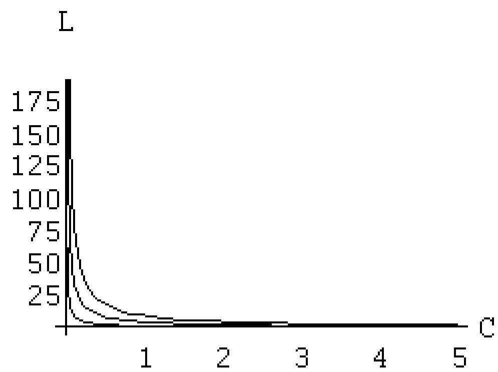
The marginal products of labour and capital are diminishing.

$$\frac{\partial^2 y}{\partial L \partial C} = \frac{\partial^2 y}{\partial C \partial L} = 0.16L^{-0.2}C^{-0.8} > 0\ .$$

The positive sign of theses "cross" derivatives tells us that capital and labour are complement in production.

6.

$$Q = 1 \ \Rightarrow \ L = \frac{1}{C},$$
$$Q = 2 \ \Rightarrow \ L = \frac{4}{C}$$
$$Q = 3 \ \Rightarrow \ L = \frac{9}{C}$$



The curves form an "isoquant map". Each isoquant gives all possible equivalent combinations of labour and capital ensuring the same quantity of production.

7.

$$Q_{r_1}(r_1, r_2) = 4 - 2r_1 + 3r_2 := 0$$
$$Q_{r_2}(r_1, r_2) = 10 + 3r_1 - 5r_2 := 0$$

$$\Rightarrow \qquad r_1 = 50, \quad r_2 = 32$$
$$Q_{r_1 r_1}(r_1, r_2) = -2, \quad Q_{r_1 r_2}(r_1, r_2) = 3, \quad Q_{r_2 r_2}(r_1, r_2) = -5\ .$$

For the factor combination $r_1 = 50,\ r_2 = 32$ the production will be maximal, since

$$Q_{r_1r_1}(r_1,r_2)\cdot Q_{r_2r_2}(r_1,r_2)-\left[Q_{r_1r_2}(r_1,r_2)\right]^2=(-2)\cdot(-5)-3^2=1>0 \text{ and } Q_{r_1r_1}(r_1,r_2)=-2<0\,.$$

The firm will then have a maximum profit of $Q(50,32)=700$.

8.

1.

$$P_{\mathrm{Pr}_1}(p_1,p_2)=p_1x_1-C_1(x_1)$$
$$=-2p_1^2-p_1p_2+104p_1+2p_2-320$$

$$P_{\mathrm{Pr}_2}(p_1,p_2)=p_2x_2-C_2(x_2)$$
$$=-3p_2^2-p_1p_2+2p_1+126p_2-360$$

$$P(p_1,p_2)=-2p_1^2-2p_1p_2-3p_2^2+106p_1+128p_2-680$$

2.

$$P_{\mathrm{Pr}_1}(p_1,p_2)=-4p_1-2p_2+106=0$$
$$P_{\mathrm{Pr}_2}(p_1,p_2)=-2p_1-6p_2+128=0$$

$$p_1=19,\quad p_2=15$$

$$P_{p_1p_1}=-4,\qquad P_{p_1p_2}=-2,\qquad P_{p_2p_2}=-6$$

Because of $P_{p_1p_1}\cdot P_{p_2p_2}-(P_{p_1p_2})^2=(-4)\cdot(-6)-(-2)^2>0,\quad P_{p_1p_2}=-4<0$ the maximum profit will be attained for $p_1=19,\quad p_2=15$; it will be equal to $P(19,\ 15)=1287$.

3.

$$P_1(p_1,\ 16)=-2p_1^2+88p_1-288=:\tilde{P}_1(p_1)$$

$$\tilde{P}_1'(p_1)=-4p\ +88=0\quad\Leftrightarrow\quad p_1=22$$

$$\tilde{P}_1''(p_1)=-4<0$$

Therefore, P_1 will be maximal for $p_1=22$; it will be equal to 680.

4.

It will be advantageous for the consumers if the producers put an end to their „price war“, since the price p_1 has increased from 19 to 22.

9.

1.

$$R(x_1,x_2) = x_1 \cdot (35 - 3x_1 + 4x_2) + x_2 \cdot (20 - 2x_1 + x_2)$$

$$= -3x_1^2 + x_2^2 + 2x_1 \cdot x_2 + 35x_1 + 20x_2 .$$

$$R(10,18) = 1094 .$$

2.

$$P(x_1,x_2) = R(x_1,x_2) - C(x_1,x_2)$$

$$= -3x_1^2 + x_2^2 + 2x_1 \cdot x_2 + 35x_1 + 20x_2 - (8.5x_1 + 6x_2 + 400)$$

$$= -3x_1^2 + x_2^2 + 2x_1 \cdot x_2 + 26.5x_1 + 14x_2 - 400$$

$$P(10,18) = 501 .$$

3.

$$P_{x_1}(x_1,x_2) = -6x_1 + 2x_2 + 26.5 ; \qquad P_{x_1}(10,18) = 2.5 .$$

An increase of x_1 by 1 unit, while holding x_2 unchanged, will lead to an approximate increase of profit by 2.5 units.

$$P_{x_2}(x_1,x_2) = 2x_1 + 2x_2 + 14 ; \qquad P_{x_2}(10,18) = 70 .$$

An increase of x_2 by 1 unit, while holding x_1 unchanged, will lead to an approximate increase of profit by 70 units.

4.

$$R_{x_1}(x_1,x_2) = -6x_1 + 2x_2 + 35 ; \qquad R_{x_1}(10,\ 18) = 11 ,$$

$$\varepsilon_{R,x_1}(x_1,x_2) = \frac{x_1}{R(x_1,x_2)} \cdot R_{x_1}(x_1,x_2) ; \qquad \varepsilon_{R,x_1}(10,\ 18) = \frac{10}{1094} \cdot 11 \approx 0.1\ \%.$$

An increase of x_1 by 1 %, while holding x_2 unchanged, will lead to an approximate increase of revenue by 0.1 %. The revenue function at this point is inelastic.

$$R_{x_2}(x_1,x_2) = 2x_1 + 2x_2 + 20 ; \qquad R_{x_2}(10,\ 18) = 76$$

$$\varepsilon_{R,x_2}(x_1,x_2) = \frac{x_2}{R(x_1,x_2)} \cdot R_{x_2}(x_1,x_2) ; \qquad \varepsilon_{R,x_2}(10,\ 18) = \frac{18}{1094} \cdot 76 \approx 1.25\ \%.$$

An increase of x_2 by 1 %, while holding x_1 unchanged, will lead to an approximate increase of revenue by 1.25 %. The revenue function at this point is elastic.

10.

$$R(p,q) = x_1 \cdot p + x_2 \cdot q$$
$$= (850 - 36p + 15q) \cdot p + (1075 + 20p - 25q) \cdot q$$
$$R(p,q) == -36p^2 - 25q^2 + 35p \cdot q + 850p + 1075q$$

$$C(x_1, x_2) = 30x_1 + 45x_2$$

$$C(p,q) = 30 \cdot (850 - 36p + 15q) + 45 \cdot (1075 + 20p - 25q)$$

$$C(p,q) = -180p - 675q + 73875$$

$$P(p,q) = R(p,q) - C(p,q)$$

$$P(p,q) = -36p^2 - 25q^2 + 35p \cdot q + 1030p + 1750q - 73875.$$

$$P_p(p,q) = -72p + 35q + 1030$$
$$P_q(p,q) = \quad 35p - 50q + 1750$$

$$\begin{cases} -72p + 35q = -1030 \\ \ \ 35p - 50q = -1750 \end{cases}$$

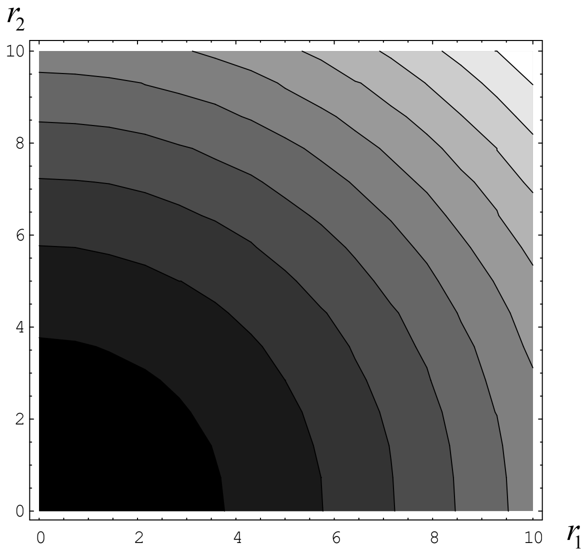
$$\Rightarrow \quad p = 47.47368421 \approx 47.47, \quad q = 68.23157895 \approx 68.23$$

Thus, because of

$$H = \begin{pmatrix} -72 & 35 \\ 35 & -50 \end{pmatrix}, \qquad \det H = (-72) \cdot (-50) + 35^2 > 0, \quad P_{p,p}(p,q) = -72 < 0$$

to maximise profit, the shoe company should sell a pair of aerobic shoes for 47.47 € and a pair of running shoes for 68.23 €. Its maximum profit will amount to 10276.58 €

11.



12.

$$P(x_1,x_2) = -3x_1^2 - 2x_2^2 - 2x_1x_2 + 160x_1 + 120x_2 - 18$$

$$P_{x_1}(x_1,x_2) = -6x_1 - 2x_2 + 160$$
$$P_{x_2}(x_1,x_2) = -2x_1 - 4x_2 + 120$$

$$\begin{cases} 6x_1 + 2x_2 = 160 \\ 2x_1 + 4x_2 = 120 \end{cases} \quad \Rightarrow \quad x_1 = x_2 = 20$$

$$P_{x_1x_1} = -6, \quad P_{x_1x_2} = P_{x_2x_1} = -2, \quad P_{x_2x_2} = -4$$

$$\det H = \det\begin{pmatrix} -6 & -2 \\ -2 & -4 \end{pmatrix} = 20 > 0, \qquad P_{x_1x_1} < 0.$$

Hence, the firm's profit will be maximised for $x_1 = x_2 = 20$ with $P(20,20) = 2782$.

13.

$$x_1 = 14 - 0.25p_1 \quad \Rightarrow \quad p_1 = 56 - 4x_1$$
$$x_2 = 24 - 0.5p_2 \quad \Rightarrow \quad p_2 = 48 - 2x_2$$

$$R(x_1,x_2) = p_1 \cdot x_1 + p_2 \cdot x_2$$

$$= (56 - 4x_1)x_1 + (48 - 2x_2)x_2$$

$$P(x_1,x_2) = R(x_1,x_2) - C(x_1,x_2)$$

$$P(x_1, x_2) = 5x_1^2 - 3x_2^2 - 5x_1 \cdot x_2 + 56x_1 + 48x_2$$

$$P_{x_1}(x_1, x_2) = -10x_1 - 5x_2 + 56$$
$$P_{x_2}(x_1, x_2) = -5x_1 - 6x_2 + 48$$

$$\begin{cases} 10x_1 + 5x_2 = 56 \\ 5x_1 + 6x_2 = 44 \end{cases} \quad \Rightarrow \quad x_1 = 2.75, \quad x_2 = 5.7$$

$$P_{x_1x_1} = -10, \quad P_{x_1x_2} = P_{x_2x_1} = -5, \quad P_{x_2x_2} = -6$$

$$\det H = \det\begin{pmatrix} -10 & -5 \\ -5 & -6 \end{pmatrix} = 35 > 0, \qquad P_{x_1x_1} < 0 .$$

Hence, the producer's profit will be maximised for $x_1 = 2.75,\ x_2 = 5.7$ with

$$P(2.75, 5.70) = 213.94 .$$

14.

$$P(x_1, x_2) = (144 - 2x_1)x_1 + (120 - x_2)x_2 - (x_1^2 + x_1x_2 + x_2^2)$$

$$P(x_1, x_2) = -3x_1^2 - 2x_2^2 - x_1x_2 + 144x_1 + 120x_2 - 35$$

$$L(x_1, x_2; \lambda) = -3x_1^2 - 2x_2^2 - x_1x_2 + 144x_1 + 120x_2 - 35 - \lambda(x_1 + x_2 - 40)$$

$$L_{x_1}(x_1, x_2; \lambda) = -6x_1 - x_2 + 144 - \lambda$$
$$L_{x_2}(x_1, x_2; \lambda) = -x_1 - 4x_2 + 120 - \lambda$$
$$L_{\lambda_1}(x_1, x_2; \lambda) = x_1 + x_2 - 40$$

(1)

$$\begin{cases} -6x_1 - x_2 + 144 - \lambda = 0 \\ -x_1 - 4x_2 + 120 - \lambda = 0 \\ x_1 + x_2 - 40 = 0 \end{cases} \qquad \Rightarrow x_1 = 18, \quad x_2 = 22, \quad \lambda = 14 .$$

(The sufficiency condition will not be tested.)

(2)

$$p_1 = 144 - 2 \cdot 18 = 108, \quad p_2 = 120 - 22 = 98.$$

(3) $P(18,\ 22) = 2861.$

15.

$$L(L,K;\lambda) = L^{0.5} \cdot K^{0.4} - \lambda(4L + 3K - 108)$$

$$L_L(L,K;\lambda) = 0.5L^{-0.5}K^{0.4} - 4\lambda = 0$$
$$L_K(L,K;\lambda) = 0.4L^{0.5}K^{-0.6} - 3\lambda = 0$$
$$L_\lambda(L,K;\lambda) = 4L + 3K - 108 = 0$$

$$\frac{0.4L^{0.5}K^{-0.6}}{0.5L^{-0.5}K^{0.4}} = \frac{-3\lambda}{-4\lambda}, \qquad 0.8K^{-1} \cdot L^{1} = 0.75, \qquad \frac{L}{K} = \frac{0.75}{0.80}, \qquad L = 0.9375K,$$

$$3K + 4 \cdot 0.9375K = 108,$$

$$K = 16, \quad L = 15, \quad \lambda = 0.097839083.$$

(The sufficiency condition will not be tested.)

16.

1.

$$L(x_1,x_2;\lambda) = 5x_1^2 + 3x_2^2 + 2x_1 \cdot x_2 + 800 - \lambda(x_1 + x_2 - 39)$$

$$L_{x_1}(x_1,x_2;\lambda) = 10x_1 + 2x_2 - \lambda = 0$$
$$L_{x_2}(x_1,x_2;\lambda) = 2x_1 + 6x_2 - \lambda = 0 \qquad \Rightarrow \quad x_1 = 13, \quad x_2 = 26, \quad \lambda=182, \quad C(13,26) = 4349.$$
$$L_\lambda(x_1,x_2;\lambda) = x_1 + x_2 = 39.$$

2.

Since $\lambda = 182$, an increased production quota will lead to additional costs of approximately 182.

17.

$$L(x_1,x_2;\lambda) = -3x_1^2 - 2x_2^2 - 2x_1 \cdot x_2 + 110x_1 + 140x_2 - \lambda(x_1 - 2x_2)$$

$$L_{x_1}(x_1,x_2;\lambda) = -6x_1 - 2x_2 + 110 - \lambda = 0$$
$$L_{x_2}(x_1,x_2;\lambda) = -2x_1 - 4x_2 + 140 + 2\lambda = 0$$
$$L_\lambda(x_1,x_2;\lambda) = x_1 - 2x_2 = 0.$$

$$\Rightarrow \quad x_1 = 20, \quad x_2 = 10, \quad \lambda=30, \quad C(20,10) = 1800.$$

18.

$$L_{x_1}(x_1,x_2;\lambda) = 0.60x_1^{-0.40} \cdot x_2^{0.25} - 8\lambda = 0$$
$$L_{x_2}(x_1,x_2;\lambda) = 0.25x_1^{0.60} \cdot x_2^{-0.75} - 5\lambda = 0$$
$$L_\lambda(x_1,x_2;\lambda) = 8x_1 + 5x_2 - 680 = 0$$

$$\frac{0.6x_1^{-0.4}x_2^{0.25}}{0.25x_1^{0.6}x_2^{-0.75}} = \frac{8\lambda}{5\lambda}$$

$$2.4x_1^{-1}x_2 = \frac{8}{5} \quad \Rightarrow \quad x_2 = \frac{2}{3}x_1$$

Substitute in $L_\lambda(x_1, x_2; \lambda)$: $x_1 = 60, \quad x_2 = 40$, and $L_{x_1}(x_1 = 60, x_2 = 40) = 0 \Rightarrow \lambda \approx 0.037.$

Because of $\lambda \approx 0.037$ an increase of the existing budget by 1 unit will lead to an approximate increase of utility by 0.037.

19.

1.

$$L(x_1, x_2; \lambda) = x_1^2 \cdot x_2^2 - \lambda(x_1 + 2x_2 - 20)$$

$$L_{x_1}(x_1, x_2; \lambda) = 2x_1 \cdot x_2^2 - \lambda = 0$$
$$L(x_1, x_2; \lambda) = 2x_1^2 \cdot x_2 - 2\lambda = 0$$
$$L_\lambda(x_1, x_2; \lambda) = x_1 + 2x_2 - 20 = 0$$

$$\begin{cases} 2x_1 \cdot x_2^2 = -\lambda \\ 2x_1^2 \cdot x_2 = -2\lambda \end{cases}$$

$$\frac{2x_1^2 \cdot x_2}{2x_1 \cdot x_2^2} = \frac{-2\lambda}{-\lambda}$$

$$\frac{x_1^2 \cdot x_2}{x_1 \cdot x_2^2} = 2 \quad \Rightarrow \quad x_1 = 2x_2$$

$$2x_2 + 2x_2 = 20, \quad x_2 = 5, \quad x_1 = 10, \quad \lambda = 500$$

Therefore

$$x_1 = 10, \quad x_2 = 5, \quad \lambda = 500.$$

2.

An increase of the right-hand side of the constraint from 20 to 21 will lead to an approximate increase of the value of the objective function by 500 units.

20.

$$U(x_1, x_2^{0.5}) = x_1^{0.5} \cdot x_2^{0.5} \to \max$$

subject to

$$2x_1 + 3x_2 = 100.$$

$$L(x_1,x_2,\lambda)=x_1^{0.5}x_2^{0.5}-\lambda\left(2x_1+3x_2-100\right)\to\max$$

$$L_{x_1}(x_1,x_2,\lambda)=0.5x_1^{-0.5}x_2^{0.5}-2\lambda$$
$$L_{x_2}(x_1,x_2,\lambda)=0.5x_1^{0.5}x_2^{-0.5}-3\lambda$$
$$L_{\lambda}(x_1,x_2,\lambda)=-2x_1-3x_2+100$$

$$\begin{cases}0.5x_1^{-0.5}x_2^{0.5}-2\lambda=0\\0.5x_1^{0.5}x_2^{-0.5}-3\lambda=0\\-2x_1-3x_2+100=0\end{cases}$$

$$\frac{0.5x_1^{-0.5}x_2^{0.5}}{0.5x_1^{0.5}x_2^{-0.5}}=\frac{2\lambda}{3\lambda}\quad\Leftrightarrow\quad\frac{x_1^{-0.5}x_2^{0.5}}{x_1^{0.5}x_2^{-0.5}}=\frac{2}{3}\quad\Leftrightarrow\quad\frac{x_2}{x_1}=\frac{2}{3}\quad\Leftrightarrow\quad x_2=\frac{2}{3}x_1$$

Substituting the last relation into the budget constraint, we obtain:

$$2x_1+3\cdot\frac{2}{3}x_1=100\quad\Leftrightarrow\quad x_1=25,\quad x_2=\frac{50}{3},\quad\lambda=\frac{1}{2\sqrt{6}}=0.2041241452.$$

$$L_{x_1x_1}(x_1,x_2,\lambda)=-0.25x_1^{-1.5}x_2^{0.5}$$

$$L_{x_1x_2}(x_1,x_2,\lambda)=0.25x_1^{-0.5}x_2^{-0.5}=L_{x_2x_1}(x_1,x_2,\lambda)$$

$$L_{x_2x_2}(x_1,x_2,\lambda)=-0.25x_1^{0.5}x_2^{-1.5}$$

$$\bar{H}(x_1,x_2,\lambda):=\begin{pmatrix}0&2&3\\2&-0.25x_1^{-1.5}x_2^{0.5}&0.25x_1^{-0.5}x_2^{-0.5}\\3&0.25x_2^{-0.5}x_1^{-0.5}&-0.25x_2^{0.5}x_2^{-1.5}\end{pmatrix}$$

$$\bar{H}\left(x_1=25,x_2=\frac{50}{3}\right)\approx\begin{pmatrix}0&2&3\\2&-0.0082&0.0122\\3&0.0122&-0.0184\end{pmatrix}$$

$$\det\bar{H}=0.2938>0\cdot$$

Therefore, the Lagrange function assumes its relative maximum at $\begin{pmatrix}25&\frac{50}{3}&0.2041\end{pmatrix}^T$.

Thus, purchasing 25 units of G_1 and $\frac{50}{3}$ units of G_2 will maximise the consumer's utility. Increasing the budget of \$100 units by \$1 dollar will lead to an increase of utility by about 0.2041241452 units.

21

1.

a) $$P(p_1,p_2)=\left(21-\frac{1}{10}p_1\right)p_1+\left(50-\frac{2}{5}p_2\right)p_2-\left[2000+10\left(21-\frac{1}{10}p_1+50-\frac{2}{5}p_2\right)\right]$$

$$P(p_1,p_2)=-\frac{1}{10}p_1^2-\frac{2}{5}p_2^2+22p_1+54p_2-2710$$

$$P_{p_1}(p_1,p_2)=-\frac{1}{5}p_1+22:=0 \quad \Rightarrow \quad p_1=110$$

$$P(p_1,p_2)=-\frac{4}{5}p_2+54:=0 \quad \Rightarrow \quad p_2=67.50$$

$$\left\langle \det H=\det\begin{pmatrix}-\frac{1}{5} & 0\\ 0 & -\frac{4}{5}\end{pmatrix}=\frac{4}{25}>0 \;\wedge\; P_{p_1p_1}(p_1,p_2)=-\frac{1}{5}<0 \right\rangle$$

$$\Rightarrow$$

The profit will be maximised for $p_1=110$ and $p_2=67.50$. $P(p_1=110,p_2=67.5)=322.50$

b)

If the producer does not discriminate, $p_1=p_2$. Thus,

$$\tilde{P}(p_1)=\left(21-\frac{1}{10}p_1\right)p_1+\left(50-\frac{2}{5}p_1\right)p_1-\left[2000+10\left(21-\frac{1}{10}p_1+50-\frac{2}{5}p_1\right)\right]$$

$$\tilde{P}(p_1)=-\frac{1}{2}p_1^2+76p_1-2710$$

$$\frac{d\tilde{P}(p_1)}{dp_1}=-p_1+76:=0, \quad p_1=76, \quad \frac{d^2\tilde{P}(p_1)}{d^2p_1}=-1<0.$$

Therefore, the profit will be maximised for $p_1=p_2=76=:p$, $\tilde{P}(76)=178$.

2.

Profit differential = 322.5 – 178 = 144.5