Das Leontief-Modell mit Intervallkoeffizienten

D. 1. (*Leontief-Modell mit Intervallkoeffizienten)*

$$Ax + y = x$$

mit

$$A = \begin{pmatrix} [a_{11}^-, a_{11}^+] & [a_{12}^-, a_{12}^+] & \dots & [a_{1n}^-, a_{1n}^+] \\ [a_{21}^-, a_{21}^+] & [a_{22}^-, a_{22}^+] & \dots & [a_{2n}^-, a_{2n}^+] \\ & \dots & & \\ [a_{n1}^-, a_{n1}^+] & [a_{n2}^-, a_{n2}^+] & \dots & [a_{nn}^-, a_{nn}^+] \end{pmatrix}, \quad y = \begin{pmatrix} [y_1^-, y_1^+] \\ [y_2^-, y_2^+] \\ \dots \\ [y_n^-, y_n^+] \end{pmatrix}, \quad x = \begin{pmatrix} [x_1^-, x_1^+] \\ [x_2^-, x_2^+] \\ \dots \\ [x_n^-, x_n^+] \end{pmatrix},$$

also

$$\begin{aligned} &[a_{11}^-, a_{11}^+][x_1^-, x_1^+] + [a_{12}^-, a_{12}^+][x_2^-, x_2^+] + \dots + [a_{1n}^-, a_{1n}^+][x_n^-, x_n^+] + [y_1^-, y_1^+] = [x_1^-, x_1^+] \\ &[a_{21}^-, a_{21}^+][x_1^-, x_1^+] + [a_{22}^-, a_{22}^+][x_2^-, x_2^+] + \dots + [a_{2n}^-, a_{2n}^+][x_n^-, x_n^+] + [y_2^-, y_2^+] = [x_2^-, x_2^+] \\ &\dots \\ &\dots \\ &\dots \\ &[a_{n1}^-, a_{n1}^+][x_1^-, x_1^+] + [a_{n2}^-, a_{n2}^+][x_2^-, x_2^+] + \dots + [a_{nn}^-, a_{nn}^+][x_n^-, x_n^+] + [y_n^-, y_n^+] = [x_n^-, x_n^+] \end{aligned}$$

BS.

Gegeben sei das Leontief-Modell mit

$$A = \begin{pmatrix} [0.20, 0.30] & [0.10, 0.15] \\ [0.05, 0.08] & [0.25, 0.35] \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Gesucht sei $x = \begin{pmatrix} [x_1^-, x_1^+] \\ [x_2^-, x_2^+] \end{pmatrix}$.

Lösung:

1. Nach der Zerlegungsmethode:

$$A^- = \begin{pmatrix} 0.20 & 0.10 \\ 0.05 & 0.25 \end{pmatrix}, \qquad A^+ = \begin{pmatrix} 0.30 & 0.15 \\ 0.08 & 0.35 \end{pmatrix}$$

$$E - A^{-} = \begin{pmatrix} 080 & -0.10 \\ -0.05 & 0.75 \end{pmatrix}, \qquad E - A^{+} = \begin{pmatrix} 0.70 & -0.15 \\ -0.08 & 0.65 \end{pmatrix}$$

$$\begin{pmatrix} 080 & -0.10 \\ -0.05 & 0.75 \end{pmatrix} \begin{pmatrix} x_1^- \\ x_2^- \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x^{-} = \begin{pmatrix} \frac{10}{7} \\ \frac{10}{7} \end{pmatrix}$$

$$\begin{pmatrix} 070 & -0.15 \\ -0.08 & 0.65 \end{pmatrix} \begin{pmatrix} x_1^- \\ x_2^- \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x^{+} = \begin{pmatrix} \frac{800}{443} \\ \frac{780}{443} \end{pmatrix}$$

Die Lösung der obigen Subsysteme ergibt:

$$\begin{pmatrix} \left[x_1^-, x_1^+ \right] \\ \left[x_2^-, x_2^+ \right] \end{pmatrix} = \begin{pmatrix} [1.42857, 1.80587] \\ [1.42857, 1.76072] \end{pmatrix}$$

2. Berechnung mit den Intervallmittelwerten

$$\bar{A} = \begin{pmatrix} 0.250 & 0.125 \\ 0.065 & 0.300 \end{pmatrix}, \quad E - \bar{A} = \begin{pmatrix} 0.750 & -0.125 \\ -0.065 & 0.700 \end{pmatrix}$$

$$\begin{pmatrix} 0.750 & -0.125 \\ -0.065 & 0.700 \end{pmatrix} \begin{pmatrix} x_1^- \\ x_2^- \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \bar{x} = \begin{pmatrix} \frac{1320}{827} \\ \frac{1304}{827} \end{pmatrix} \approx \begin{pmatrix} 0.596131 \\ 0.576784 \end{pmatrix}$$

3. Nach dem Jacobi-Verfahren

$$x^{(k)} = D^{-1} \left(U + O \right) x^{(k-1)} + D^{-1} y, \quad k = 1, 2, \ldots$$

$$D = \begin{pmatrix} [0.20, 0.30] & [0.00, 0.00] \\ [0.00, 0.00] & [0.25, 0.35] \end{pmatrix}, \qquad U = \begin{pmatrix} [0.00, 0.00] & [0.00, 0.00] \\ [0.05, 0.08] & [0.00, 0.00] \end{pmatrix},$$

$$O = \begin{pmatrix} [0.00, 0.00] & [0.10, 0.15] \\ [0.00, 0.00] & [0.00, 0.00] \end{pmatrix}$$

Sei

$$\begin{pmatrix} \left[x_1^{-(0)}, x_1^{+(0)}\right] \\ \left[x_2^{-(0)}, x_2^{+(0)}\right] \end{pmatrix} = \begin{pmatrix} [0.00, 0.00] \\ [0.00.0.00] \end{pmatrix}$$

Iterationen nach Jacobi

Iteration	x_1	x_2
0	$[0.000000, 0.000000]$	$[0.000000, 0.000000]$
1	$[1.250000, 1.428571]$	$[1.333333, 1.538462]$
2	$[1.410256, 1.717949]$	$[1.428571, 1.688406]$
3	$[1.428571, 1.780952]$	$[1.428571, 1.745206]$
4	$[1.428571, 1.799680]$	$[1.428571, 1.758514]$
5	$[1.428571, 1.804742]$	$[1.428571, 1.760381]$
6	$[1.428571, 1.805725]$	$[1.428571, 1.760658]$
7	$[1.428571, 1.805904]$	$[1.428571, 1.760714]$
8	$[1.428571, 1.805936]$	$[1.428571, 1.760721]$
9	$[1.428571, 1.805943]$	$[1.428571, 1.760722]$
10	$[1.428571, 1.805944]$	$[1.428571, 1.760722]$
…		
25	$[1.428571, 1.805869]$	$[1.428571, 1.760722]$

4. Nach Gauß-Seidel-Verfahren

$$x^{(k)} = (D-U)^{-1} O.x^{(k-1)} + y, \qquad k = 1, 2, \ldots$$

Die Matrizen D, U, O wie bei Jacobi.

Iterationen nach Gauß-Seidel

Iteration	x_1	x_2
0	$[0.000000, 0.000000]$	$[0.000000, 0.000000]$
1	$[1.250000, 1.428571]$	$[1.416667, 1.714286]$
2	$[1.427083, 1.795918]$	$[1.428472, 1.759498]$
3	$[1.428559, 1.805607]$	$[1.428571, 1.760690]$
4	$[1.428571, 1.805862]$	$[1.428571, 1.760721]$
5	$[1.428571, 1.805869]$	$[1.428571, 1.760722]$
6	$[1.428571, 1.805869]$	$[1.428571, 1.760722]$
7	$[1.428571, 1.805869]$	$[1.428571, 1.760722]$
…		
25	$[1.428571, 1.805869]$	$[1.428571, 1.760722]$

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