Part II
Analysis in Economics

D. 2. 1. (***Function***)
Let $D \subseteq R^n$. A *function* f of n *variables*, denoted by $f : D \to R$, is a rule that assigns to each n-tuple $(x_1, x_2, ..., x_n) \in D$ a unique real number denoted by $f(x_1, x_2, ..., x_n)$. The set D is called the *domain* of f and its *range* is the set of values that f takes on, that is,
$f(D) = \{f(x_1, x_2, ..., x_n) \,|\, (x_1, x_2, ..., x_n) \in D\}$

Ex. 2. 1. (***Some Examples of Economic Functions***)
1. A Cobb-Douglas Function:

$$y(L, K) = 2 \cdot L^{0.4} \cdot K^{0.6} .$$

Here are:

L: Labour,
K: Capital.

2. A total cost function:

$$C(r_1, r_2) = r_1 + 4r_2 + r_1 \cdot r_2$$

Here are:

C: Factor costs,
r_1, r_2: Inputs.

3. A utility function:

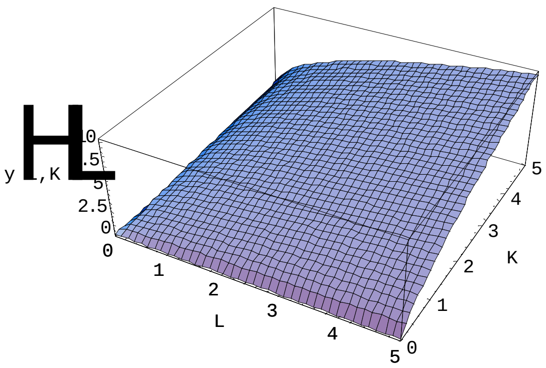
$$U(x_1, x_2) = 128x_1 - 10x_2^2 .$$

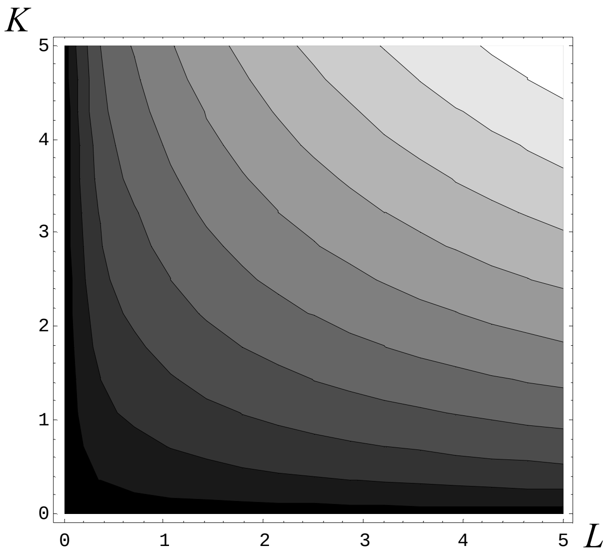
Here are:

U: Utility,
x_1, x_2: Consumed amounts of two products.

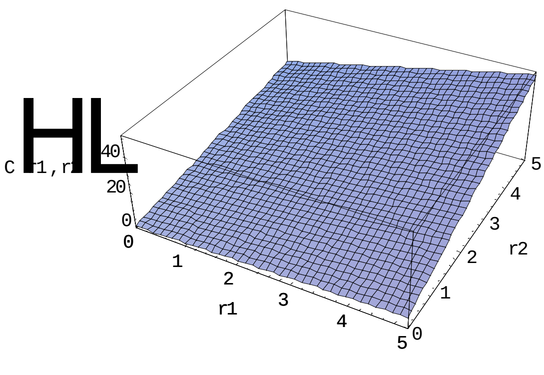
Ex. 2. 2.
Sketch the graph and the contour map of the functions 1-3 in Ex. 2. 1.:

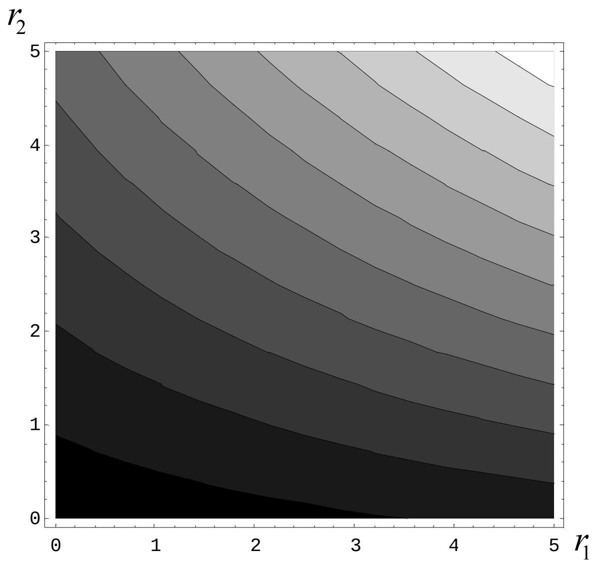
Solution:
1. $y(L, K) = 2 \cdot L^{0.4} \cdot K^{0.6}$



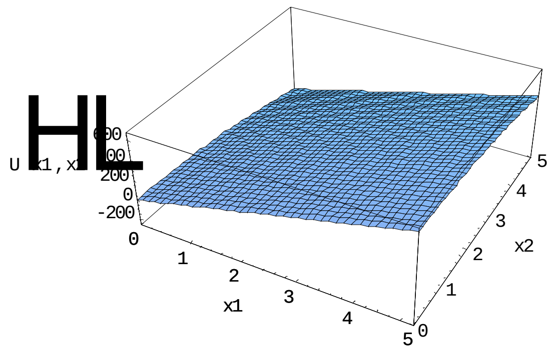


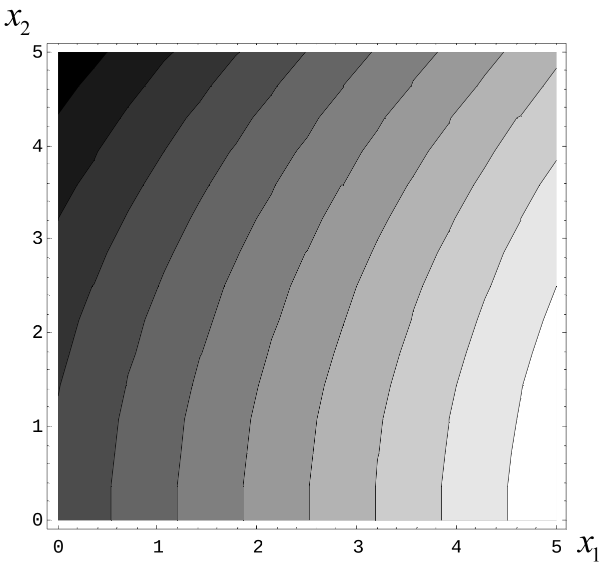
2. $C(r_1, r_2) = r_1 + 4r_2 + r_1 \cdot r_2$





3. $U(x_1, x_2) = 128x_1 - 10x_2^2$





D. 2. 2. (***Partial Derivatives of a Function of Two Variables***)
If f is a function of two variables, then its *partial derivatives* are the functions f_x and f_y defined by

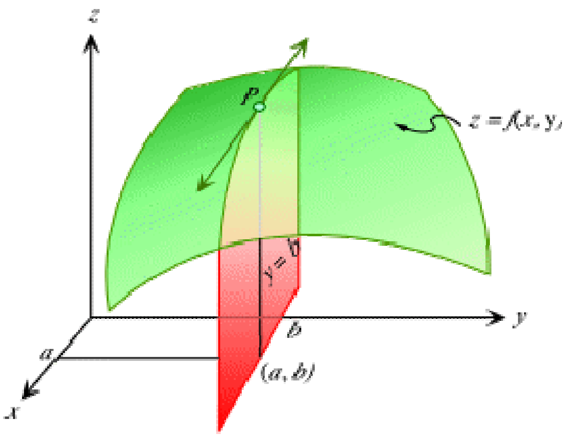
$$f_{x(x,y)} := \lim_{h \to o} \frac{f(x+h,y) - f(x,y)}{h},$$
$$f_{y(x,y)} := \lim_{h \to o} \frac{f(x,y+h) - f(x,y)}{h}.$$

R. 2. 1. (***Notations for Partial Derivatives***)
Let $z = f(x,y)$. Then

$$f_{x(x,y)} = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x,y) = \frac{\partial z}{\partial x},$$
$$f_{y(x,y)} = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x,y) = \frac{\partial z}{\partial y}.$$

R. 2. 2. (***Rule for Finding*** f_x and f_y)
(i) To find f_x keep y as constant and differentiate $f(x,y)$ with respect to x.
(ii) To find f_y keep x as constant and differentiate $f(x,y)$ with respect to y.

R. 2. 3. (***Geometric Interpretation of Partial Derivatives***)
Let $z = f(x,y)$. Then, taking the partial derivative f_x and evaluating it at $(a,\ b)$ amounts to holding y constant at $y = b$ and finding the rate of change of f at $x = a$. Thus, the partial derivative is the slope of the tangent line to this curve at the point where $x = a$ and $y = b$, along the plane $y = b$. (See the figure below.)



Ex. 2 . 3.
Find the partial derivatives of the function

$$y(L,K) = 2 \cdot A^{0.4} \cdot K^{0.6}$$

at $L = 100$ and $K = 80$ and interpret the results.

Solution:

$$y_L(L,K) = 0.8 \cdot A^{-0.6} \cdot K^{0.6}; \qquad y_L(100,80) = 0.699751727 \approx 0.70.$$

Increasing $L = 100$ by one unit while keeping $K = 80$ constant will lead to an approximate increase of the output by 0.70 units.

$$y_K(L,K) = 1.2 \cdot A^{0.4} \cdot K^{-0.4}; \qquad y_K(100,80) = 0.502791789.$$

Increasing $K = 80$ by one unit while keeping $L = 100$ constant will lead to an approximate increase of the output by 0.50 units.

D. 2. 3. (***Partial Derivatives of a Function of*** *n* ***Variables***)
Let $(x_1, x_2, ..., x_n) \in R^n$ and f be a function of n variables $x_1, x_2, ..., x_n$. The partial derivative of f with respect to x_i, $i = 1, 2, ..., n$, is hat function denoted by f_{x_i}, such that its function value at any point P in the domain of f is given by

$$f_{x_i}(x_1,...,x_n) := \lim_{h \to 0} \frac{f(x_1,...,x_{i-1},x_i,x_{i+1},...,x_n) - f(x_1,...,x_n)}{h}$$

if this limit exists, and it is called the $i-th$ *partial derivative of f.*

R. 2. 4. (***Higher Order Partial Derivatives***)
If $z = f(x,y)$ is a function of two variables x and y, then f_x and f_y are also functions of two variables and we hall call them *first order partial derivatives* (or simply *first partial derivatives*). If it is possible to differentiate each of these partial derivatives with respect to x or y, then this will result in four *second partial derivatives* (or simply *second partial derivatives*), namely,

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(f_x(x,y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2},$$
$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(f_x(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x},$$
$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(f_x(x,y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y},$$
$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(f_x(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y^2}.$$

The two second partial derivatives $\frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y)$ and $\frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y)$ are called *mixed partial derivatives* of f (or simply *mixed partials* of f).

T. 2. 1.
Let f be a function of two variables x and y. If f, f_x, f_y, f_{xy} and f_{yx} are continuous on an open Region, then $f_{xy} = f_{yx}$ throughout this region.

Ex. 2 . 4.
Find the partial derivatives of the function f if

$$f(L,K) = 2 \cdot A^{0.4} \cdot K^{0.6}.$$

Solution:

$$f_L(L,K) = 0.8 \cdot A^{-0.6} \cdot K^{0.6}, \qquad f_K(L,K) = 1.2 \cdot A^{0.4} \cdot K^{-0.4},$$

$$f_{LL}(L,K) = -0.48 \cdot A^{-1.6} \cdot K^{0.6}, \qquad f_{KK}(L,K) = -0.48 \cdot A^{0.4} \cdot K^{-01.4}$$

$$f_{LK}(L,K) = 0.48 \cdot A^{-0.6} \cdot K^{-0.4} = f_{KL}(L,K)$$

D. 2. 4. (***Limit of*** $f(x,y)$)
Let f be a function of two variables defined throughout the interior of a circle with centre (a,b), except possibly at (a,b) itself. The expression

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

means that $\forall \varepsilon > 0,\ \exists \delta > 0$ such that

$$|f(x,y) - l| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

R. 2. 5. (***Two-Path Rule***)
If $\lim_{(x,y)\to(a,b)} f(x,y) = l_1$ along a path C_1 and $\lim_{(x,y)\to(a,b)} f(x,y) = l_2$ along a path C_2 such that $l_1 \neq l_2$, then $\lim_{(x,y)\to(a,b)} f(x,y) = l$ does not exist.

T. 2. 2. (***The Algebra of Limits***)
Let f and g be functions of two variables x and y for which

$$\lim_{(x,y)\to(a,b)} f(x,y) = l_f \qquad \text{and} \qquad \lim_{(x,y)\to(a,b)} g(x,y) = l_g$$

where $l_f,\ l_g \in R$. Then

1. $\lim_{(x,y)\to(a,b)} \left[f(x,y) + g(x+y) \right] = l_f + l_g$
2. $\lim_{(x,y)\to(a,b)} \left[f(x,y) - g(x+y) \right] = l_f - l_g$
3. $\lim_{(x,y)\to(a,b)} k \cdot \left[f(x,y) \right] = k \cdot l_f$ (k is constant)

4. $\lim_{(x,y)\to(a,b)} f(x,y)\cdot g(x,y) = l_f \cdot l_g$

5. $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{l_f}{l_g}$ (provided $l_g \neq 0$).

D. 2. 5. (***Continuity***)
Let f be a function of two variables x and y defined on a disc with centre (a,b). Then f is said to be *continuous at the point* (a,b) if

1. f is defined at (a,b),
2. $\lim_{(x,y)\to(a,b)} f(x,y) = l$ exists.
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

A function is said to be *continuous* if it is continuous at every point of its domain.

T. 2. 3.
If f and g are functions which are continuous at the point (a,b), then

1. $f+g$ is continuous at (a,b),
2. $f-g$ is continuous at (a,b),
3. $f\cdot g$ is continuous at (a,b),
4. $\dfrac{f}{g}$ is continuous at (a,b) provided that $g(a,b)\neq 0$.

T. 2. 4.
Let f be a function of two variables and g a function of one variable. If f is continuous at (a,b) and g is continuous at $f(a,b)$, then the composite function $h = g\circ f$ defined by $h(x,y) = g(f(x,y))$ is continuous at (a,b).

D. 2. 6. (***Partial Differentials***)
Let $(x_1, x_2, ..., x_n) \in R^n$ and f be a function of n variables $x_1, x_2, ..., x_n$. The *partial differential of* f with respect to x_i, $i = 1, 2, ..., n,$ is defined by

$$df_{x_i} := f_{x_i}\cdot dx_i, \quad i = 1,\ 2, ..., n.$$

R. 2. 6.

$$df_{x_i} \approx \Delta f_{x_i}, \quad i = 1,\ 2, ..., n.$$

Ex. 2. 5.
Consider a firm that uses capital (K) and labour (L) to produce a good according to the following production function

$$Y(L,K) = \sqrt{L\cdot K},$$

where the current level of capital and labour are given by $K = 400$ and $L = 100$, respectively. Suppose that the management of the firm wants to increase only the total stock by 0.6 units. Approximate the change in production using the partial differential.

Solution:

$$dK_K\left(L,K,dK\right) = Y_K\left(L,K\right)\cdot dK \ ,$$

$$dK_K\left(L,K,dK\right) = \frac{1}{2}\sqrt{\frac{L}{K}}\cdot dK \ ,$$

$$dK_K\left(L=1,K=4\right) = \frac{1}{2}\sqrt{\frac{100}{400}}\cdot 0.6$$

$$= \frac{1}{2}\sqrt{\frac{1}{4}}\cdot 0.6 = 0.15$$

<u>D. 2. 7.</u> (*Total Differential*)
Let $(x_1, x_2, ..., x_n) \in R^n$ and f be a function of n variables $x_1, x_2, ..., x_n$. The *total differential of* f is defined by

$$df := \sum_{i=1}^{n} f_{x_i} \cdot dx_i .$$

<u>R. 2. 7.</u>

$$df \approx \Delta f .$$

<u>Ex. 2. 6.</u>
Consider the informations in Ex. 2. 5. Suppose that the management of the firm wants to increase the total stock by 0.6 units and the labour by 0.1 units. Approximate the change in production using the total differential.

Solution:

$$dY(L,K) = Y_A(L,K)\cdot dL + Y_K(L,K)\cdot dK$$

$$dY(L,K) = \left(\frac{1}{2}\sqrt{\frac{K}{L}}\cdot dA + \frac{1}{2}\sqrt{\frac{L}{K}}\cdot dK\right)_{|L=1,K=4,dL=0.1,dK=0.6}$$

$$= \frac{1}{2}\sqrt{\frac{1}{4}}\cdot 0.6 + \frac{1}{2}\sqrt{\frac{400}{100}}\cdot 0.1$$

$$0.15 + 0.10 = 0.175 .$$

<u>T. 2. 5.</u> (*Implicit Differentiation*)
If an equation $F(x, y) = 0$ determines implicitly a differentiable function f of one variable x such that $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} .$$

D. 2. 8. (***Partial Elasticity***)
Let f be a partially differentiable function of n variables $x_1, x_2, ..., x_n$. The partial *elasticity of f* with respect to x_i is defined by

$$\varepsilon_{f,x_i}(x_1, x_2, ..., x_n) := \frac{x_i}{f(x_1, x_2, ..., x_n)} \cdot f_{x_i}, \quad i = 1,\ 2, ..., n.$$

R. 2. 8.
The elasticity ε_{f,x_i} gives the approximate percentage increase of production in reaction to a one percent increase of the factor of x_i.

Ex. 2. 7
Find and interpret the partial elasticity's of the production function

$$y(L,K) = 2 \cdot L^{0.2} \cdot K^{0.8}$$

at the point $(20,\ 10)$.

Solution:

$$\begin{aligned}\varepsilon_{y,L}(L,K) &= \frac{L}{y(L,K)} \cdot y_L \\ &= \frac{L}{2 \cdot L^{0.2} \cdot K^{0.8}} \cdot 0.4 \cdot L^{-0.8} \cdot K^{0.8} = 0.2\end{aligned}$$

An increase of labour by 1% (*irrespective of the production level*) leads to an approximate increase of production by 0.2%

$$\begin{aligned}\varepsilon_{y,K}(L,K) &= \frac{K}{y(L,K)} \cdot y_K \\ &= \frac{K}{2 \cdot L^{0.2} \cdot K^{0.8}} \cdot 1.6 \cdot L^{0.2} \cdot K^{-0.2} = 0.8\end{aligned}$$

An increase of capital by 1% (*irrespective of the production level*) leads to an approximate increase of production by 0.8%

Hence, the exponents in the Cobb-Douglas production are the partial elasticities of production with respect to labour and capital.

D. 2. 9. (***Homogeneity***)
Let f be a function of n variables for which $(tx_1, tx_2, ..., tx_n) \in D,\ t > 0$. f is called *homogeneous of degree k* if

$$f(tx_1, tx_2, ..., tx_n) = t^k \cdot f(x_1, x_2, ..., x_n),\ \forall (x_1, x_2, ..., x_n) \in D.$$

Ex. 2. 8.
1.
The production function

$$f(x_1,x_2)=8x_1+9x_2$$

is homogeneous of degree 1 because:

$$f(tx_1,tx_2)=8tx_1+9tx_2=t\left(8x_1+9x_2\right)=t\cdot f\left(x_1,x_2\right).$$

2.
The production function

$$f(x_1,x_2)=x_1^2+x_1x_2+x_2^2$$

is homogeneous of degree 2 because:

$$f(tx_1,tx_2)=(tx_1)^2+t^2x_1x_2+(tx_2)^2=t^2\left(x_1^2+x_1x_2+x_2^2\right)=t^2\cdot f\left(x_1,x_2\right).$$

3.
The production function

$$f(x_1,x_2)=x_1^3+2x_1x_2+x_2^3$$

is not homogeneous because t cannot be completely factored out:

$$f(tx_1,tx_2)=(tx_1)^3+2t^2x_1x_2+(tx_2)^3.$$

D. 10. 10. (***Returns to Scale***)
A production function exhibits *constant returns to scale* if when all inputs are increased by a given proportion k, output increases by the same proportion.
If output increases by a proportion greater than k, there are *increasing returns to scale*; and if output increases by a smaller proportion than k, there are *diminishing returns to scale*.

D. 2. 11. (***Relative Extrema of Functions of Two Variables***)
Let f be a function of two variables. We say that f has a *relative maximum* at the point (x_0,y_0) (or $f(x_0,y_0)$ is a *relative maximum* of f) if there is some disc D with centre (x_0,y_0) such that

$$f(x,y)\le f(x_0,y_0),\ \forall(x,y)\in D.$$

If

$$f(x,y)\ge f(x_0,y_0),\ \forall(x,y)\in D,$$

then $f(x_0,y_0)$ is called a *relative minimum* of f.

<u>D. 2. 12.</u> (*Absolute Extrema of Functions of Two Variables*)
If the inequality $f(x,y) \le f(x_0,y_0)$ holds for all points (x,y) in the domain of f, then $f(x_0,y_0)$ is called an *absolute maximum* of f. Likewise $f(x_0,y_0)$ is called an *absolute minimum* of f if $f(x,y) \ge f(x_0,y_0)$ holds for all points (x,y) in the domain of f.

<u>T. 2. 11.</u> (*A Necessary Condition for Relative Extrema*)
Let $z = f(x,y)$ be a function of two variables. If f has a relative extremum (either a relative maximum or a relative minimum) at (x_0,y_0) and $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ both exist, then

$$f_x(x_0,y_0) = 0 \quad \wedge \quad f_y(x_0,y_0) = 0.$$

<u>D. 2. 13.</u> (*Critical Points*)
Let f be a function of two variables x and y. The point (x_0,y_0) is called a *critical point* of f if either

1. $f_x(x_0,y_0) = 0 \quad \wedge \quad f_y(x_0,y_0) = 0$, or
2. $f_x(x_0,y_0) = 0$ or $f_y(x_0,y_0) = 0$ does not exist.

<u>D. 2. 14.</u> (*Saddle Point*)
Let f be a function of two variables x and y. We say that the function f has a *saddle point* on its graph at $(x_0,y_0,f(x_0,y_0))$ if (x_0,y_0) is a critical point of f and f does not have a local extremum at (x_0,y_0).

<u>T. 2. 7.</u> (*Second Partials Test*)
Let f be a function of two variables x and y. Suppose f has continuous second partial derivatives in some open disc with centre (x_0,y_0) and $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$. Let

$$D = D(x_0,y_0) := f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) - \left[f_{xy}(x_0,y_0)\right]^2.$$

Then

1. If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, $f(x_0,y_0)$ is a relative maximum.
2. If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, $f(x_0,y_0)$ is a relative minimum.
3. If $D < 0$, f has a saddle point at $(x_0,y_0,f(x_0,y_0))$.
4. If $D = 0$, the test fails.

<u>Ex. 2. 9.</u>
A firm's production Q depending on two input amounts r_1 and r_2 is given by the following function

$$Q(r_1,r_2) = 440 + 4r_1 + 10r_2 - r_1^2 + 3r_1 \cdot r_2 - 2.5r_2^2.$$

Determine the factor combination for which the production will be maximal. How much will it be for this combination?

Solution

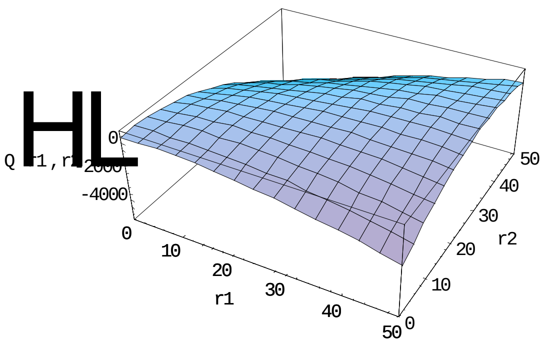
$$Q_{r_1}(r_1, r_2) = 4 - 2r_1 + 3r_2 := 0$$
$$Q_{r_2}(r_1, r_2) = 10 + 3r_1 - 5r_2 := 0$$

$$\Rightarrow \quad r_1 = 50, \quad r_2 = 32$$
$$Q_{r_1 r_1}(r_1, r_2) = -2, \quad Q_{r_1 r_2}(r_1, r_2) = 3, \quad Q_{r_2 r_2}(r_1, r_2) = -5 .$$

For the factor combination $r_1 = 50,\ r_2 = 32$ the production will be maximal, since

$$Q_{r_1 r_1}(r_1, r_2) \cdot Q_{r_2 r_2}(r_1, r_2) - \left[Q_{r_1 r_2}(r_1, r_2)\right]^2 = (-2)\cdot(-5) - 3^2 = 1 > 0 \text{ and } Q_{r_1 r_1}(r_1, r_2) = -2 < 0 .$$

The firm will then have a maximum output oft of $Q(50, 32) = 700$.



<u>R. 2. 9.</u> (*Optimisation with Constraints as Equations*)
Suppose $f:\ R^n \to R$ and $g:\ R^n \to R^m$. Consider the constrained optimisation problem:

$$\max_{x \in R^n} f(x) \text{ s.t. } g(x) = 0.$$

If x^* is a solution to this problem, then there exist *Lagrange multipliers* $\left(\lambda_1, \lambda_2, ..., \lambda_n\right) =: \lambda$ such that with the *Lagrangean*

$$L\left(x; \lambda\right) = f(x) - \sum_{i=1}^{m} \lambda_i \cdot g_i\left(x\right)$$

the following conditions are fulfilled:

$$L_{x_j}\left(x^*; \lambda^*\right) = 0, \quad j = 1, 2, ..., n$$
$$L_{\lambda_i}\left(x^*; \lambda^*\right) = 0, \quad i = 1, 2, ..., m.$$

R. 2. 10.
The Lagrange multipliers λ_i approximates the marginal impact on the objective function caused by a small change in the constant of the constraint i.

R. 2. 11.
Optimisation problems with constraints as equations can in the simples cases be also solved by the *elimination method*.

Ex. 2. 10.
What combination of goods G_1 and G_2 should a firm produce to minimise costs when the joint cost function is

$$C(x_1,x_2)=6x_1^2+10x_2^2-x_1x_2+30 \quad (x_i,\ i=1,2:\ \text{amount of } G_i)$$

and the firm has a production quota

$$x_1+x_2=34\,?$$

Estimate the effect on costs if the production quota is reduced by 1 unit.

1. Solution by the elimination method:

$$x_2=34-x_1$$
$$\tilde{C}(x_1)=6x_1^2+10\left(34-x_1\right)^2-x_1\left(34-x_1\right)+30$$
$$\tilde{C}(x_1)=17x_1^2-714x_1+11590$$
$$\tilde{C}'(x_1)=34x-714$$
$$34x_1-714=0 \quad \Rightarrow \quad x_1=21, \qquad \tilde{C}''(x_1)=34>0,$$

Hence $\tilde{C}(x_1)$ assumes its relative minimum at $x_1=21$.

$$x_2=34-x_1=34-21=13.$$

$C\left(21,13\right)=4093.$

2. Solution by the method of Lagrange multipliers:

$$L\left(x_1,x_2;\lambda\right)=6x_1^2+10x_2^2-x_1x_2+30-\lambda\left(x_1+x_2-34\right)$$

$$L_{x_1}\left(x_1,x_2;\lambda\right)=12x_1-x_2-\lambda=0$$
$$L_{x_2}\left(x_1,x_2;\lambda\right)=-x_1+20x_2-\lambda=0$$
$$L_{\lambda}\left(x_1,x_2;\lambda\right)=x_1+x_2-33=0$$

$$\Rightarrow \quad x_1 = 21, \quad x_2 = 13, \quad \lambda = 239.$$

(The sufficiency condition will not be tested.)

With $\lambda = 239$, a decrease in the constant (the production quota) will lead to a cost increase of approximately 239.

(Last updated: 22.09.2011)